COMPLEXITY AND NONEXTENSIVE STATISTICAL MECHANICS
THEORY, EXPERIMENTS,
OBSERVATIONS AND COMPUTER SIMULATIONS

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S. Umarov, C. T., M. Gell-Mann and S. Steinberg, cond-mat/0603593, 0606038, 0606040
UBIQUITOUS LAWS IN COMPLEX SYSTEMS

ORDINARY DIFFERENTIAL EQUATIONS

PARTIAL DIFFERENTIAL EQUATIONS
(Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)

STOCHASTIC DIFFERENTIAL EQUATIONS
(Langevin, multiplicative noise)

CENTRAL LIMIT THEOREMS
(Gauss, Levy-Gnedenko)

NONLINEAR DYNAMICS
(Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)

SUPERSTATISTICS
(Other generalizations)

THERMODYNAMICS

AGING (metastability, glass, spin-glass)

LONG-RANGE INTERACTIONS
(Hamiltonians, coupled maps)

IMAGE PROCESSING

SIGNAL PROCESSING
(ARCH, GARCH)

GLOBAL OPTIMIZATION
(Simulated annealing)

SUPERSTATISTICS
(Other generalizations)

CENTRAL LIMIT THEOREMS
(Gauss, Levy-Gnedenko)

NONLINEAR DYNAMICS
(Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)

q-ALGEBRA

CORRELATIONS IN PHASE SPACE

GEOMETRY
(Scale-free networks)

FURTHER APPLICATIONS
(Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)

ENTROPY $S_q$
(Nonextensive statistical mechanics)

q-TRIPLET
J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

Page 35:

In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite value, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.
This is mainly because entropy is an additive quantity as the other ones. In other words, the entropy of a system composed of several independent parts is equal to the sum of entropy of each single part. [...] Therefore one considers all possible internal determinations as equally probable. This is indeed a new hypothesis because the universe, which is far from being in the same state indefinitely, is subjected to continuous transformations. We will therefore admit as an extremely plausible working hypothesis, whose far consequences could sometime not be verified, that all the internal states of a system are a priori equally probable in specific physical conditions. Under this hypothesis, the statistical ensemble associated to each macroscopic state $A$ turns out to be completely defined.
The values of \( p_i \) are determined by the following dogma: if the energy of the system in the \( i \)-th state is \( E_i \) and if the temperature of the system is \( T \) then:

\[
p_i = \frac{e^{-E_i/kT}}{Z(T)}, \quad \text{where} \quad Z(T) = \sum_i e^{-E_i/kT}
\]

(this last constant is taken so that \( \sum_i p_i = 1 \)).

This choice of \( p_i \) is called the Gibbs distribution. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as "deep and incompletely clarified".
**ENTROPIC FORMS**

<table>
<thead>
<tr>
<th>BG entropy ($q = 1$)</th>
<th>Concave</th>
<th>Extensive</th>
<th>Lesche-stable</th>
<th>Finite entropy production per unit time</th>
<th>Pesin-like identity (with largest entropy production)</th>
<th>Composable</th>
<th>Topsoe-factorizable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i = \frac{1}{W}$ ($\forall i$)</td>
<td>$k \ln W$</td>
<td>$-k \sum_{i=1}^{W} p_i \ln p_i$</td>
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<tr>
<td><strong>Nonextensive entropy</strong></td>
<td></td>
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</tr>
<tr>
<td>($q = 2$)</td>
<td>($q \neq 1$)</td>
<td>$k \frac{W^{1-q} - 1}{1 - q}$</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Possible generalization of Boltzmann-Gibbs statistical mechanics

$S_q(N,t)$ versus $t$
LOGISTIC MAP:

\[ x_{t+1} = 1 - a \ x_t^2 \quad (0 \leq a \leq 2; \ -1 \leq x_t \leq 1; \ t = 0,1,2,...) \]

(strong chaos, i.e., positive Lyapunov exponent)

We verify

\[ K_1 = \lambda_1 \quad (\text{Pesin–like identity}) \]

where

\[ K_1 \equiv \lim_{t \to \infty} \frac{S_1(t)}{t} \]

and

\[ \xi(t) \equiv \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} = e^{\lambda_1 t} \]
(weak chaos, i.e., zero Lyapunov exponent)

\[ S_q(t) = 1 - a \ x_i^2 \]

\[ a = 1.4011552 \]

\[ N = W = 2.5 \times 10^6 \]

\[ \# \text{realizations} = 15115 \]

E. Mayoral and A. Robledo, Phys. Rev. E 72, 026209 (2005), and references therein
THE CASATI-PROSEN TRIANGLE MAP:
G. Casati and T. Prosen,

“While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary.”

\[ y_{t+1} = y_t + \alpha \ \text{sgn}(x_t) + \beta \pmod{2} \]
\[ x_{t+1} = x_t + y_{t+1} \pmod{2} \]

(\(\alpha\) and \(\beta\) independent irrationals)

e.g., \((\alpha,\beta) = ((1/2)(\sqrt{5}-1)-(1/e), (1/2)(\sqrt{5}-1)+(1/e))\)

This map is conservative, mixing, ergodic and nevertheless with zero Lyapunov exponent!

Furthermore \( \xi \equiv \lim_{\Delta X(0) \to 0} \frac{\Delta X(t)}{\Delta X(0)} \propto t \)
CASATI-PROSEN TRIANGLE MAP [Casati and Prosen, Phys Rev Lett 83, 4729 (1999) and 85, 4261 (2000)]
(two-dimensional, conservative, mixing, ergodic, vanishing maximal Lyapunov exponent)

NONEXTENSIVITY OF THE CASATI-PROSEN MAP:

Answer to the above equation:

\[ \xi = \left[ 1 + (1-q) \lambda_q t \right]^{1/(1-q)} \]

It is not necessary: a meaningful statistical description is possible with zero Lyapunov exponent!

[Essentially because an integrable system has zero Lyapunov exponent but the opposite is not true]

In general, \( \xi = \left[ 1 + (1-q) \lambda_q t \right]^{1/(1-q)} \)

hence, \( \xi \propto t \Rightarrow q = 0 \)

Consistently, we expect

\[
(i) \quad S_q(t) \equiv \frac{1 - \sum_{i=1}^{W} [p_i(t)]^q}{q-1} \propto t \quad \text{only for } q = 0
\]

\[
(ii) \quad K_q \equiv \lim_{t \to \infty} \frac{S_q(t)}{t} = \lambda_q \quad \text{for } q = 0
\]
**CASATI-PROSEN TRIANGLE MAP** [Casati and Prosen, Phys Rev Lett 83, 4729 (1999) and 85, 4261 (2000)]
(two-dimensional, conservative, mixing, ergodic, **vanishing maximal Lyapunov exponent**)

\[ W = 4000 \times 4000 \text{ cells} \]
\[ N = 1000 \text{ initial conditions randomly chosen in one cell} \]
\[ \text{Average done over 100 initial cells} \]

\[ [q = 0 \rightarrow \text{linear correlation} = 0.99993] \]

Also \( \xi = e^{\lambda_0 t} \)

with \( \lambda_0 = \lim_{n \to \infty} \frac{S_0(n)}{n} = 1 \)

\( q - \text{generalization of Pesin (- like) theorem} \)

$S_q(N, t) \text{ versus } N$
HYBRID PASCAL - LEIBNITZ TRIANGLE

(N = 0) 
1 × \frac{1}{1}

(N = 1) 
1 × \frac{1}{2} \quad 1 × \frac{1}{2}

(N = 2) 
1 × \frac{1}{3} \quad 2 × \frac{1}{6} \quad 1 × \frac{1}{3}

(N = 3) 
1 × \frac{1}{4} \quad 3 × \frac{1}{12} \quad 3 × \frac{1}{12} \quad 1 × \frac{1}{4}

