Computational Complexity of the Landscape

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Outline

The string theory landscape

Computational complexity of landscape problems

The power of postselection and inflation

Cosmology and computation
The string theory landscape
Vacua of string theory (or any theory with low energy limit given by 10d supergravity)
String vacua $\Rightarrow$ compactification geometries

To zeroth order in $1/M_{Pl}$ expansion: string vacua $=$ solutions to 10d Einstein equations, with spacetime topology $\mathbb{R}^4 \times X_6$, coupled to $p$-form field strengths $F = F_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$.

Nontrivial $p$-cycles $C_i$ in $X_6$ can carry quantized magnetic $F$-flux:

$$\int_{C_i} F = N_i \in \mathbb{Z}$$

$\Rightarrow$ Energy density in effective 4d theory:

$$V(\phi, N) = V_0(\phi) + \int_{X_6(\phi)} F \wedge *F = V_0(\phi) + g_{ij}(\phi) N^i N^j$$
Choice of topology of compactification manifold + fluxes + scalar potential minima leads to huge landscape of vacua, each with different effective low energy parameters.

But: highly constrained nevertheless, and plausibly only finite number compatible with rough observational constraints.

In principle we could hope to one day classify all vacua and find unique one matching precision data, enabling us to predict everything we want to know.
Computational complexity of landscape problems
Imagine that we have a systematic classification of all vacua, and that we can compute for each vacuum every low energy quantity to very high accuracy.
Simple model for matching data in landscape

E.g. cosmological constant in Bousso-Polchinski model:

\[ \Lambda(N) = -\Lambda_0 + \sum_{ij} g_{ij} N^i N^j \]

with flux \( N \in \mathbb{Z}^K \). Example question: \( \exists N : 0 \leq \Lambda(N) < \epsilon \)?

Can be extended to more complicated models, other parameters, ...
Is this a tractable problem?

\[ \sim \] computational complexity theory
Basic complexity classes

- **P** = yes/no problems solvable in polynomial time (e.g. is \( n_1 \times n_2 = n_3 \), primality)
- **NP** = problems for which a candidate solution can be verified in polynomial time (e.g. subset sum: given finite set of integers, is there subset summing up to zero?)
- **NP-hard** = loosely: problem at least as hard as *any* NP problem, i.e. *any* NP problem can be reduced to it in polynomial time.
- **NP-complete** = NP \( \cap \) NP-hard (e.g. subset-sum, 3-SAT, traveling salesman, \( n \times n \) Sudoku, ...)

So: if *one* NP-complete problem turns out to be in \( P \), then NP \( = \) P.

Widely believed: NP \( \neq \) P, but no proof to date (Clay prize problem). Therefore: expect no P algorithms for NP-complete problems.
 Complexity of BP

- Clear: BP $\in$ NP
- Bad news: BP is NP-complete
- Proof: by mapping version of subset sum to it.
- Standard subset sum: Given $t, g_1, \ldots, g_N \in \mathbb{Z}$,

$$\exists k_i \in \{0, 1\} : \sum_{i} k_i g_i = t?$$

- Modified (bit still NPC) version: we are “promised” that

$$\sum_{i} k_i g_i = t \text{ with } k_i \in \mathbb{Z}^+ \Rightarrow k_i \in \{0, 1\}$$

- Reduction to BP: take BP with $g_{ij} = g_i \delta_{ij}, \Lambda_0 = t, \epsilon = 1$:

$$\exists N_i \in \mathbb{Z} : 0 \leq -t + \sum_{i} N_i^2 g_i < 1?$$

$\leadsto$ equivalent to modified subset sum.

(Hard part is to show that promise version of subset sum is still NP-complete.)
Exponentially many local minima for local relaxation process of $|\Lambda - \epsilon/2|$ with steps $\Delta N_i = \pm \delta_{ki}$, say for $g_{ij} \equiv g_i \delta_{ij}$:

$$|\Delta \Lambda| = g_k |1 \pm 2 N^k| > g_k.$$ 

$\Rightarrow$ any $|\Lambda - \epsilon/2| < \min_k g_k/2$ is local minimum, but if $\epsilon \ll \min_k g_k$, one generically gets stuck far from target range.
Simulated annealing

- Simulated annealing: add thermal noise to get out of local minima and gradually cool.
- Converges to Boltzmann distribution, so will always find target range, but only guaranteed in time exponential in problem size.
Prospects for solving NP-hard problems

- Parallel processing? (P) ❌
- Classical polynomial time probabilistic algorithms? (BPP) ❌
- Polynomial time quantum computing? (BQP) ❌
- Other known physical models of computation? ❌
Computing vacuum from cosmological selection principles?

Wait, I'm a theorist! I don't need experiment to find our vacuum!

▶ Can't we just *compute* our vacuum *ab initio* from cosmological selection principles?
▶ Example: Hartle-Hawking measure selects smallest positive $\Lambda$ with overwhelming probability. $\Rightarrow$ No need to match data, just find the one with smallest $\Lambda$!
▶ Problem: finding minimal $\Lambda(N)$ in BP is even harder than NP-complete! (is in DP, i.e. conjunction of NP and co-NP)
Caveats and indirect approaches

▶ NP-completeness is asymptotic, worst case notion. Particular instances may turn out easy. Cryptographic codes do get broken.

▶ Some problems may be more tractable than matching continuous parameters, e.g. matching particle spectra (??).

