Glue-mesons : their conception needs all of QCD in the infrared

The trace anomaly and modification of canonical structure

Peter Minkowski

Albert Einstein Center for Fundamental Physics - ITP, University of Bern

Abstract

The regularities implied at large distances by complete gauge invariance in QCD are shown to bear nontrivial consequences for the selection among inequivalent representations of canonical commutation (anticommutation) rules for gauge boson - and quark fields. The trace anomaly forces a modification of the gauge boson Lagrangean and by this of the entire associated canonical stucture .

Lecture prepared for the ' INTERNATIONAL SCHOOL OF SUBNUCLEAR PHYSICS, 50th Course: WHAT WE WOULD LIKE LHC TO GIVE US, Erice, 23. June - 2. July 2012, 50th Anniversary Celebrations 2011-2012-2013',
ETTORE MAJORANA FOUNDATION AND CENTRE FOR SCIENTIFIC CULTURE .

List of contents

List of contents (continued)

List of figures

1 - Introduction

1 - 1 Assembling elements of the Lagrangean density

1-1-a – Premises

We face the theoretical abstraction of QCD with $N_{fl} = 6$, representing strong interactions – adaptable to two or three light flavors (u , d , s) of quarks and antiquarks. $\leftrightarrow$

quarks : color is counted in $\pi^0 \rightarrow \gamma\gamma$ (**assuming global color- and flavor-projections to commute**) **yet see ref. [1-2001]**

spin and flavor are clearly seen in $q\overline{q}$ and $3q$, $3\overline{q}$ spectroscopy (**a pre-condition to count color**) .

(1)
$$\mathcal{L} = \left[\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}} \left\{ \begin{array}{c} \frac{i}{2}\overleftrightarrow{\partial}_\mu\ \delta_{c'\dot{c}} \\ + W^{\,r}_\mu \left(\frac{1}{2}\lambda_r\right)_{c'\dot{c}} \end{array} \right\} \gamma^{\,\mu}_{\dot{\mathcal{S}}'\mathcal{S}}\, q^{\,c}_{\mathcal{S}\,f} \atop - m_f\, \overline{q}^{\,\dot{c}}_{\dot{\mathcal{S}}\,\dot{f}}\, q^{\,c}_{\mathcal{S}\,f} \right] - \frac{1}{4\,g^2} B^{\mu\nu\,r} B^{\,r}_{\mu\nu} + \Delta\mathcal{L}$$

$W^{\,r}_\mu \equiv -v^{\,r}_\mu$: **for identification of convention for potentials**

quarks : c' , $c = 1, 2, 3$ **color** , $f = 1, \cdots, 6$ **flavor**

$\mathcal{S}', \mathcal{S} = 1, \cdots, 4$ **spin** , m_f **mass**

$\longrightarrow$

In eq. 1 , the $\mathcal{D}$ associated gauge connection fields – where $\mathcal{D} = \mathcal{D}\,(\,\mathcal{G}\,)$ denotes a general , irreducible representation of the local gauge group $\mathcal{G} = SU3_{\,c}$ – appear in the form appropriate for quarks : $\mathcal{D} = \{\,3\,\}$, and antiquarks : $\mathcal{D} = \{\,\overline{3}\,\}$ respectively

(2)
$$\begin{aligned} &(\,\mathcal{W}_{\,\mu}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta}\,(\,x\,) = W_{\,\mu}^{\,r}\,(\,x\,)\,(\,d_{\,r}\,)_{\,\alpha\beta} \\ &d_{\,r} = -\,d_{\,r}^{\,\dagger} = \tfrac{1}{i}\,J_{\,r} \in Lie\,(\,\mathcal{D}\,) \;\;;\;\; [\,d_{\,p}\,,\,d_{\,q}\,] = f_{\,pqr}\,d_{\,r} \\ &r\,,\,p\,,\,q = 1, \cdots, dim\,\mathcal{G} \;\;;\;\; \alpha\,,\,\beta = 1, \cdots, dim\,\mathcal{D} \end{aligned}$$

For $\mathcal{D}\,(\,SU3_{\,c}\,) = \left\{\,3\,(\overline{3})\,\right\}$ the representation matrices become (the Gell-Mann matrices [2-1964])