(N = 4) 
1 × \frac{1}{5} \quad 4 × \frac{1}{20} \quad 6 × \frac{1}{30} \quad 4 × \frac{1}{20} \quad 1 × \frac{1}{5}

(N = 5) 
1 × \frac{1}{6} \quad 5 × \frac{1}{30} \quad 10 × \frac{1}{60} \quad 10 × \frac{1}{60} \quad 5 × \frac{1}{30} \quad 1 × \frac{1}{6}

\sum = 1 \quad (\forall \ N)

Blaise Pascal (1623-1662)
Gottfried Wilhelm Leibnitz (1646-1716)
Daniel Bernoulli (1700-1782)
EQUIVALENTLY:

\[
\begin{array}{c|cc}
\text{A} & 1 & 2 \\
\hline
1 & p^2 + \kappa & p(1-p) - \kappa & p \\
2 & p(1-p) - \kappa & (1-p)^2 + \kappa & 1 - p \\
\end{array}
\]

\[
\begin{array}{c}
(N=2) \\
\end{array}
\]

\[
\begin{array}{c}
(N=0) \\
1 \times 1 \\
\end{array}
\]

\[
\begin{array}{c}
(N=1) \\
1 \times p & 1 \times (1-p) \\
\end{array}
\]

\[
\begin{array}{c}
(N=2) \\
1 \times [p^2 + \kappa] & 2 \times [p(1-p) - \kappa] & 1 \times [(1-p)^2 + \kappa] \\
\end{array}
\]
$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N$ ($N \to \infty$)

Leibnitz triangle

$$\left( p_{N,0} = \frac{1}{N+1} \right)$$

$N$ independent coins

$$\left( p_{N,0} = p^N \right)$$

with $p = 1/2$

Stretched exponential

$$\left( p_{N,0} = p^{N\alpha} \right)$$

with $p = \alpha = 1/2$

(All three examples strictly satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA 102, 15377 (2005)
Asymptotically scale-invariant (d=2)

\[
\begin{array}{ccc}
(N = 0) & 1 \\
(N = 1) & 1/2 & 1/2 \\
(N = 2) & 1/3 & 1/6 & 1/3 \\
(N = 3) & 3/8 & 5/48 & 5/48 & 0 \\
(N = 4) & 2/5 & 3/40 & 1/20 & 0 & 0 \\
\end{array}
\]

(It asymptotically satisfies the Leibniz rule)
$q \neq 1 \text{ SYSTEMS}$

i.e., such that $S_q(N) \propto N \ (N \to \infty)$

$q = 1 - \frac{1}{d}$

(All three examples \textit{asymptotically} satisfy the \textit{Leibnitz rule})

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA \textbf{102}, 15377 (2005)
C.T., M. Gell-Mann and Y. Sato
Europhysics News 36 (6), 186 (2005) [European Physical Society]
If $A$ and $B$ are independent, i.e., if $p_{ij}^{A+B} = p_i^A p_j^B$, then
\[ S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B) \]
whereas
\[ S_q(A + B) = S_q(A) + S_q(B) + \frac{1-q}{k_B} S_q(A) S_q(B) \neq S_q(A) + S_q(B) \quad (\text{if } q \neq 1) \]

But if $A$ and $B$ are especially (globally) correlated then
\[ S_q(A + B) = S_q(A) + S_q(B) \]
whereas
\[ S_{BG}(A + B) \neq S_{BG}(A) + S_{BG}(B) \]
NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS
(CANONICAL ENSEMBLE):

Extremization of the functional

\[ S_q[p_i] \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \]

with the constraints

\[ \sum_{i=1}^{W} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{W} p_i^q E_i = U_q \]

yields

\[ p_i = \frac{e^{-\beta_q(E_i - U_q)}}{\mathcal{Z}_q} \]

with \( \beta_q = \frac{\beta}{\sum_{i=1}^{W} p_i^q} \), \( \beta \equiv \text{energy Lagrange parameter} \), and \( \mathcal{Z}_q \equiv \sum_{i=1}^{W} e^{-\beta_q(E_i - U_q)} \).
We can rewrite

\[ p_i = \frac{e^{-\beta_q' E_i}}{Z_q} \]

with

\[ \beta_q' \equiv \frac{\beta_q}{1 + (1-q)\beta_q U_q} \]

and

\[ Z_q' \equiv \sum_{i=1}^{w} e^{-\beta_q' E_i} \]

And we can prove

(i) \[ \frac{1}{T} = \frac{\partial S_q}{\partial U_q} \]

with \[ T \equiv \frac{1}{k \beta} \]

(ii) \[ F_q \equiv U_q - TS_q = -\frac{1}{\beta} \ln_q Z_q \]

where \[ \ln_q Z_q = \ln_q Z_q' - \beta U_q \]

(iii) \[ U_q = -\frac{\partial}{\partial \beta} \ln_q Z_q \]

(iv) \[ C_q \equiv T \frac{\partial S_q}{\partial T} = \frac{\partial U_q}{\partial T} = -T \frac{\partial^2 F_q}{\partial T^2} \]

(i.e., the Legendre structure of Thermodynamics is \( q \)-invariant!)
**SOME FORM-INVARIANT RELATIONS** (arbitrary $q$)

**CLAUSIUS INEQUALITY AND BOLTZMANN H-THEOREM**
(macroscopic time irreversibility)
Abe and Rajagopal, Phys Rev Lett 91 (2003)

$$\beta \delta Q_q \leq \delta S_q ; \quad q \frac{dS_q}{dt} \geq 0$$

**EHRENFEST THEOREM** (correspondence principle)
Plastino and Plastino, Phys Lett A 177 (1993) 177

$$\frac{d}{dt} \langle \hat{O} \rangle_q = \frac{i}{\hbar} \left\langle \left[ \hat{H}, \hat{O} \right] \right\rangle_q$$

**FACTORIZATION OF LIKELIHOOD FUNCTION**
(Einstein’s 1910 reversal of Boltzmann’s formula; thermodynamically independent systems)
Caceres and Tsallis, unpublished (1993); Chame and Mello,

$$W_q(A + B) = W_q(A) W_q(B)$$

**ONSAGER RECIPROCITY THEOREM**
(microscopic time reversibility)
Chame and Mello, Phys Lett A 228 (1997) 159

$$L_{jk} = L_{kj}$$

**KRAMERS AND KRONIG RELATION** (causality)
Rajagopal, Phys Rev Lett 76 (1996) 3469

**PESIN EQUALITY**
(mixing; Kolmogorov-Sinai entropy and Lyapunov exponent)
Tsallis, Plastino and Zheng, Chaos, Solitons and Fractals 8 (1997) 885;

$$K_q = \begin{cases} \hat{\lambda}_q & \text{if } \hat{\lambda}_q > 0 \\ 0 & \text{otherwise} \end{cases}$$
Recent minireviews:
(Europhysics News, Nov-Dec 2005, European Physical Society)

http://www.europhysicsnews.com

Full bibliography:
(28 August 2006: 1953 manuscripts)

http://tsallis.cat.cbpf.br/biblio.htm
**A PRIORI CALCULATION OF $q$**

From deterministic, microscopic-like, dynamics:

- Low-dimensional (d=1, 2) dissipative maps
  
  e.g., $q = 0.2445 \ldots$ (z=2 logistic map universality class)

