PRECISE α_s: WHY and HOW

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SFB TR9

I. Status of α_s

II. Implications:

  1) Stability of the SM

  2) Beyond: MSSM and GUTS

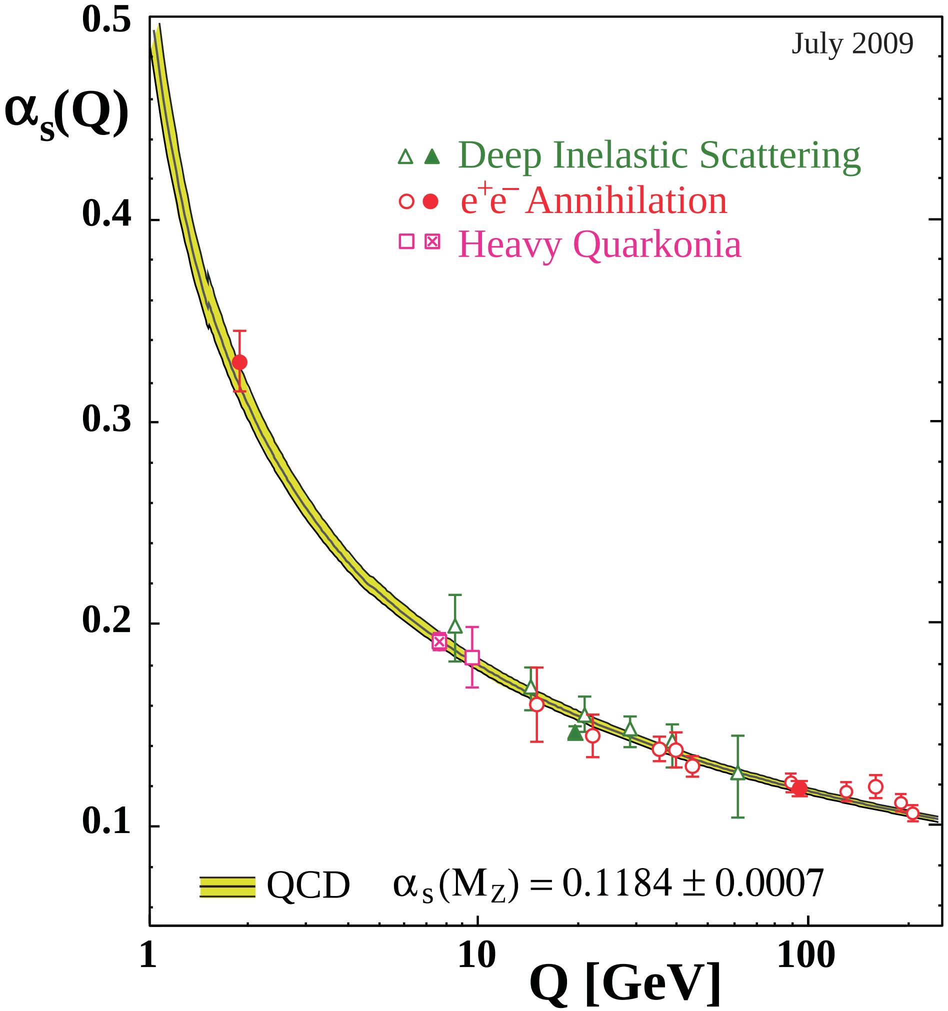
III. Methods for N^{3}LO

IV. Results: $R = \sigma(e^+e^- \to \text{had})/\sigma(e^+e^- \to \mu^+\mu^-)$ at 5 loops

V. Results: Sum Rules for DIS and the Generalized Crewther Relation

VI. Summary

I. STATUS of α_s



running of α_s!
precise value (2009)

$\alpha_s = 0.1184 \pm 0.0007$

but:

Large individual derivations for α_s

determined from closely related (or identical!) observables:

Selected examples:

α_s from τ: (N^{3}LO, contour improved vs. fixed order)

0.1202 ± 0.0019 (Chetyrkin+...)
0.1204 ± 0.0016 (Pich+...)
$0.1185^{+0.0014}_{-0.0009}$ (Beneke+...)

α_s from event shapes (N^{2}LO), e.g.

0.1175 ± 0.0025 (Gehrmann+..., three-jet rate)
0.1153 ± 0.0029 (Gehrmann+..., event shapes)
0.1135 ± 0.0012 (Abbate+..., SCET)

DIS (moments, PDF) (N^{2}LO)

0.1134 ± 0.0020 (Blümlein+...)
0.1171 ± 0.0014 (MRST)

Lattice (N^2LO)

0.1183 ± 0.0007	HPQCD :	staggered fermions
$0.1205\,(8)\,(5)\left(\begin{array}{c}+0\\-17\end{array}\right)$	PACS-CS :	Schödinger functional

partially large spread, conflicting opinions
on validity of approximation and methods