(3)
$$\begin{aligned} &\left(\,d_{\,r}\,(3) = \tfrac{1}{i}\,\tfrac{1}{2}\,\lambda_{\,r}\,\right)_{\,\alpha\beta} \;;\; r = 1, \cdots, 8\,;\; (\,\alpha\,,\,\beta\,) \leftrightarrow \left(\,c^{\prime}\,,\,\dot{c}\,\right) = 1, \cdots, 3 \\ &d_{\,r}\,(\overline{3}) = \overline{d}_{\,r}\,(3) \end{aligned}$$

with the *conventional* **normalization conditions :** $-\;tr\,d^{\,r}\,d^{\,s} = \tfrac{1}{2}\,\delta^{\,rs}$

The quantity proportional to the gauge potentials $W_{\,\mu}^{\,r}$ for the $\overline{q}\,q$ in eq. 1 is thus identified as

(4)
$$\left[\,W_{\,\mu}^{\,r}\,\left(\,\tfrac{1}{2}\lambda_{\,r}\,\right)_{\,c^{\prime}\dot{c}} = i\,(\,\mathcal{W}_{\,\mu}\,(\,\mathcal{D} = \{\,3\,\}\,)\,)_{\,c^{\prime}\dot{c}}\,\right](\,x\,)$$

Here we postpone the discussion of complete connections and extend the QCD Lagrangean density to include the term quadratic in the field strengths $B_{\,\mu\nu}^{\,r}$ and $\Delta\,\mathcal{L}$ in eq. 1, pertinent to Fermi gauges. $\longrightarrow$

gauge bosons : $\mathcal{L}_B = -\frac{1}{4g^2} B^{\mu\nu\, r} B^{\,r}_{\mu\nu}$

$$B^{\,r}_{\mu\nu} = \partial_\mu W^{\,r}_\nu - \partial_\nu W^{\,r}_\mu + f_{rst} W^{\,s}_\mu W^{\,t}_\nu \longleftarrow (W^{\,r}_\mu \equiv -v^{\,r}_\mu)$$

(5) $$r, s, t = 1, \cdots, dim\,(G = SU3_c) = 8$$

Lie algebra labels, $\left[\frac{1}{2}\lambda^r, \frac{1}{2}\lambda^s\right] = i f_{rst} \frac{1}{2}\lambda^t$

perturbative rescaling :

$$W^{\,r}_\mu = g W^{\,r}_{\mu\, pert} , \; B^{\,r}_{\mu\nu} = g B^{\,r}_{\mu\nu\, pert}$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.

Completing $\Delta\,\mathcal{L}$ in Fermi gauges

$$\Delta\,\mathcal{L} = \left\{ \begin{array}{c} -\frac{1}{2\eta g^2} (\partial_\mu W^{\mu\, r})^2 \\ +\partial^\mu \overline{c}^{\,r} (D_\mu c)^r \end{array} \right\} \; ; \; \eta \; : \textbf{ gauge parameter}$$

(6)

ghost fermion fields : $c\,,\, \overline{c} \;\; ; \;\; (D_\mu c)^r = \partial_\mu c^r + f_{rst} W^{\,s}_\mu c^t$

gauge fixing constraint : $C^{\,r} = \partial_\mu W^{\mu\, r}$

$\longrightarrow$

perturbative renormalization rescaling

We begin with the renormalization of external operaters and gauge boson fields, denoting unrenormalized fields and operators by the suffix $^{(0)}$

(7)
$$J_\alpha = (Z_J)^{-1} J_\alpha^{(0)} \quad , \quad \mathcal{O} = (Z_{\mathcal{O}})^{-1} \mathcal{O}^{(0)}$$
$$g = (Z_3)^{3/2} (Z_1)^{-1} g^{(0)} \quad , \quad \eta = (Z_3)^{-1} \eta^{(0)}$$
$$W^{r}_{\mu\, pert} = (Z_3)^{-\frac{1}{2}} \left(W^{r}_{\mu\, pert} \right)^{(0)}$$

and continue with the ghost fields c^r , $\overline{c}^s$ as defined in $\Delta \mathcal{L}$ in eq. 6