- Low-dimensional (d=2) conservative maps
  
  e.g., $q = 0$ (Casati-Prosen triangle map)

- Quantum transport in optical lattice
  
  $q = 1 + (44 E_r / U_o)$

- Long-range many-body classical Hamiltonians
  
  Towards $q = f(\alpha/d)$ \(\lim_{t \to \infty} \lim_{N \to \infty} \neq \lim_{N \to \infty} \lim_{t \to \infty}\) [metaequilibrium equilibrium(BG)]

From stochastic, mesoscopic-like, dynamics:

- Langevin equation with multiplicative noise
  
  $q = (3 M + \gamma) / (M + \gamma)$

- Langevin equation with colored dichotomic noise
  
  $q = [1 - 2 (\gamma / \lambda)] / [1 - (\gamma / \lambda)]$

- Nonlinear Fokker-Planck equation (correlated anomalous diffusion)
  
  $q = 2 - \nu$

- Fractional Fokker-Planck equation (Lévy anomalous diffusion)
  
  $q = (3 + \gamma_t) / (1 + \gamma_t)$ \{asymptotic behavior\}

- Nonlinear fractional Fokker-Planck equation
  
  $q = (3 + \gamma) / (1 + \gamma) = (5 + 2 \nu) / 3$ \{asymptotic behavior\}

- Fluctuating temperature (superstatistics)
  
  $q = [\alpha (n + 2) + 1] / [\alpha n + 1]$

- Lattice Lotka-Volterra model (d-dimensional growth)
  
  $q = 1 - 1/d$

- Boltzmann d-dimensional Bravais lattice models for the Navier – Stokes equations for incompressible fluids
  
  e.g., $q = 1 - 2/d$ (single speed, single mass)

- Bubbling fluidized beds
  
  $q = 1 + 1 / [\tau - (1/2)]$

- Growth of many-body scale-free networks
  
  $q = [2 m (2 - r) + 1 - p - r] / [m (3 - 2 r) = 1 - p - r]$
It can be proved that

\[ K_q = \lambda_q \] (q–generalized Pesin–like identity)

where

\[ K_q \equiv \lim_{t \to \infty} \sup \left\{ \frac{S_q(t)}{t} \right\} \]

and

\[ \xi(t) \equiv \sup \left\{ \lim_{\Delta x(0) \to 0} \frac{\Delta x(t)}{\Delta x(0)} \right\} = e^{\lambda_q t} \]

with

\[ \frac{1}{1-q} = \frac{1}{\alpha_{\text{min}}} - \frac{1}{\alpha_{\text{max}}} = \frac{\ln \alpha_F}{\ln 2} \quad \text{and} \quad \lambda_q = \frac{1}{1-q} \]

\[
\begin{bmatrix}
x_{t+1} = 1-a | x_t |^z & \Rightarrow & \frac{1}{1-q(z)} = \frac{1}{\alpha_{\text{min}}(z)} - \frac{1}{\alpha_{\text{max}}(z)} = (z-1) \frac{\ln \alpha_F(z)}{\ln 2}
\end{bmatrix}
\]
Generic pitchfork bifurcations:
\[ x_{t+1} = x_t + b \, \text{sign}(x_t) \, |x_t|^z \ (z > 1; \ b > 0) \]

Generic tangent bifurcations:
\[ x_{t+1} = x_t + b \, |x_t|^z \ (z > 1; \ b > 0) \]

*The fixed point map is a q-exponential with*
\[ q = z \]

*and the sensitivity to the initial conditions is a q_{sen} exponential with*
\[ q_{sen} = 2 - \frac{1}{q} \]

*Example: The \( \xi \)-logistic family of maps*
\[ x_{t+1} = 1 - a \, |x_t|^\xi \ (\xi > 1; \ 0 \leq a \leq 2; \ \xi > 1) \]

*has*
\[ z = 3 \ \text{for pitchfork bifurcations} \ (\forall \xi), \ \text{hence} \ q = 3 \ \text{and} \ q_{sen} = \frac{5}{3} ; \]
\[ z = 2 \ \text{for tangent bifurcations} \ (\forall \xi), \ \text{hence} \ q = 2 \ \text{and} \ q_{sen} = \frac{3}{2} . \]

DEFINITIONS:

$q$–logarithm:

\[ \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \quad (\ln_1 x = \ln x) \]

$q$–exponential:

\[ e_q^x \equiv [1 + (1-q)x]^{1-q} \quad (e_1^x = e^x) \]

(if \quad 1 + (1-q)x > 0; \quad \text{vanishes otherwise})
q-GAUSSIANS: \( p_q(x) \propto e_{q}^{-(x/\sigma)^2} = \frac{1}{\left[1+(q-1)(x/\sigma)^2\right]^{\frac{1}{q-1}}} \) (\( q < 3 \))
**q - CENTRAL LIMIT THEOREM:** (conjecture)

\[
\frac{\partial p(x,t)}{\partial t} = D \frac{\partial^{\gamma}[p(x,t)]^{2-q}}{\partial |x|^\gamma} \quad (0 < \gamma \leq 2; \quad q < 3)
\]

- Globally correlated variables; finite q-variance; q-Gaussian attractor
- Independent variables; finite variance; Gaussian attractor
- Independent variables; divergent variance; Levy attractor

C.T., Milan J. Math. 73, 145 (2005)
q - CENTRAL LIMIT THEOREM (q-product and de Moivre-Laplace theorem):

The q-product is defined as follows:

\[ x \otimes_q y \equiv \left[ x^{1-q} + y^{1-q} - 1 \right]^{1/(1-q)} \]

Properties:

i) \( x \otimes_1 y = x y \)

ii) \( \ln_q (x \otimes_q y) = \ln_q x + \ln_q y \)

[whereas \( \ln_q (x \cdot y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y) \)]


The de Moivre-Laplace theorem can be constructed with

\[ p_{N,0} = p^N \quad \text{with} \quad p = 1/2 \]

and

Leibnitz rule
q - CENTRAL LIMIT THEOREM: (numerical indications)

We q - generalize the de Moivre–Laplace theorem with

\[
\frac{1}{p_{N,0}} = \left( \frac{1}{p} \right) \otimes_q \left( \frac{1}{p} \right) \otimes_q \ldots \left( \frac{1}{p} \right) \quad (N \text{ terms})
\]

i.e.,

\[
p_{N,0} = \left[ N p^{q-1} - (N - 1) \right]^{1/(q-1)} \quad (\text{with } p = 1/2)
\]

\[\text{(}q = 3/10)\]

[Hence \( q \rightarrow 2 - q \) (additive duality) and \( q \rightarrow 1/q \) (multiplicative duality) are involved]

**q - GENERALIZED CENTRAL LIMIT THEOREM:**  
(mathematical proof)

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

\[
F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix_\xi} \otimes_q f(x) \, dx = \int_{-\infty}^{\infty} e_q^{[f(x)]^{1-q}} f(x) \, dx \quad \text{(nonlinear!)}
\]

q-correlation:

Two random variables \(X\) [with density \(f_X(x)\)] and \(Y\) [with density \(f_Y(y)\)] are said q-correlated if

\[
F_q[X+Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi),
\]

i.e., if

\[
\int_{-\infty}^{\infty} dz \, e_q^{iz_\xi} \otimes_q f_{X+Y}(z) = \left[ \int_{-\infty}^{\infty} dx \, e_q^{ix_\xi} \otimes_q f_X(x) \right] \otimes_q \left[ \int_{-\infty}^{\infty} dy \, e_q^{iy_\xi} \otimes_q f_Y(y) \right],
\]

with \(f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, h(x, y) \delta(x+y-z) = \int_{-\infty}^{\infty} dx \, h(x, z-x) = \int_{-\infty}^{\infty} dy \, h(z-y, y)\)

where \(h(x, y)\) is the joint density.