II. Implications of precise value of α_s

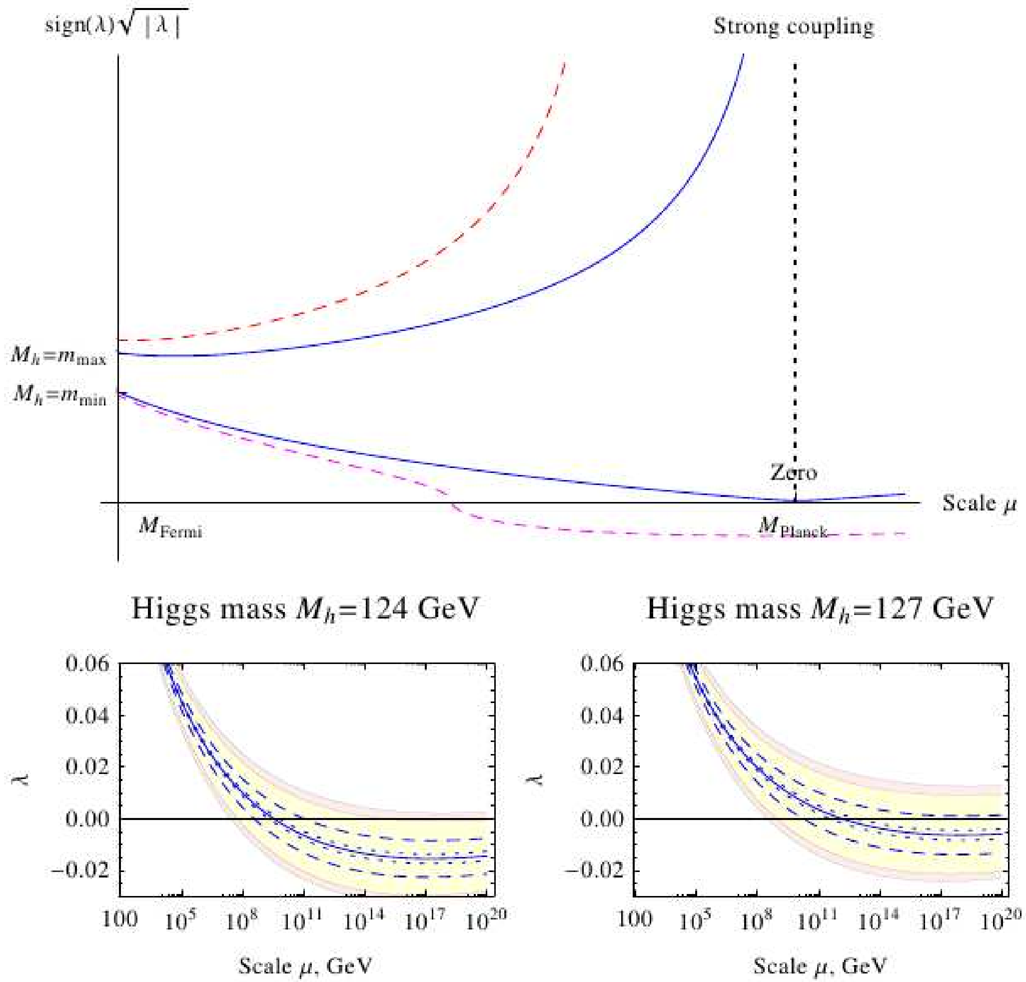
1) Stability of SM for very high energies (Planck scale?)

Scylla & Charybdis

Landau pole

unstable vacuum

Renormalization group: Higgs self-coupling $\lambda(\mu)$

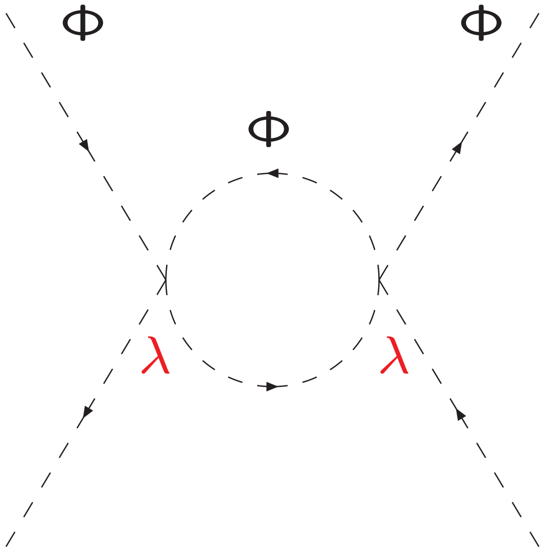


recent work:

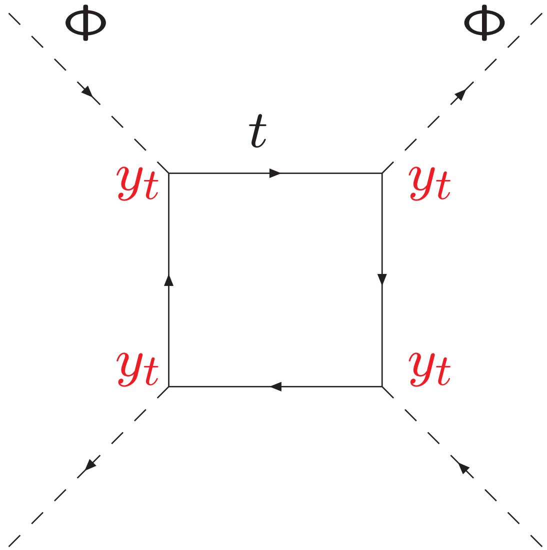
Three-loop β_λ:
Chetyrkin+Zoller 1205.2892

phenomenological analysis and two-loop matching:
Shaposhnikov+... 1205.6497
Degrassi+... 1205.2892

Stability $\Rightarrow$ λ positive for μ below M_{Planck} $\Rightarrow$ lower limit on M_H



positive contribution to β_λ



negative contribution to β_λ

$+\ldots$

Ingredients and recent improvements:

β-function: two-loop $\Rightarrow$ three-loop running

parameters: one-loop $\Rightarrow$ two-loop matching

[matching: relate observables to $\overline{\text{MS}}$ parameters $(M_H, G_F, \ldots \longrightarrow \lambda, \ldots)$
or between different (effective) theories, for example
(QCD: $n_f = 5 \Rightarrow n_f = 6$) or (SM $\longrightarrow$ MSSM $\longrightarrow$ GUT)]