▶ String theory may have much more (as yet hidden) structure and underlying simplicity than current landscape models suggest.

▶ As in statistical mechanics, one could hope to compute probabilities on low energy parameter space without need for exact construction of corresponding microstates. Measure e.g. determined by cosmological dynamics (but: much work still needed to understand this in string theory).

▶ Already without dynamics, number distribution estimates together with experimental input could lead to virtual exclusion of certain future measurable properties.
Flattening the landscape?

- Controlled constructions of string vacua are tricky and explicit examples sparse \(\rightarrow\) maybe overestimate size?
- Unexplored stability issues: huge number of potential decay channels, classical and quantum!
- Hint (?): most of the landscape is strongly coupled mess with tons of exotic stuff, weird particles, crazy interactions,... Why don’t we see any of that?
Experimental surprises

Universe as Doughnut: New Data, New Debate

Is infinity an illusion? Some scientists believe that space is like a hall of mirrors in which light travels around a small universe in an apparently endless chain of images.
The power of postselection and inflation
Standard tractable complexity classes

- **BPP** (Bounded Probabilistic Polynomial): Yes/no problems solvable in $P$ time with classical randomized algorithm and probability correct $> 2/3$.

  Equivalently: Nondeterministic Turing machine halting in $P$ time with $> 2/3$ paths rejecting if answer is no and $> 2/3$ accepting if answer is yes.

- **BQP** (Bounded Quantum Polynomial): Same but with quantum computer.
“Anthropic” complexity classes

- **PostBPP** (Bounded Probabilistic Polynomial with Postselection): Same as BPP but with probabilities conditioned on some output bit: if output bit = 0, we shoot ourselves, if = 1, we don’t ↼ we only require $P(\text{correct}) > \frac{2}{3}$ given that we are alive in the end.

  ![Diagram of PostBPP](chart)

- **PostBQP**: Same as PostBPP but with quantum computer.
“Inflationary” complexity classes

- **BPPpath** (BPP with variable path length)

  Equivalent to nondeterministic Turing machine with variable computation path lengths halting in P time with > 2/3 paths rejecting if answer is no and > 2/3 accepting if answer is yes. *(No probabilistic interpretation in terms of randomized algorithm!)*

  “Inflationary” because this allows exponential amplification of final weight of favored computation branches.
Power of anthropic and inflationary computing

Some mathematical theorems:

- PostBQP = PP [Aaronson]. Here PP (Probabilistic Polynomial) = class of problems solvable with classical randomized algorithm and probability of correctness $> 1/2$. Is huge class.
- Believed: NP $\subset$ PP $\subset$ PSPACE. (Examples of class-complete problems resp. subset sum, permanent, Go.)
- BPPpath = PostBPP (elementary!)
- PostBQP = problems which can be solved by quantum computer in P time after modification laws of quantum mechanics, either by allowing nonunitary evolution with re-normalization at the end (this is similar to BPPpath), or by allowing nonlinear evolution, or by replacing $P \sim |\psi|^2$ by $P \sim |\psi|^p$, $p \neq 2$. [Aaronson]
Cosmology and computation
Remarkable interplay between physics and computation: laws of physics seem to be “designed” exactly in such way that all attempts to solve NP problems in P time by physical systems (real or imagined) are thwarted.

Examples: giving up any of the following fundamental principles would make NP problems in principle solvable in P time: linearity, unitarity and $P \sim |\psi|^2$ in quantum mechanics, finiteness of speed of light, finiteness of $M_{Pl}$, second law of thermodynamics.

So it appears a fundamental “meta-principle” of physics could be stated as: “Hard is hard”, or more prosaically: “The fundamental principles of physics are such that no operationally meaningful system can be constructed, even in principle, which effectively solves an NP-hard problem in time growing polynomially with the input size.”
Application to early universe cosmology? S & V!

Since we know next to nothing about the principles of early universe cosmology (quantum creation of universe, eternal inflation, and other confusing swamps of ideas ...), perhaps the “Hard is hard” meta-principle can give us some guidance (or even better: interesting paradoxes to think about).

▶ Example 1: There can be no “one step” dynamical mechanism on general landscapes selecting, say, the smallest $\Lambda$ vacuum when starting in some metastable high $\Lambda$ state.

▶ Example 2: Adding anthropic postselection on the other hand allows finding suitable or even optimal universes in polynomial number of steps.
Example 3: Assume fundamental “single causal patch” description of universe is in some effective way described by eternal inflation picture, with some prescription for computing time evolution of probabilities on parameter space. Notorious problem: probability measure highly ambiguous. \( \Rightarrow \) Test proposals for compatibility with \( H = H_0 \). \( \Rightarrow \) e.g. equal comoving time volume weighting excluded.
Toy example

Take e.g. “landscape”

$$\Lambda(\vec{m}) = \mu^4 e^{-N(\vec{m} \cdot \vec{g})^2}$$

with $\vec{m} \in \{0, 1\}^N$, $\vec{g} \in \mathbb{Z}^N$ and tunneling allowed between “vacua” labeled by $\vec{m}$.

$\Lambda = \mu^4$  \hspace{1cm} $\Lambda = \mu^4 e^{-N}$

$$P(\vec{m}, t) \sim Vol(\vec{m}, t) \sim \exp[\Lambda(\vec{m})t]$$

$\Rightarrow$ Peaks exponentially strong in poly time on minimal sum.

$\Rightarrow$ Solves subset sum problem in P time $\times$
Conclusions

Life is hard!