\[
\left\{ \begin{array}{ll}
q\text{-correlation means independence} & \text{if } q = 1, \text{ i.e., } h(x, y) = f_X(x) f_Y(y) \\
\text{global correlation} & \text{if } q \neq 1, \text{ hence } h(x, y) \neq f_X(x) f_Y(y)
\end{array} \right.
\]
\[ q - \text{FourierTransform} \left[ \frac{\sqrt{\beta}}{C_q} e_q^{-\beta \ t^2} \right] = e_q^{-\beta_1 \ \omega^2} \]

where \[ q_1 = \frac{1+q}{3-q} \]

and \[ \beta_1 = \frac{3-q}{8 \beta^{2-q} C_q^{2(1-q)}} \]

with \[ C_q = \begin{cases} \sqrt{\pi} & \text{if } q = 1 \\ \frac{2\sqrt{\pi} \Gamma \left( \frac{1}{q-1} \right)}{(3-q) \sqrt{(1-q)} \Gamma \left( \frac{3-q}{2(1-q)} \right)} & \text{if } q < 1 \\ \frac{\sqrt{\pi} \Gamma \left( \frac{3-q}{2(q-1)} \right)}{\sqrt{q-1} \Gamma \left( \frac{1}{q-1} \right)} & \text{if } 1 < q < 3 \end{cases} \]
\[ G_q(t) \]

\[ q = 1.5 \]
\[ \beta = 0.5 \]

\[ \ln_q \left( \frac{G(t)}{G(0)} \right) \]

\[ t^2 \]

\[ F_q \left[ G_b(\omega) \right] \]

\[ q_1 = \frac{5}{3} \]
\[ \beta_1 = 0.119 \]

\[ \ln_q \left[ F_b \left[ G(\omega) \right] \right] \]

\[ \omega^2 \]
Closure:

The $q$-Fourier transform of a $q$-Gaussian is a $z(q)$-Gaussian with

$$z(q) = \frac{1+q}{3-q} \in (-\infty,3)$$

Iteration:

$$q_n \equiv z_n(q) \equiv z(z_{n-1}(q)) = \frac{2q + n(1-q)}{2 + n(1-q)} \quad (n = 0, \, \pm 1, \, \pm 2, \ldots; \, q_0 = q)$$

(the same as in R.S. Mendes and C.T. [Phys Lett A 285, 273 (2001)] when calculating marginal probabilities!)

hence

(i) $q_n(1) = 1 \, (\forall n), \quad q_{\pm n}(q) = 1 \, (\forall q),$

(ii) $q_{n-1} = 2 - \frac{1}{q_{n+1}},$

(the same as in L.G. Moyano, C.T. and M. Gell-Mann (2005)!)  
(the same as in A. Robledo [Physica D 193, 153 (2004)] for pitchfork and tangent bifurcations!)

(iii) $n = 2m = 0, \, \pm 2, \, \pm 4, \ldots \text{ yields } q_{(m)} = q_{2m} = \frac{q + m(1-q)}{1 + m(1-q)}$

(the same obtained in C.T., M. Gell-Mann and Y. Sato [Proc Natl Acad Sci (USA) 102, 15377 (2005)], by combining only additive and multiplicative dualities, and which was conjectured to be a possible explanation for the NASA-detected $q$-triangle for $m = 0, \, \pm 1$!)
\[ \frac{\alpha}{1 - q_{\alpha,n}} = \frac{\alpha}{1 - q} + n \]

\[ (n = 0, \pm 1, \pm 2, \ldots) \]
A random variable $X$ is said to have a $(q, \alpha)$-stable distribution $L_{q,\alpha}(x)$ if its $q$-Fourier transform has the form $a \, e_{q}^{-b \, |\xi|^\alpha}$ ($a > 0, \, b > 0, \, 0 < \alpha \leq 2$)

i.e., if

$$F_{q}[L_{q,\alpha}](\xi) \equiv \int_{-\infty}^{\infty} e^{i \xi x} \otimes_{q} L_{q,\alpha}(x) \, dx = \int_{-\infty}^{\infty} e_{q}^{(L_{q,\alpha}(x))^{-q}} L_{q,\alpha}(x) \, dx = a \, e_{q}^{-b \, |\xi|^\alpha}$$

$L_{1,2}(x) \equiv G(x)$ (Gaussian)

$L_{1,\alpha}(x) \equiv L_{\alpha}(x)$ ($\alpha$ - stable Levy distribution)

$L_{q,2}(x) \equiv G_{q}(x)$ ($q$ - Gaussian)


cond-mat/0606038
cond-mat/0606040
**CENTRAL LIMIT THEOREMS:** \( N^{1/\alpha(2-q)} \) - **SCALED ATTRACTOR** \( F(x) \) **WHEN SUMMING** \( N \to \infty \) 

**q - CORRELATED IDENTICAL RANDOM VARIABLES WITH SYMMETRIC DISTRIBUTION** \( f(x) \)

<table>
<thead>
<tr>
<th>( q = 1 )</th>
<th>( q = 1 ) (i.e., ( Q = 2q - 1 \neq 1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[independent]</td>
<td>[globally correlated]</td>
</tr>
<tr>
<td>( \sigma_Q &lt; \infty ) ((\alpha = 2))</td>
<td>( \sigma_Q \to \infty ) ((0 &lt; \alpha &lt; 2))</td>
</tr>
<tr>
<td>( \mathbb{F}(x) = \text{Gaussian } G(x), ) ( \text{with same } \sigma_1 \text{ of } f(x) )</td>
<td>( \mathbb{F}(x) = G_{3q-1}(x) = \frac{3q-1}{q+1} \text{-Gaussian,} ) ( \text{with same } \sigma_Q \left[ \equiv \int dx , x^2[f(x)]^Q / \int dx , [f(x)]^Q \right] \text{ of } f(x) )</td>
</tr>
<tr>
<td>( \text{Classic CLT} )</td>
<td>( \mathbb{F}(x) = G_{3q-1}(x) = \frac{3q-1}{q+1} \text{-Gaussian,} ) ( \text{with same } \sigma_Q \left[ \equiv \int dx , x^2[f(x)]^Q / \int dx , [f(x)]^Q \right] \text{ of } f(x) )</td>
</tr>
<tr>
<td>( \text{with } \lim_{q \to 1} x_c(q,2) = \infty )</td>
<td>( \mathbb{F}(x) = G_{3q-1}(x) = \frac{3q-1}{q+1} \text{-Gaussian,} ) ( \text{with same } \sigma_Q \left[ \equiv \int dx , x^2[f(x)]^Q / \int dx , [f(x)]^Q \right] \text{ of } f(x) )</td>
</tr>
<tr>
<td>( \mathbb{F}(x) = \text{Levy distribution } L_\alpha(x), ) ( \text{with same } \mid x \mid \to \infty )</td>
<td>( \mathbb{F}(x) = L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1},\alpha} ) ( \text{stable distribution}, ) ( \text{with } L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1},\alpha} \sim f(x) \sim C_q^{(L)} / \mid x \mid^{(1+\alpha) / (1+\alpha q-\alpha)} ) ( \text{or} )</td>
</tr>
<tr>
<td>( \text{asymptotic behavior} ) ( \left{ \begin{align*} \approx G(x) &amp; \quad \text{if } \mid x \mid &lt;&lt; x_c(1,\alpha) \ \sim f(x) &amp; \sim C_\alpha / \mid x \mid^{1+\alpha} \quad \text{if } \mid x \mid &gt;&gt; x_c(1,\alpha) \end{align*} \right. )</td>
<td>( \mathbb{F}(x) = L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1},\alpha} ) ( \text{stable distribution}, ) ( \text{with } L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1},\alpha} \sim f(x) \sim C_q^{(*)} / \mid x \mid^{2(\alpha + q - 1) / \alpha (q-1)} )</td>
</tr>
<tr>
<td>( \text{with } \lim_{\alpha \to 2} x_c(1,\alpha) = \infty )</td>
<td>( S. U m a r o v, ~ C. T., ~ M. G e l l-M a n n ) ( \text{and } S. ~ S t e i n b e r g ) ( (2006) ) ( [\text{cond-mat/0603593}] )</td>
</tr>
</tbody>
</table>
WHAT IS IT $q$–CORRELATION?:

It appears to be (no proof available yet)

$$\int dx_N h(x_1, x_2, \ldots, x_N) = h(x_1, x_2, \ldots, x_{N-1})$$

i.e., scale invariance!
**BOLTZMANN-GIBBS STATISTICAL MECHANICS**
(Maxwell 1860, Boltzmann 1872, Gibbs ≤ 1902)

Entropy
\[ S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i \]

Internal energy
\[ U_{BG} = \sum_{i=1}^{W} p_i E_i \]

Equilibrium distribution
\[ p_i = e^{-\beta E_i} / Z_{BG} \quad \left( Z_{BG} = \sum_{j=1}^{W} e^{-\beta E_j} \right) \]

Paradigmatic differential equation
\[ \frac{dy}{dx} = ay \quad \begin{cases} y(0) = 1 \end{cases} \Rightarrow \quad y = e^{ax} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( a )</th>
<th>( y(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilibrium distribution</td>
<td>( E_i )</td>
<td>(-\beta)</td>
</tr>
<tr>
<td>Sensitivity to initial conditions</td>
<td>( t )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>Typical relaxation of observable ( O )</td>
<td>( t )</td>
<td>(-1/\tau)</td>
</tr>
</tbody>
</table>

\( S_{BG} \to \) extensive, concave, Lesche-stable, finite entropy production
NONEXTENSIVE STATISTICAL MECHANICS

Entropy

\[ S_q = k \left( 1 - \sum_{i=1}^{W} p_i^q \right) / (q - 1) \]

Internal energy

\[ U_q = \sum_{i=1}^{W} p_i^q E_i / \sum_{j=1}^{W} p_j^q \]

Stationary state distribution

\[ p_i = e_{-\beta_q(E_i-U_q)} / Z_q \]

Paradigmatic differential equation

\[ \frac{dy}{dx} = ay^q \]
\[ y(0) = 1 \]

\[ y = e_{a x} \equiv \left[ 1 + (1 - q) ax \right]^{1/(1-q)} \]

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( a )</th>
<th>( y(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stationary state distribution</td>
<td>( E_i )</td>
<td>( -\beta_{q_{stat}} )</td>
<td>( Z_{q_{stat}} p(E_i) )</td>
</tr>
<tr>
<td>Sensitivity to initial conditions</td>
<td>( t )</td>
<td>( \lambda_{q_{sen}} )</td>
<td>( \xi = e^{\lambda_{q_{sen}} t} )</td>
</tr>
<tr>
<td>Typical relaxation of observable O</td>
<td>( t )</td>
<td>( -1 / \tau_{q_{rel}} )</td>
<td>( \Omega = e^{-t/\tau_{q_{rel}}} )</td>
</tr>
</tbody>
</table>

\( S_q \rightarrow \text{extensive, concave, Lesche-stable, finite entropy production} \)


Fig. 2. The triangle of the basic values of $q$, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leq 1$, $q_{rel} \geq 1$ and $q_{stat} \geq 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent $\alpha$ and the dimension $d$, it could be that $q_{stat}$ decreases from a value above unity (e.g., 2 or $\frac{3}{2}$) to unity when $\alpha/d$ increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for $q_{sen}$ and $q_{rel}$, but only those associated with a Hamiltonian are expected to also have a well defined value for $q_{stat}$. 
SOLAR WIND: Magnetic Field Strength


[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]

\[ q_{\text{sen}} = -0.6 \pm 0.2 \]

\[ q_{\text{rel}} = 3.8 \pm 0.3 \]

\[ q_{\text{stat}} = 1.75 \pm 0.06 \]
Playing with additive duality \((q \to 2 - q)\)
and with multiplicative duality \((q \to 1/q)\)
(and using numerical results related to the \(q\)–generalized central limit theorem)

we conjecture

\[
q_{rel} + \frac{1}{q_{sen}} = 2 \quad \text{and} \quad q_{stat} + \frac{1}{q_{rel}} = 2
\]

hence

\[
1 - q_{sen} = \frac{1 - q_{stat}}{3 - 2q_{stat}}
\]

hence only one independent!

Burlaga and Vinas (NASA) most precise value of the \(q\)–triplet is

\[
q_{stat} = 1.75 = 7/4
\]

hence

\[
q_{sen} = -0.5 = -1/2 \quad \text{(consistent with } q_{sen} = -0.6 \pm 0.2 !)\]

and

\[
q_{rel} = 4 \quad \text{(consistent with } q_{rel} = 3.8 \pm 0.3 !)\]

C.T., M. Gell-Mann and Y. Sato

Proc Natl Acad Sc USA 102, 15377 (2005)
The solution of
\[
\frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} \quad [p(x,0) = \delta(0)] \quad (q < 3)
\]
is given by
\[
p(x,t) \propto \left[1 + (1-q) x^2 / \left(\Gamma t\right)^{2/(3-q)}\right]^{1/(1-q)} \equiv e_q^{-x^2 / \left(\Gamma t\right)^{2/(3-q)}} \quad (\Gamma \propto D)
\]
hence
\[
x^2 \text{ scales like } t^\gamma \quad (e.g., \quad \langle x^2 \rangle \propto t^\gamma)
\]
with
\[
\gamma = \frac{2}{3-q}
\]
(e.g., \( q = 1 \Rightarrow \gamma = 1 \), i.e., normal diffusion)

Hydra viridissima: A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada

Physica A 293, 549 (2001)

q = 1.5
slope $\gamma = 1.24 \pm 0.1$

hence $\gamma = \frac{2}{3-q}$ is satisfied
Defect turbulence:

\( q \approx 1.5 \) and \( \gamma \approx 4/3 \) are consistent with \( \gamma = \frac{2}{3-q} \)

XY FERROMAGNET WITH LONG-RANGE INTERACTIONS:

A. Rapisarda and A. Pluchino,
Europhys News 36, 202
(European Physical Society, Nov/Dec 2005)
XY FERROMAGNET WITH LONG-RANGE INTERACTIONS:


Fig. 2 – Normalized PDFs for (a) vertical and (b) horizontal displacements. The symbols indicate the orifice aperture in the silo: circles for 3.8\,d and squares for 11\,d. The dotted line is a gaussian and the continuous line is Eq. (5).