Stability $\simeq$ running λ restricted to remain positive

borderline case: $\lambda(\mu_{Planck}) = 0$ and $\beta_\lambda(\lambda(\mu_{Planck})) = 0$

$$\Rightarrow M_H(min) = \left[128.95 + \frac{(M_t/\text{GeV} - 172.9)}{1.1} \cdot 2.2 - \frac{(\alpha_s - 0.1184)}{0.0007} \cdot 0.56\right] \text{GeV}$$

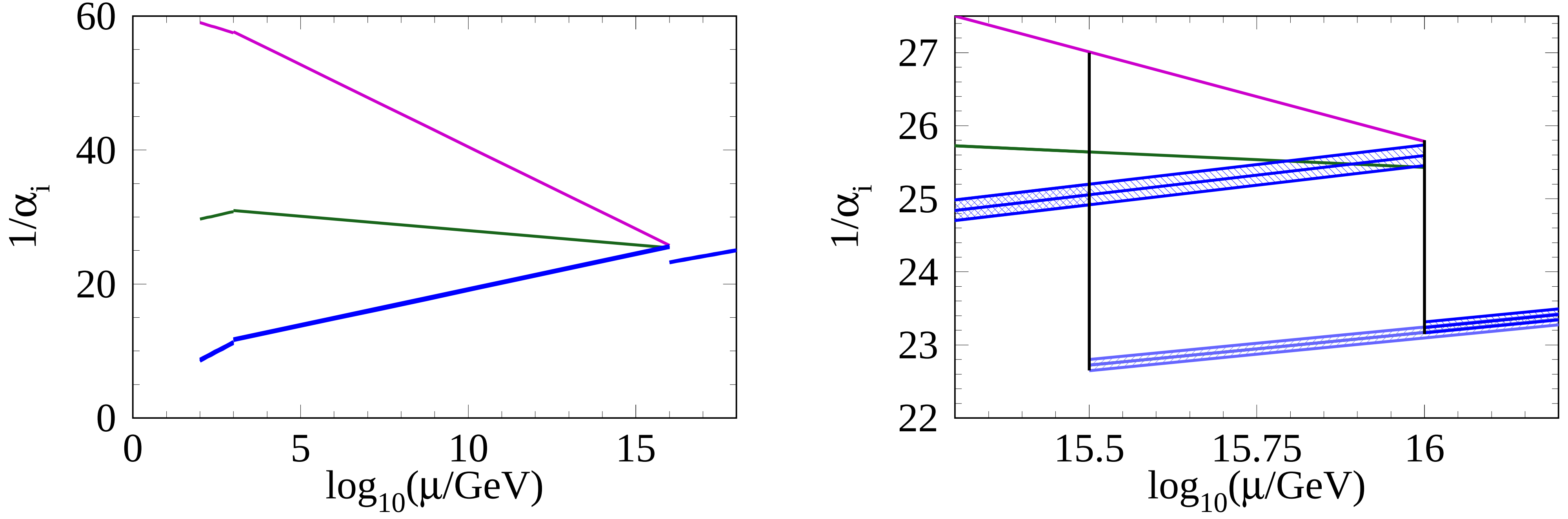
(Shaposhnikov)

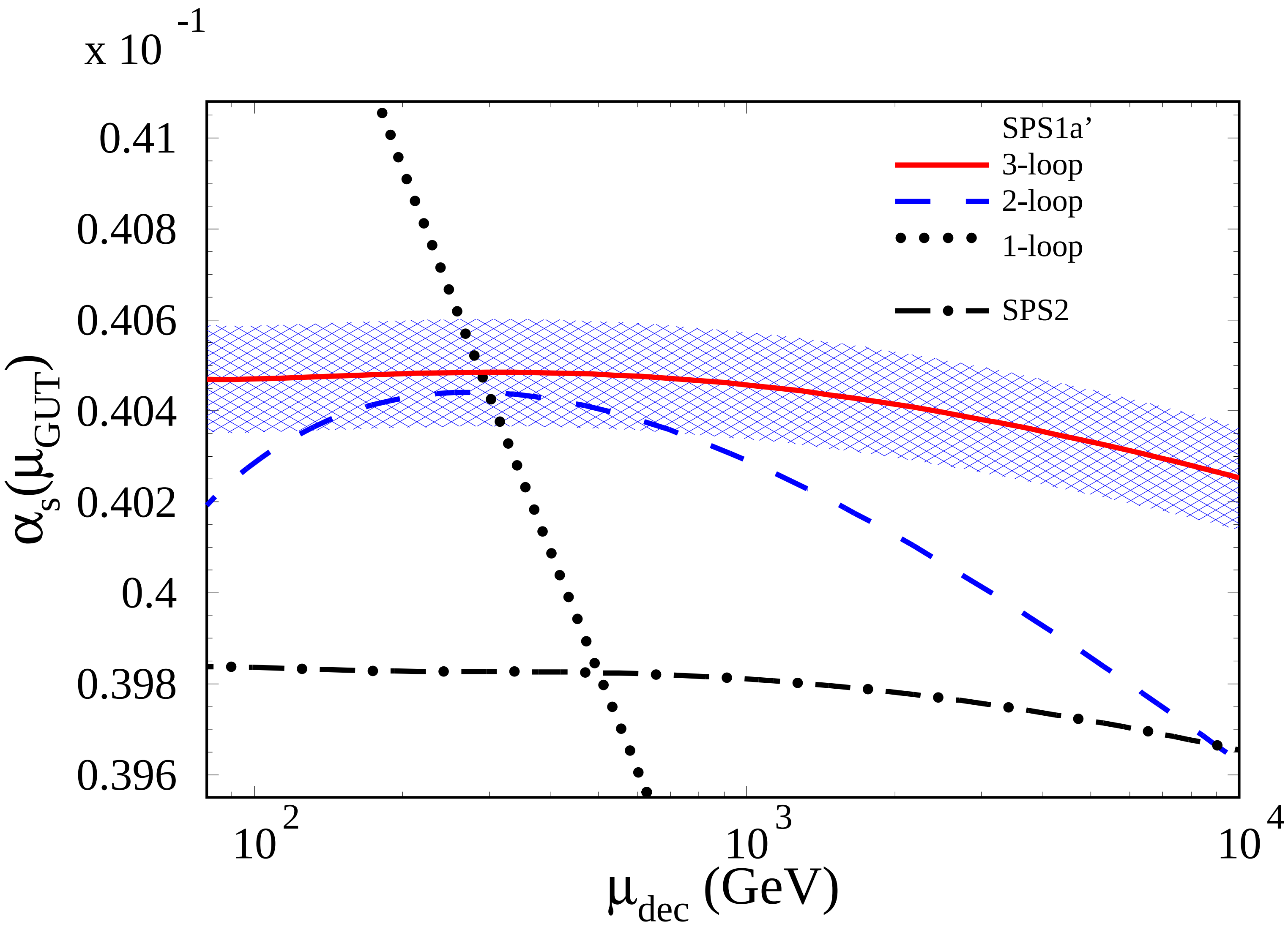
coincidence or deeper connection?
sensitivity to M_t and $\boxed{\alpha_s}$!