(8)
$$c^r = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (c^r)^{(0)} \quad , \quad \overline{c}^r = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (\overline{c}^r)^{(0)}$$
$$g = (Z_3)^{1/2} \widetilde{Z}_2 \left(\widetilde{Z}_1 \right)^{-1} g^{(0)}$$

The renormalization constant $Z_1 = Z_1(W-3)$ refers to the 3 gauge boson vertex of the derivative type $W W \partial W$ as shown in figure 1, while $\widetilde{Z}_1 = \widetilde{Z}_1(\overline{c}\, c\, W)$ refers to the ghost-W vertex, shown in figure 2, while the 4 gauge boson vertex involves the analogous renormalization constant $Z_{1(4W)}$ and the $\overline{q}\, q\, W$ vertex $Z_{1(\overline{q}qW)}$, shown in figures 3 and 4 respectively below . →

4 irreducible vertex diagrams

W-3 : (i)

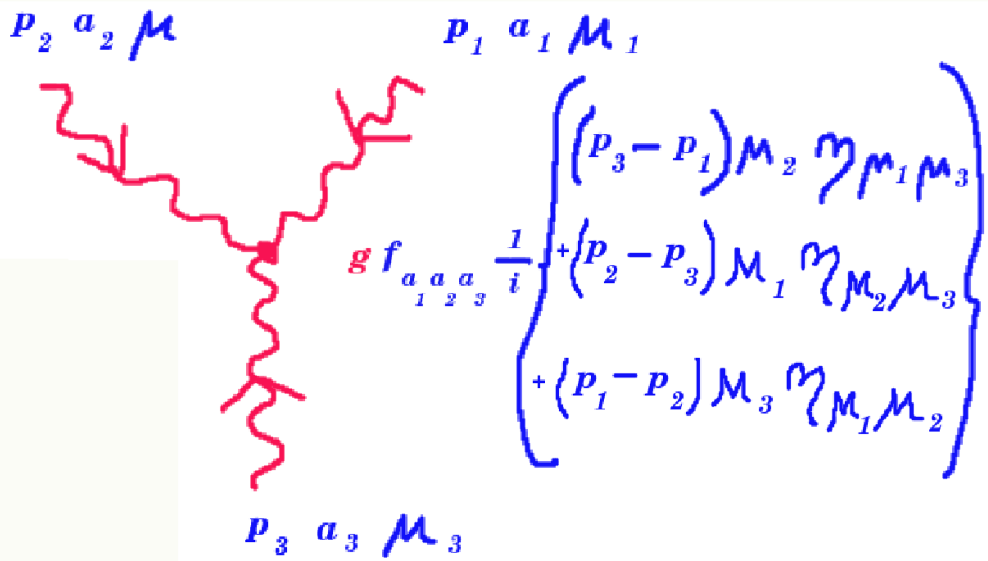