Fig. 4 – Normalized PDFs for (a) vertical and (b) horizontal displacements for the fully developed discharge regime. Both cases fit well to a gaussian profile. Nevertheless, some skewness, similar to the experimental results (particularly in the case of the 11\,d orifice) can be observed for the vertical direction. It is probably related to the difficulty of adequately defining the mean value of the vertical velocity.
slope $\gamma = \frac{4}{3}$

hence $\gamma = \frac{2}{3 - q}$ is satisfied
COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:


(i) The distribution of atomic velocities is a $q$-Gaussian;

(ii) $q = 1 + \frac{44 E_R}{U_0}$  

where  $E_R \equiv$ recoil energy  

$U_0 \equiv$ potential depth
Experimental and computational verifications

Computational verification: quantum Monte Carlo simulations

\[ q = 1 + \frac{44E_R}{U_0} \]

Experimental verification:

(a) \( W(p) \)

(b) \( \omega_N/(2\pi) \) kHz vs. \( p/p_r \)

(b) \( q \) vs. \( U_0/E_r \)

(Computational verification: quantum Monte Carlo simulations)

(Experimental verification)
HADRONIC JETS FROM ELECTRON-POSITRON ANNIHILATION:


Fig. 1. Transverse momentum distribution. The distribution \((1/\sigma)d\sigma/dp_t\) of the transverse momentum \(p_t\) of charged hadrons with respect to jet axis (defined in these experimental results as the sphericity axis) is sketched for four different experiments, whose center-of-mass energies vary from 14 and 34 GeV (TASSO) up to 91 and 161 GeV (DELPHI). The Hagedorn predicted exponential behavior is shown by the dotted line. We can see that the deviation of the exponential behavior increases when the energy increases. The continuous lines are obtained from our Eq. (3) and agree very well with the experimental data. The inset shows the transverse momentum distribution for small values of \(p_t\).
Dynamical correlations as origin of nonextensive entropy

T. Kodama\textsuperscript{1}, H.-T. Elze\textsuperscript{1}, C. E. Aguiar\textsuperscript{1} and T. Koide\textsuperscript{2}(*)

\textsuperscript{1} Instituto de Física, Universidade Federal do Rio de Janeiro
C.P. 68528, 21945-970 Rio de Janeiro, RJ, Brazil
\textsuperscript{2} Institut für Theoretische Physik, University of Frankfurt - Frankfurt, Germany

(Phenomenological model for collisions in a diluted gas with probability $r$ of forming clusters of $q$ correlated particles)
Connections with asymptotically scale-free networks
GEOGRAPHIC PREFERENTIAL ATTACHMENT GROWING NETWORK: 
THE NATAL MODEL


(1) Locate site $i=1$ at the origin of say a plane

(2) Then locate the next site with

$$P_G \propto 1/r^{2+\alpha_G} \quad (\alpha_G \geq 0)$$

($r \equiv$ distance to the baricenter of the pre-existing cluster)

(3) Then link it to only one of the previous sites using

$$p_A \propto k_i / r_i^{\alpha_A} \quad (\alpha_A \geq 0)$$

($k_i \equiv$ links already attached to site $i$)

($r_i \equiv$ distance to site $i$)

4) Repeat
\( (\alpha_G = 1; \alpha_A = 1; N = 250) \)
\[
P(k)/P(0) = e_q^{-k/\kappa} \\
\equiv 1/[1 + (q - 1)k/\kappa]^{1/(q-1)}
\]

D.J.B. Soares, C. T., A.M. Mariz and L.R. Silva
Europhys Lett 70, 70 (2005)
Barabasi-Albert universality class

\[ q = 1 + \left( \frac{1}{3} \right) e^{-0.526 \alpha_A} \]

\( \forall \alpha_G \)

\[ \kappa = 0.083 + 0.092 \alpha_A \]

\( \forall \alpha_G \)
ASTROPHYSICS
**FLUXES OF COSMIC RAYS**


---

![Graph of cosmic ray flux vs. energy](image)

- **Boltzmann–Gibbs**
  - $1/\beta = 1.67 \times 10^6$ eV
  - $A = 3.0 \times 10^{-13}$

- **Knee**
  - $[\approx 1 \text{ particle / (m}^2 \text{ year)]}$
  - $E_{\text{crossover}} = 8.32 \times 10^{15}$

- **Ankle**
  - $[\approx 1 \text{ particle / (km}^2 \text{ year)]}$

- **Values**
  - $q = 1.225$
  - $1/\beta_q = 9.615 \times 10^7$ eV
  - $q^- = 1.185$
  - $1/\beta_q = 1.562 \times 10^9$ eV
  - $A = 5.3 \times 10^{-13}$
COSMIC MICROWAVE BACKGROUND RADIATION: TEMPERATURE FLUCTUATIONS
(Band Q: 22.8 GHz) (Band V: 60.8 GHz) (Band W: 93.5 GHz)

(Data after using Kp0 mask) \( q = 1.045 \pm 0.005 \) (99 \% confidence level)

$q = 1.045 \pm 0.005$ (99\% confidence level)

GRavitational emission from a Black Hole

[Oliveira and Damiao Soares, Phys. Rev. D 70, 084041(2004)]

\[ \Delta \equiv \text{fraction of mass extracted} \equiv \frac{M_{\text{init}} - M_{\infty}}{M_{\text{init}}} \]

\[ y \equiv \text{dimensionless initial mass} \propto M_{\text{init}} \]

\[ \ln \Delta \]

\[ \Delta = \frac{C_0 (1 - y)^\alpha}{\left[1 + (q-1)\lambda_2 (1 - y)\right]^{1/(q-1)}} \]

3rd initial data family: \( q = 1.45 \)

1st initial data family: \( q = 1.73 \)
Connections with conservative and Hamiltonian systems
\[ V(\vec{r}) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \]

\[ (A > 0, \quad \alpha \geq 0) \]

integrable if \( \alpha / d > 1 \) (short-ranged)
non-integrable if \( 0 \leq \alpha / d \leq 1 \) (long-ranged)
MANY COUPLED STANDARD MAPS:

\[ p_i(t+1) = p_i(t) + \frac{a}{2\pi} \sin[2\pi\theta_i(t)] + \]
\[ \frac{b}{2\pi \tilde{N}} \sum_{j=1}^{N} \sin\left\{2\pi \left[\theta_i(t) - \theta_j(t)\right]\right\} \quad \text{(mod 1)} \]

\[ \theta_i(t+1) = \theta_i(t) + p_i(t+1) \quad \text{(mod 1)} \]

with
\[ \tilde{N} \equiv \frac{N^{1-\alpha/d} - 1}{1 - \alpha / d} ; a \geq 0 ; b \geq 0 ; \alpha \geq 0 ; d = 1 \]
FIG. 4: Upper panel: Temperature evolution for $\alpha = 2$ and $\alpha = 0.6$ and four system sizes $N = 100, 400, 1000, 4000$. Initial conditions correspond to $p_0 = 0.3$ and $\delta p = 0.05$. Fixed constants are $a = 0.05$ and $b = 2$. For $\alpha = 2$ the four curves coincide almost completely, all having a very fast relaxation to $T_{BG}$. For $\alpha = 0.6$ the same sizes are shown, growing in the direction of the arrow. Left bottom panel: crossover time $t_c$ vs. $N$, showing a power-law dependence $t_c \sim N^{\beta(\alpha)}$ with $\beta(\alpha) > 0$. Right bottom panel: $\beta$ vs $\alpha$ shows that for long-range interactions the QS state life-time diverges in the thermodynamic limit. Note that when $\alpha = 0$, $\beta = 1$, and hence $t_c \propto N$. 