2) Unification in the MSSM

three-loop running $\Rightarrow$ reduction of uncertainties (Mihaila, Steinhauser), fundamental SUSY parameters unknown

uncertainty from $\alpha_s >$ uncertainty from theory.





matching: uncertainty from $\alpha_s >$ uncertainty from theory

III. Hadron production at e^+e^--colliders at N^3LO

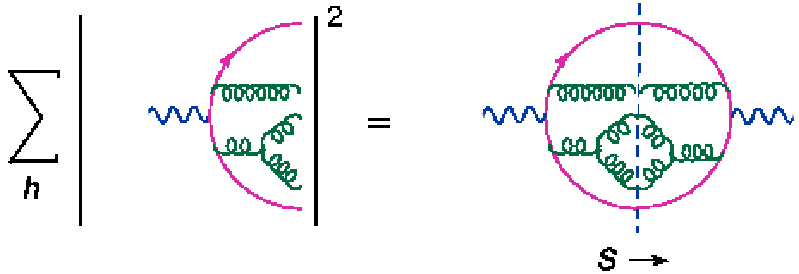
1. Methods

"gold plated" (Bjorken, 1979) QCD observables:

$$\mathrm{R_Z} = \Gamma(\mathrm{Z_0} \to \text{hadrons})/\Gamma(\mathrm{Z_0} \to \mu^+\mu^-)$$

$$\mathrm{R}_\tau = \Gamma(\tau \to \text{hadrons} + \nu_\tau)/\Gamma(\tau \to \mathrm{l} + \bar{\nu}_\mathrm{l} + \nu_\tau)$$

$$\mathrm{R(s)} = \sigma_{\mathrm{tot}}(\mathrm{e^+e^-} \to \text{hadrons})/\sigma(\mathrm{e^+e^-} \to \mu^+\mu^-)$$

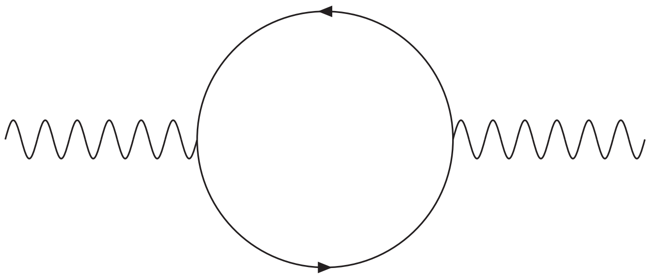


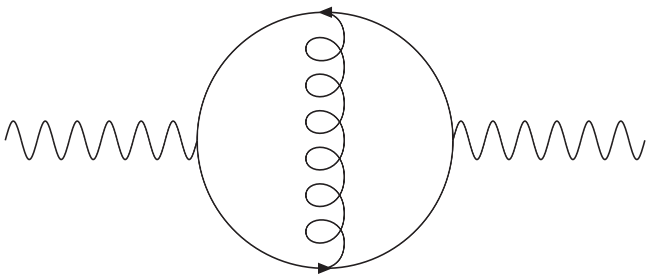
(via unitarity) $R(s) \approx \Im\,\Pi(s - i\delta)$

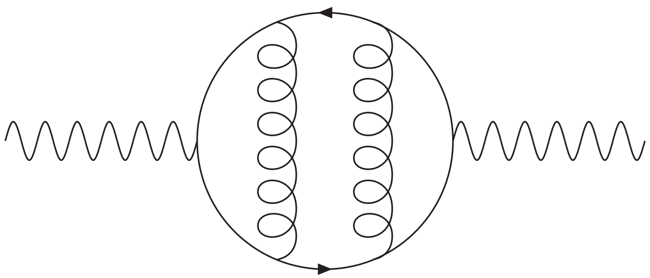
$$\Pi(Q^2) \sim \int e^{iqx}\langle 0|T[\; j^v_\mu(x) j^v_\mu(0)\;]|0\rangle dx$$

$$R(s) \leftrightarrow D(Q) \quad \Longleftarrow \text{Adler function} \equiv Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\, ds$$

$$R(s) = 1 + \sum_{i\geq 1} r_i\, a_s(s)^i, \quad D = 1 + \sum_{i\geq 1} d_i\, a_s(Q)^i, \quad (a_s \equiv \alpha_s/\pi, \mu = Q, Q^2 \equiv -q^2)$$

Im  α_s^0, 1 loop

Im  α_s^1, 2 loop

Im  α_s^2, 3 loop

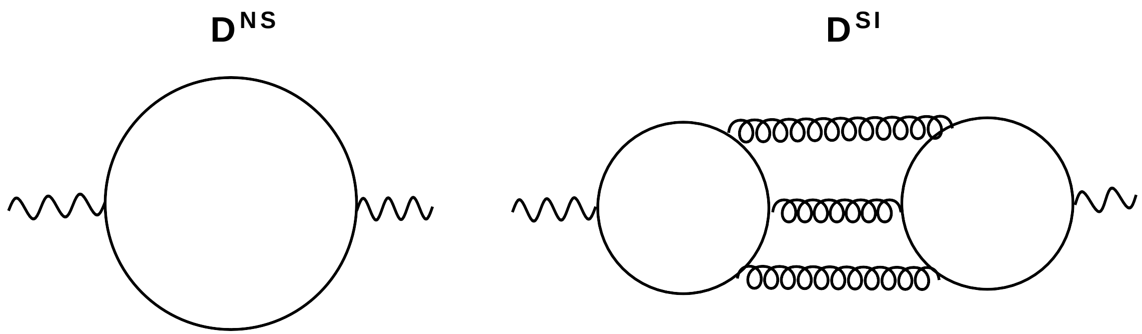
• status of theory (in the massless limit) •

$$R^{NS} = 3\sum_i Q_i^2\left(1 + \frac{\alpha_s}{\pi} + \#\left(\frac{\alpha_s}{\pi}\right)^2 + \#\left(\frac{\alpha_s}{\pi}\right)^3 + \boxed{\#\left(\frac{\alpha_s}{\pi}\right)^4} + \cdots\right)$$

parton model	QED Källen+ Sabry 1955	Chetyrkin, Kataev, Tkachov; Dine, Sapirstein; Celmaster 1979	Gorishny, Kataev, Larin; Surguladze, Samuel 1991 Chetyrkin /gen. gauge/ 1996	Baikov, Chetyrkin, JK 2008,2010

$$R^{SI} = (\sum_i Q_i)^2\left(\#\left(\frac{\alpha_s}{\pi}\right)^3 + \boxed{\#\left(\frac{\alpha_s}{\pi}\right)^4} + \cdots\right)$$

Baikov, Chetyrkin, JK, Rittinger 2012



Lots of technicalities

Correlator of two currents $\mathbf{j} = \bar{\mathbf{q}}\,\boldsymbol{\Gamma}\,\mathbf{q}$ and $\mathbf{j}^\dagger$

$$\boldsymbol{\Pi}^{\mathbf{jj}}(\mathbf{q}^2 = -\mathbf{Q}^2) = \mathbf{i}\int \mathrm{d}\mathbf{x}\mathbf{e}^{\mathbf{iqx}}\langle \mathbf{0}|\mathbf{T}[\ \mathbf{j}(\mathbf{x})\mathbf{j}^\dagger(\mathbf{0})\ ]|\mathbf{0}\rangle$$

related to the corresponding absorptive part $R(s)$ through

$$R^{jj}(s) \approx \Im\,\Pi^{jj}(s - i\delta)$$

RG equation ($a_s \equiv \alpha_s/\pi$)

$$\Pi^{jj} = Z^{jj} + \Pi^{B}(-Q^2, \alpha_s^B)$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right)\Pi = \gamma^{jj}(a_s)$$

extremely useful for determining the absorptive part of Π^{jj}

For Π at $(\mathbf{L+1})$ loop

$$\frac{\partial}{\partial \log(\mu^2)}\mathbf{\Pi} = \gamma^{\mathbf{jj}}(\mathbf{a_s}) - \left(\beta(\mathbf{a_s})\frac{\partial}{\partial \mathbf{a_s}}\right)\mathbf{\Pi}$$

anom.dim. at a_s^L
(L+1) loop integrals

L-loop integrals only contribute
due to the factor of $\beta(\mathbf{a_s})$

- to find Log-dependent part of Π at (L+1)-loops one needs (L+1)-loop anomalous dimension γ^{jj} and L-loop Π (BUT! including its constant part)

- (L+1) loop anom.dim. reducible to L-loop p-integrals

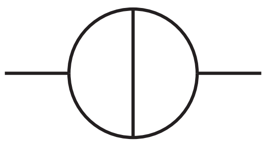
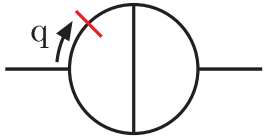
Strategy

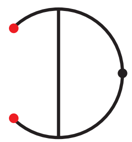
α_s^4 requires absorptive part of 5-loop correlator

$\widehat{=}$ divergent part $(1/\epsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

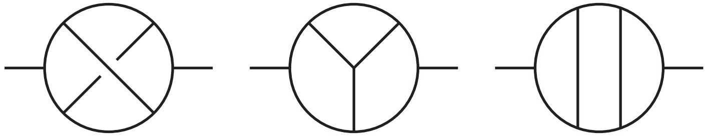
systematic, automatized algorithm (Chetyrkin)

div  $\widehat{=} \int dq^2$ 

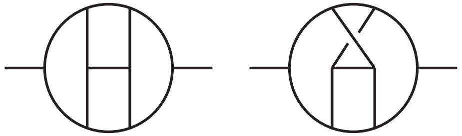
requires 

B finite part of 4-loop massless propagators difficult!
compare 3- and 4-loop calculation

3 topologies without insertions

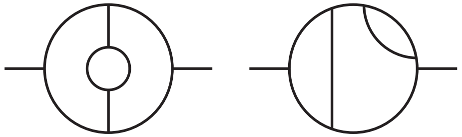


11 topologies without insertion



. . .