$d$-DIMENSIONAL CLASSICAL INERTIAL XY FERROMAGNET:


(We illustrate with the XY (i.e., $n=2$) model; the argument holds however true for any $n>1$ and any $d$-dimensional Bravais lattice)

$$H = K + V = \frac{1}{2I} \sum_{i=1}^{N} L_i^2 + \frac{J}{\mathcal{A}} \sum_{i,j} \frac{1 - \cos(\vartheta_i - \vartheta_j)}{r_{ij}^\alpha}$$

$I > 0$, $J > 0$

with $\mathcal{A} \equiv \sum_{j=1}^{N} r_{ij}^{-\alpha} \propto \begin{cases} N^{1-\alpha/d} & \text{if} \quad 0 \leq \alpha/d < 1 \\ \ln N & \text{if} \quad \alpha/d = 1 \\ \text{constant} & \text{if} \quad \alpha/d > 1 \end{cases}$

and periodic boundary conditions.

[The HMF model corresponds to $\alpha = 0$, $\forall d$]

FIG. 3. $\tilde{\lambda}_N^{\text{max}}$ versus $N$ (log-log plot) for typical values of $\alpha$ and $\frac{E_N}{N^{\nu^*}} = 5$. The full lines are the best fittings with the forms $(a - \frac{b}{N})/(N^{\nu^*})^\gamma$. Consequently, $\tilde{\lambda}_N^{\text{max}} \propto N^{-\kappa(\alpha)}$ where $\kappa(\alpha) = (1 - \alpha)c$ for $0 \leq \alpha < 1$ and $\kappa(\alpha) = 0$ for $\alpha > 1$; for $\alpha = 1$, $\tilde{\lambda}_N^{\text{max}}$ is expected to vanish as a power of $1/\ln N$. Inset: $\kappa$ versus $\alpha$ (related random matrices arguments will be detailed elsewhere).

C. Anteneodo and C. T.

A. Campa. A. Giansanti, D. Moroni and C. T.
Phys Lett A 286, 251 (2001)
V. Latora, A. Rapisarda and C. T., Phys Rev E 64, 056134 (2001)
$2\langle K(t) \rangle / N$

$U=0.69$

$N=500$

$QSS\;\text{regime}$

$(a)$

$(b)$

$T$

$U$

$0.5 \quad 0.6 \quad 0.7 \quad 0.8$

$0.4 \quad 0.5 \quad 0.525$

$0.475$

$0.45$

$10^{1} \quad 10^{2} \quad 10^{3} \quad 10^{4} \quad 10^{5} \quad 10^{6}$

$BC$
In contact with a thermostat (canonical ensemble):

**FIG. 1:** Sketch of the interactions considered in our canonical setup. Dashed lines mimic the interactions between the HMF- (full circles) and the TB- (empty circles) spins. Full (dotted) lines represent the HMF (TB) couplings.

F. Baldovin and E. Orlandini
AGING \[ \bar{x} \equiv (\bar{\theta}, \bar{L}) \]
Montemurro, Tamarit and Anteneodo, PRE (2003)

e=0.69

\[ C(t; t_w) \]

\[ e=0.69, N=1000 \]

\[ t \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]

\[ 10^4 \]

\[ 10^5 \]

\[ q=2.35 \]

\[ e=5.0, N=1000 \]

\[ t/t_w^{0.9} \]

\[ 10^{-1} \]

\[ 10^{0} \]

\[ 10^{1} \]

\[ 10^{2} \]

\[ q=2.35 \]
\( \alpha = 0 \) model:

\[ q_{rel} \approx 2.35 \ ? \]

\[ q_{stat} \approx 1.5 \ ? \]

F.A. Tamarit and C. Anteneodo
Europhysics News 36 (6), 194 (2005) [European Physical Society]
FIG. 1. Temperature evolution of an isolated $N$-rotor system (Eq. (1)) in grey line, and cold (hot) $M$-rotor subsystem in black line (circles). Inset: magnification of the crossover between $T_{QSS}$ and $T_{BG}$.

FIG. 2. Temperature evolution of an $N$-rotor thermostat (Eq. (1)) in grey line, and of an $M$-rotor thermometer (Eq. (3)) in black line. After $t_{contact}$ the systems interact through $H_{int}$. Inset: magnification of the thermostat temperature minimum (see text for details).

L.G. Moyano, F. Baldovin and C. T., cond-mat/0305091
Nonextensive statistical mechanics appears to be consistent with the 0\textsuperscript{th}, 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd} and 4\textsuperscript{th} principles of thermodynamics, hence the thermodynamical principles appear to be stronger than the role attributed to them by Boltzmann - Gibbs statistical mechanics.
ECONOMICS
\[ dv = -\gamma \left( v - \frac{\alpha + 1}{\beta} \right) dt + \sqrt{2v \frac{\gamma}{\beta}} dW_t, \]

\[ P(v) = \frac{1}{Z} \left( \frac{v}{\theta} \right)^\alpha \exp_q \left( -\frac{v}{\theta} \right) \]

\[ e^x_q = \left[ 1 + (1 - q) x \right]^{\frac{1}{1-q}} \quad (e^x_1 \equiv e^x) \]

**Figure 6.** In panel (a) open symbols represents the PDF for the ten-high 1 minute traded volume stocks in NYSE exchange; solid symbols represent the PDF obtained for the numerical realization depicted in panel (b) and line the theoretical PDF Eq. (28). Parameters are \( q = 1.17, \alpha = 1.79, \lambda = 1.42 \) and \( \delta = 3.09 \).
STOCK VOLUMES:

\[ P(v) = \frac{1}{Z} \left( \frac{v}{\theta} \right)^{-\alpha-2} \exp_q \left[ -\frac{\theta}{v} \right] \]

J de Souza, SD Queiros and LG Moyano, physics/0510112 (2005)
q-GENERALIZED BLACK-SCHOLES EQUATION:

L Borland and J-P Bouchaud, cond-mat/0403022 (2004)
L Borland, Europhys News 36, 228 (2005)
See also H Sakaguchi, J Phys Soc Jpn 70, 3247 (2001)

[REMARK: Student t-distributions are the particular case of q-Gaussians when $q = \frac{n+3}{n+1}$ with $n$ integer]
Earthquakes

Data from

P. Bak, K. Christensen, L. Danon and T. Scanlon,

\[ y = y_0[1+(q-1)x/T]^{1/(1-q)} \]

\[ q = 1 + 1/b \]

\( N(M > m) \) (earthquakes per year)

\( q = 2.05 \quad T = 26 \quad y_0 = 21400 \)

power law fit \( b = 0.95 \)

Magnitude \( m = \log_{10}(S) \)
TIME INTERVALS BETWEEN EARTHQUAKES
Southern California data [S. Abe and N. Suzuki (2004)]

Calm periods (stationary states) between major earthquakes, i.e., excluding the Omori-regime periods (nonstationary states)

\[ P(t > \tau) = \frac{1}{[1 + (q - 1)\tau / \tau_0]^{1/(q-1)}} \]

**After March 6, 1998**

\( q = 1.10 \)
\( \tau_0 = 1830 \text{ s} \)

**After June 9, 1998**

\( q = 1.08 \)
\( \tau_0 = 2410 \text{ s} \)

**After May 16, 1999**

\( q = 1.05 \)
\( \tau_0 = 2330 \text{ s} \)

Influence of the threshold \( m_{th} \)
AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:

Aging:

![Graph showing activity vs time and aftershock frequency vs time. The graph on the left shows a histogram of activity over time, with a peak around 1000 time units. The graph on the right shows a log-log plot of aftershock frequency against time, with a linear fit indicating an Omori law with τ=1.02.]
AGING IN THE NEWMAN MODEL FOR COHERENT NOISE:

Model:

Aging:
Omori law with $\tau=1.02$
\[ C(n + n_w, n_w) \equiv \frac{\langle t_{n+n_w} t_{n_w} \rangle - \langle t_{n+n_w} \rangle \langle t_{n_w} \rangle}{\sigma^2_{n+n_w} \sigma^2_{n_w}} \]

"Natural time" suggested in
P.A. Varotsos, N.V. Sarlis and E.S. Skordas,
$N=10000$
$a=0.2\;f=0.01\;s_1=1$

$C(n+n_w, n_w)$

- $n_w=250$
- $n_w=500$
- $n_w=1000$
- $n_w=2000$
- $n_w=5000$

natural time $n$
MODEL FOR EARTHQUAKES (OMORI REGIME):

S. Abe, U. Tirnakli and P.A. Varotsos
Europhysics News 36 (6), 206 (2005) [European Physical Society]
$N = 10000$

$a = 0.2 ; f = 0.01 ; s_1 = 1$

\[
\ln_q \left[ C(n+n_w, n_w) \right]
\]

- $n_w = 1000$
- $n_w = 2000$
- $n_w = 5000$

\[
n / n_w^{1.05}
\]
EARTHQUAKES:

NEWMAN MODEL (average over 100,000 realizations)

OLAMI-FEDER-CHRISTENSEN MODEL (average over 20,000 realizations)

$e_{2.98} (0.7 \frac{n}{n_W^{1.05}})$

$e_{2.9} (0.6 \frac{n}{n_W^{1.05}})$

GENERALIZED SIMULATED ANNEALING AND RELATED ALGORITHMS
q-GENERALIZED SIMULATED ANNEALING (GSA):


**Visiting algorithm:**

Boltzmann machine → Gaussian

Generalized machine → $q_V$ - Gaussian

**Acceptance algorithm:**

Boltzmann machine → Boltzmann weight

Generalized machine → $q_A$ - exponential weight

**Cooling algorithm:**

Boltzmann machine → $\frac{T(t)}{T(1)} = \frac{\ln 2}{\ln(1 + t)}$

Generalized machine → $\frac{T(t)}{T(1)} = \frac{2^{q_V-1} - 1}{(1 + t)^{q_V-1} - 1}$

[Typical values: $1 < q_V < 3$ and $q_A < 1$]
q-GENERALIZED SIMULATED ANNEALING (GSA):

Illustration: \[ E(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} (x_i^2 - 8)^2 + 5\sum_{i=1}^{4} x_i \]

(15 local minima and one global minimum)

\(q_V = 1 \Rightarrow \text{mean convergence time} \approx 50000\)
q-GENERALIZED PIVOT METHOD:


(Branin function) (Lennard-Jones clusters)

Number of function calls

Recently: M.A. Moret, P.G. Pascutti, P.M. Bisch, M.S.P. Mundim and K.C. Mundim
Classical and quantum conformational analysis using Generalized Genetic Algorithm
Physica A 363, 260 (2006) (presumably better than both!)
IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]

Original image

Canny edge detector

q = 1.5

q = 1
(Jensen-Shannon)
IMAGE EDGE DETECTION [A. Ben Hanza, J. Electronic Imaging 15, 013011 (2006)]

Original image

Canny edge detector

q = 1.5

q = 1
(Jensen-Shannon)
Fig. 4. Influence of parameter $q$ in natural images: $q = 0.5$, $q = 1.0$ (classical entropic segmentation) and $q = 3.0$.

M.P. de Albuquerque, I.A. Esquef, A.R.G. Mello and M.P. de Albuquerque
Fig. 3. Example of entropic segmentation for mammography image with an inhomogeneous spatial noise. Two image segmentation results are presented for $q = 1.0$ (classic entropic segmentation) and $q = 4.0$. 
than_{q}
HOW MANY PHYSICAL UNIVERSAL CONSTANTS?

\[ \nu = \text{number of independent physical universal constants in contemporary physics} \]

\[
\frac{1}{\nu} = \frac{1}{3} \left[ \frac{1}{3} + \frac{1}{4} + \frac{1}{\infty} \right] = \frac{7}{36} \]

Nino  Constantino  Gerardus

\[ \nu = \frac{36}{7} = 2 + \left( 3 + \frac{1}{7} \right) = 2 + \pi = 2 + 180^0 = 2 + 1 = 3 \]

\[ \nu = 3 \text{ !!!} \]

NINO WAS RIGHT!!!
Merging probability \( p_{ij} \propto \frac{1}{d_{ij}^\alpha} \) \( (\alpha \geq 0) \)

\( d_{ij} \equiv \) shortest path (chemical distance) connecting nodes \( i \) and \( j \) on the network

\( \alpha = 0 \) and \( \alpha \to \infty \) recover the random and the neighbor schemes respectively

Fig. 1: Snapshot of a non-growing dynamic network with $q$-exponential degree distribution for $N = 256$ nodes and a linking rate of $\bar{r} = 1$, for details see [8, 9]. The shown network is small to make connection patterns visible.
\[(\alpha \rightarrow \infty ; \quad < r > = 8)\]

\[Z_q(k) = \ln_q [P(>k)] = \frac{[P(>k)]^{1-q} - 1}{1-q}\]

[optimal \(q_c = 1.84\)]

$P(\geq k) = e_{qc}^{-(k-2)/\kappa}$ \((k = 2, 3, 4, \ldots)\)

*linear correlation* \(\in [0.999901, 0.999976]\)

\[(r = 2)\]

**SANTOS THEOREM:** RJV Santos, J Math Phys 38, 4104 (1997)  
(q - generalization of Shannon 1948 theorem)

**IF** \( S(\{p_i\}) \) continuous function of \( \{p_i\} \)

**AND** \( S(p_i = 1/W, \forall i) \) monotonically increases with \( W \)

**AND**  \[
\frac{S(A+B)}{k} = \frac{S(A)}{k} + \frac{S(B)}{k} + (1-q) \frac{S(A)}{k} \frac{S(B)}{k} \quad \text{(with } p_{ij}^{A+B} = p_i^A p_j^B \text{)}
\]

**AND** \( S(\{p_i\}) = S(p_L, p_M) + p_L^q S(\{p_i/p_L\}) + p_M^q S(\{p_m/p_M\}) \quad \text{(with } p_L + p_M = 1 \text{)} \)

**THEN AND ONLY THEN**

\[
S(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad \left( q = 1 \Rightarrow S(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i \right)
\]

**CE SHANNON (The Mathematical Theory of Communication):**

"This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications."
**ABE THEOREM:**  S Abe, Phys Lett A 271, 74 (2000)

(q - generalization of Khinchin 1953 theorem)

**IF**  
\[ S(\{p_i\}) \text{ continuous function of } \{p_i\} \]

**AND**  
\[ S(p_i = 1/W, \forall i) \text{ monotonically increases with } W \]

**AND**  
\[ S(p_1, p_2, \ldots, p_W, 0) = S(p_1, p_2, \ldots, p_W) \]

**AND**  
\[ \frac{S(A + B)}{k} = \frac{S(A)}{k} + \frac{S(B | A)}{k} + (1-q) \frac{S(A)}{k} \frac{S(B | A)}{k} \]

**THEN AND ONLY THEN**

\[ S(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \]

\[ q = 1 \Rightarrow S(\{p_i\}) = -k \sum_{i=1}^{W} p_i \ln p_i \]

The possibility of such theorem was conjectured by AR Plastino and A Plastino (1996, 1999).