14 topologies with+without insertions



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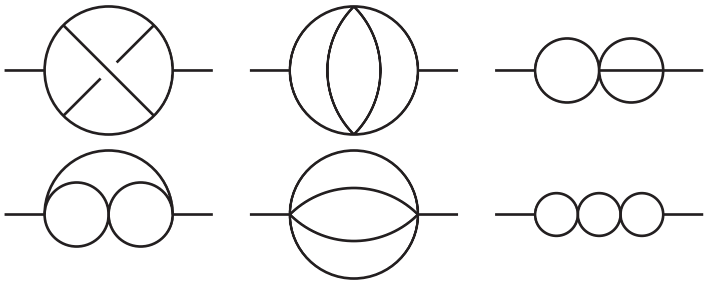
~150 topologies with+without insertions

. . .

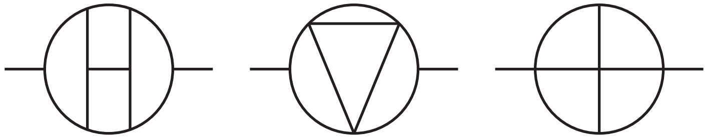
reduction to master integrals: MINCER

6 master integrals



reduction to master integrals ???

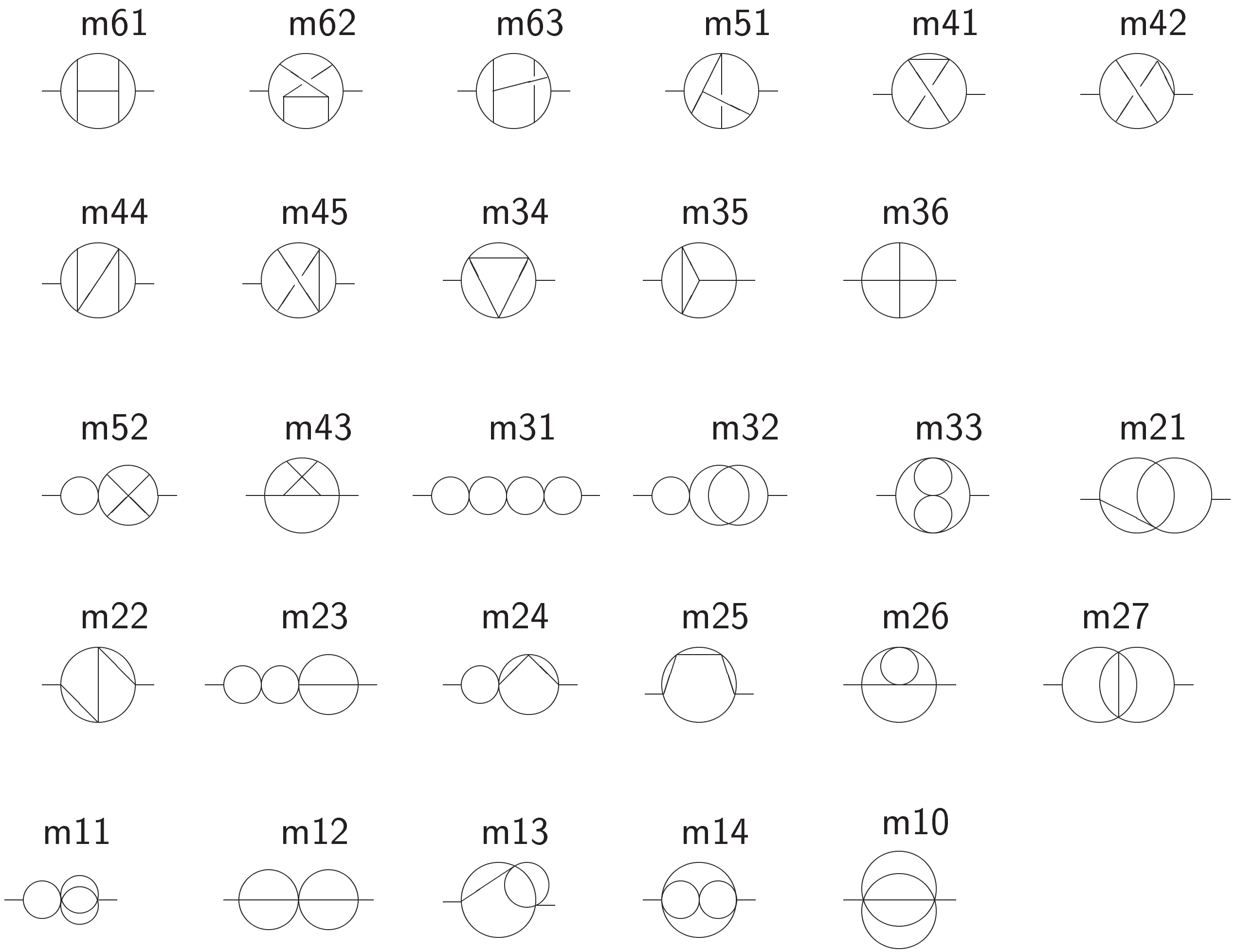
28 master integrals



. . .

All relevant Master Integrals solved (2004)

(method: “glue and cut” (Chetyrkin, Tkachov))



C **Baikov: recursion relations can be solved "mechanically" in the limit of large dimension d:**

consider amplitude f:

f(topology, power of prop, d)
$= \sum_{\alpha=\text{masters}} C^{(\alpha)}$(topology, power of prop, d) $\star f^{(\alpha)}(d)$

$f^{(\alpha)}$: 28 masters, analytically solved

$C^{(\alpha)}$: rational function $\frac{P^n(d)}{Q^m(d)}$, to be calculated;
$m + n \approx 60$ corresponds to ~ 60 coefficients

expand $C^{(\alpha)}$:

$C^{(\alpha)} = \sum_k c_k^{(\alpha)}$(topology, power of prop)$(1/d)^k + \ldots$

sufficiently many terms $c_k^{(\alpha)} \Rightarrow C^{(\alpha)}$

additional information on structure of $P^n(d)$, $Q^m(d)$ may lead to drastic reduction of hardware requirements:
originally ~ 60 numbers
additional information on structure of $Q^m(d)$ and using already calculated integrals
$\Rightarrow \; (m+n)_{\text{eff}} \approx 20$

evaluation of $c_k^{(\alpha)}$:

handling of polynomials of 9 variables of degree 2 k

$\frac{(9+2k)!}{9!\,(2k)!}$ terms $\qquad 2\,k = 40 \;\Rightarrow\; 2 \cdot 10^9$ terms
(200 GB storage, 1 TB for operation))

months of runtime using PARFPORM

IV. Results: R(s) at 5 loops

recall $D(Q^2) \equiv -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi = \int_0^\infty ds \frac{Q^2}{(Q^2+s)^2} R(s)$

(Adler function, μ independent)

$$\begin{aligned} D(q^2) = 1 + a_s + a_s^2\,(-0.1153\, n_f + 1.968) \\ + a_s^3\,(0.08621\, n_f^2 - 4.216\, n_f + 18.24) \\ + a_s^4\,(-0.010\, n_f^3 + 1.88\, n_f^2 - 34.4\, n_f + 135.8) \end{aligned}$$

impact on α_s from Z-decays

$$R(s) = D(s) - \pi^2 \beta_0^2 \left\{ \frac{d_1}{3} a_s^3 + \left(d_2 + \frac{5\beta_1}{6\beta_0} d_1 \right) a_s^4 \right\}$$

$$\Rightarrow \delta\alpha_s(M_Z) = 0.0005$$

and complete elimination of theoretical uncertainty.

$$\alpha_s(M_Z)^{\text{NNNLO}} = 0.1190 \pm 0.0026$$

impact on α_s from τ-decays

$$\frac{\Gamma(\tau\to h_{s=0}\nu)}{\Gamma(\tau\to l\bar{\nu}\nu)} = |V_{ud}|^2 S_{\rm EW} 3\,(1 + \delta_P + \underbrace{\delta_{\rm EW}}_{\rm small} + \underbrace{\delta_{\rm NP}}_{0.003\pm0.003})$$

$R_\tau = 3.471 \pm 0.011$ (Davier, Höcker, Zhang; ALEPH, OPAL, CLEO,...)

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm theo}$$

$$\alpha_s(M_Z) = 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm theo} \pm 0.0003_{\rm evol}$$

consistent with α_s from Z

$\delta\alpha_s$ from τ dominated by theory.

$\delta\alpha_s$ from Z dominated by statistics. First and only N^{3}LO results

$$\alpha_s(M_z) = \begin{cases} 0.1190 \pm 0.0026 & \text{from } Z \\ 0.1202 \pm 0.0019 & \text{from } \tau \end{cases}$$

combined $\alpha_s(M_Z) = 0.1198 \pm 0.0015$

Are these results reliable?

Independent checks!

V. Generalized Crewther Relation for D^{NS}

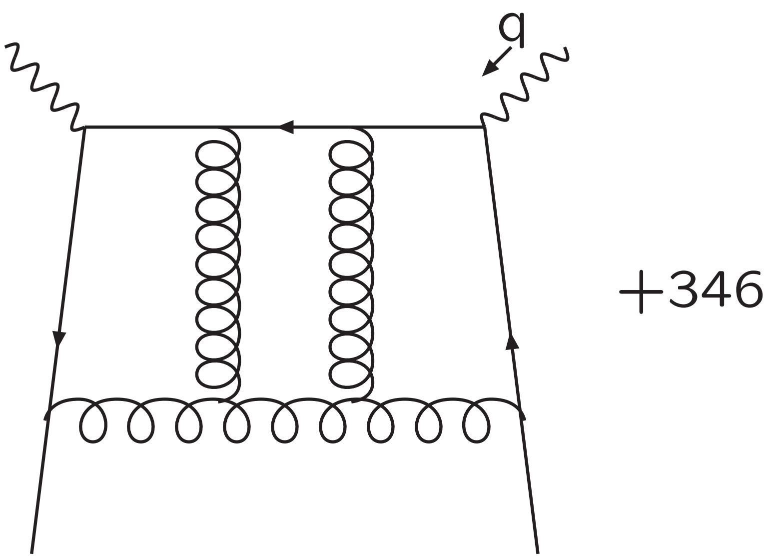
To check reduction to masters, a second, independent calculation in $\mathcal{O}(\alpha_s^4)$, for a general gauge group is required!

Perturbative factor $C^{Bjp}(a_s)$ in Bjorken sum rule:

$$\int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}\left|\frac{g_A}{g_V}\right| C^{Bjp}(a_s)$$

Unambiguous QCD predictions confrontable with data.

Typical diagrams at α_s^3



(Generalized) Crewther relation for D^{NS}

$$\mathbf{C^{Bjp}(a_s)\, D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \ldots\Big]}$$

with $\frac{\beta(\alpha_s)}{\alpha_s} \equiv -\beta_0 a_s + \ldots, \quad \beta_0 = \frac{11}{12} C_A - \frac{T_f\, n_f}{3}$

conformal limit: $\beta = 0 \Rightarrow C^{Bjp} D^{NS} = 1$

deviations: $\sim$ violation of conformal symmetry

$\sim \beta-$ function

define $$D(Q^2) = d_R \left(1 + \sum_i d_i a_s^i(Q^2) \right)$$

$$C^{Bjp}(Q^2) = 1 + \sum_i c_i a_s^i(Q^2)$$

$$
\begin{aligned}
d_4 &= \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16}-\frac{1}{4}\zeta_3-\frac{5}{4}\zeta_5\right]+n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right]+C_F^4\left[\frac{4157}{2048}+\frac{3}{8}\zeta_3\right] \\
&+ C_F^3T_f\left[\frac{1001}{384}+\frac{99}{32}\zeta_3-\frac{125}{4}\zeta_5+\frac{105}{4}\zeta_7\right]+C_F^2T_f^2\left[\frac{5713}{1728}-\frac{581}{24}\zeta_3+\frac{125}{6}\zeta_5+3\zeta_3^2\right] \\
&+ C_FT_f^3\left[-\frac{6131}{972}+\frac{203}{54}\zeta_3+\frac{5}{3}\zeta_5\right] \\
&+ C_F^3C_A\left[-\frac{253}{32}-\frac{139}{128}\zeta_3+\frac{2255}{32}\zeta_5-\frac{1155}{16}\zeta_7\right] \\
&+ C_F^2T_fC_A\left[\frac{32357}{13824}+\frac{10661}{96}\zeta_3-\frac{5155}{48}\zeta_5-\frac{33}{4}\zeta_3^2-\frac{105}{8}\zeta_7\right] \\
&+ C_FT_f^2C_A\left[\frac{340843}{5184}-\frac{10453}{288}\zeta_3-\frac{170}{9}\zeta_5-\frac{1}{2}\zeta_3^2\right] \\
&+ C_F^2C_A^2\left[-\frac{592141}{18432}-\frac{43925}{384}\zeta_3+\frac{6505}{48}\zeta_5+\frac{1155}{32}\zeta_7\right] \\
&+ C_FT_fC_A^2\left[-\frac{4379861}{20736}+\frac{8609}{72}\zeta_3+\frac{18805}{288}\zeta_5-\frac{11}{2}\zeta_3^2+\frac{35}{16}\zeta_7\right] \\
&+ C_FC_A^3\left[\frac{52207039}{248832}-\frac{456223}{3456}\zeta_3-\frac{77995}{1152}\zeta_5+\frac{605}{32}\zeta_3^2-\frac{385}{64}\zeta_7\right]
\end{aligned}
$$

similar result for c_4

$$\mathbf{C^{Bjp}(a_s)\, D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \dots\Big]}$$

implies 6 constraints on 12 color structures

$$C_F^4\,,\ C_F^3 C_A\,,\ C_F^2 C_A^2\,,\ C_F C_A^3\,,\ C_F^3 T_F n_f\,,\ C_F^2 C_A T_F n_f\,,$$
$$C_F C_A^2 T_F n_f\,,\ C_F^2 T_F^2 n_f^2\,,\ C_F C_A T_F^2 n_f^2\,,\ C_F T_F^3 n_f^3\,,\ d_F^{abcd} d_A^{abcd}\,,\ n_f d_F^{abcd} d_F^{abcd}$$

appearing at $\mathcal{O}(\alpha_s^4)$ in the difference

$$\mathbf{D^{NS} - 1/C^{Bjp}}$$

All 6 constraints are met identically! (which means $6 \cdot 7 = 42$ separate constraints on coefficients of $\zeta_3, \zeta_3^2, \dots$

similar result for singlet terms

VI. SUMMARY

- Precise result for α_s is crucial for many applications outside QCD

  $\rightarrow$ stability of SM

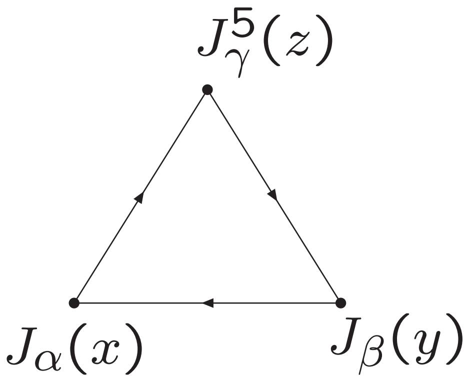
  $\rightarrow$ beyond SM

- advanced theoretical methods for multiloop calculations play a crucial role

- interesting connection between structural aspects of QCD and multiloop calculations.

Generalized Crewther Relation

consider $C_{\alpha\beta\gamma} = \langle T\, J_\alpha(x) J_\beta(y) J^5_\gamma(z) \rangle$

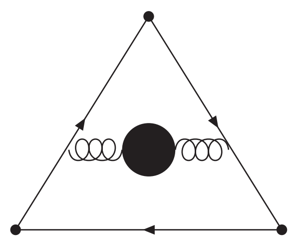


structure fixed by scale and
conformal invariance ($m \cdot a = 0$)
(plus current conservation)

normalization fixed by anomaly: $\frac{\partial}{\partial z_\gamma} J^5_\gamma = \ldots$;
result remains valid for "quenched QED" (photonic corrections only)!

but:

modification for QCD (or QED with fermions)
β function $\neq 0$;
scale invariance broken; starting from $\mathcal{O}(\alpha^{\in})$



$$C_{\alpha\beta\gamma} = C^{\text{conformal}}_{\alpha\beta\gamma} + \frac{\beta(\alpha)}{\alpha} \cdot K_{\alpha\beta\gamma}$$

consider $J_\alpha(x) J_\beta(0) \longrightarrow D^R_{\alpha\beta}(x) I + C^{Bj}_{\alpha\beta\gamma}(x) J^{5\mu}(0)$ for $x \to 0$
(perturbative expansion of Adler function
+ perturbative expansion of Bjorken SR)
insert into $\langle T\,(J\,J\,J^5)\rangle$ for $x \ll z$

$$\longrightarrow C^{Bj}_{\alpha\beta\gamma} \langle T\,(\underbrace{J^{5\mu}(0) J^5_\gamma(z)}_{D^{Adler}})\rangle \sim C^{Bj} \cdot D^{Adler} \Rightarrow C^{Bj} D^{Adler} = 1 + \frac{\beta(\alpha)}{\alpha} K$$

conf limit