Fig. 1 : $Z_1 \leftrightarrow$ **W-3 vertex of the type** $W\,W\,\partial\,W$

→

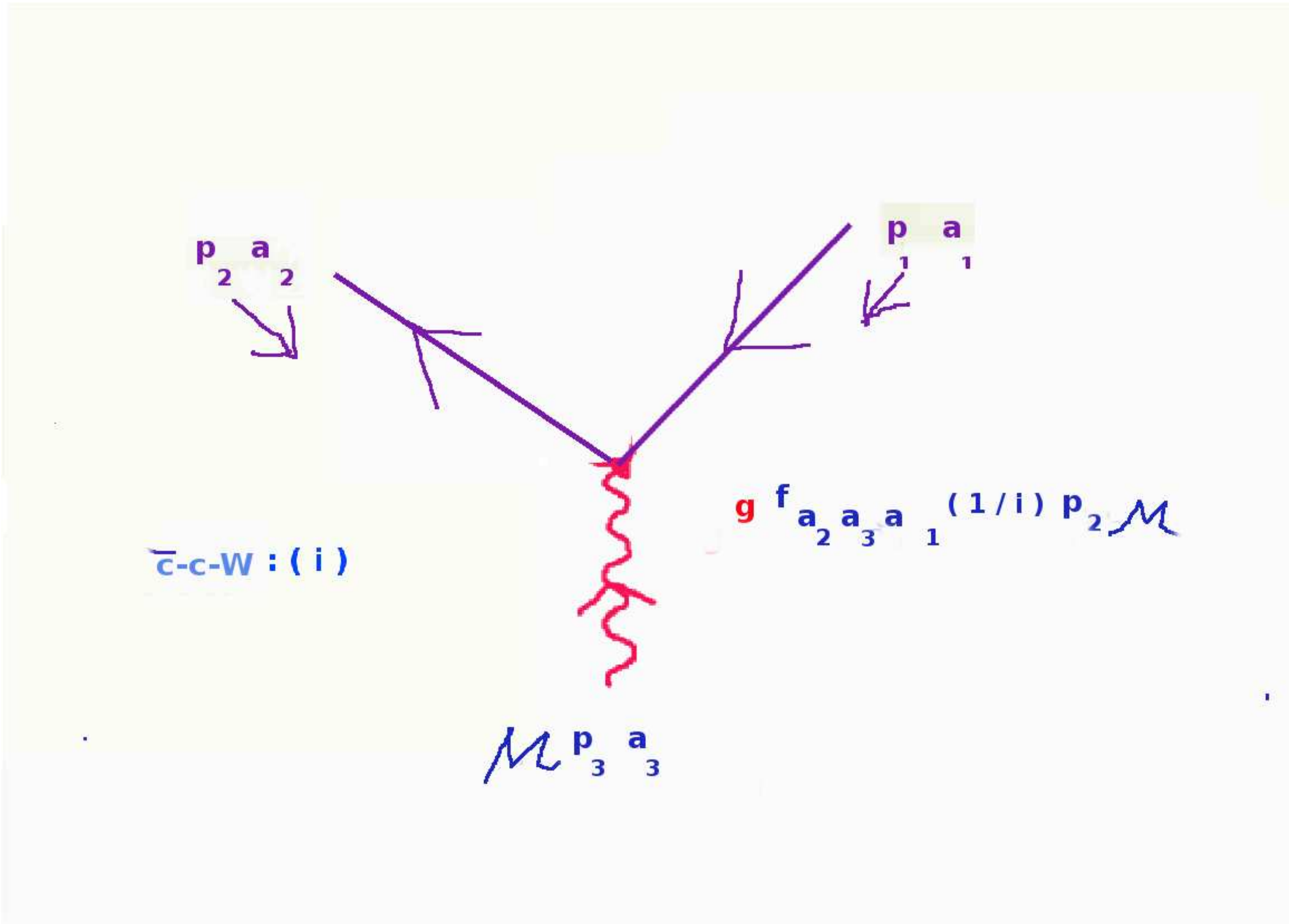

Fig. 2 : $\widetilde{Z}_1 \leftrightarrow$ cbar-c W-3 vertex of the type $\partial\, \overline{c}\, c\, W$

$\longrightarrow$

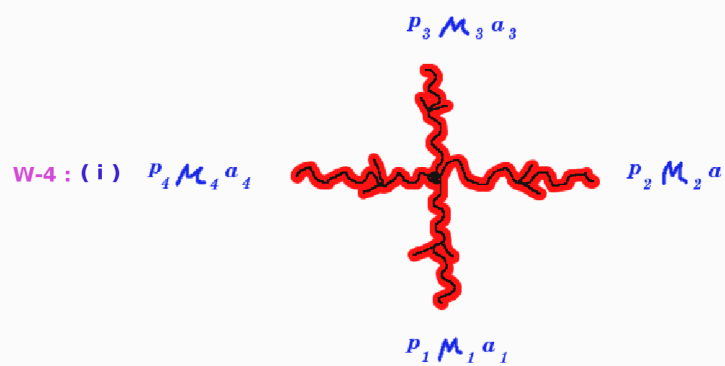

Fig. 3 : $Z_{1(4W)} \leftrightarrow$ **W-4 vertex for gauge fields**

$$g^2 \left(\begin{array}{rl} r_{[a_1 a_2][a_3 a_4]} & \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \end{array} \right] \\ + r_{[a_1 a_3][a_2 a_4]} & \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \\ + r_{[a_1 a_4][a_2 a_3]} & \left[\begin{array}{r} \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \end{array} \right)^{\to r}$$

$$r_{[a_1 b_1][a_2 b_2]} = f_{d a_1 b_1} f_{d a_2 2_2}$$

$\rightarrow$

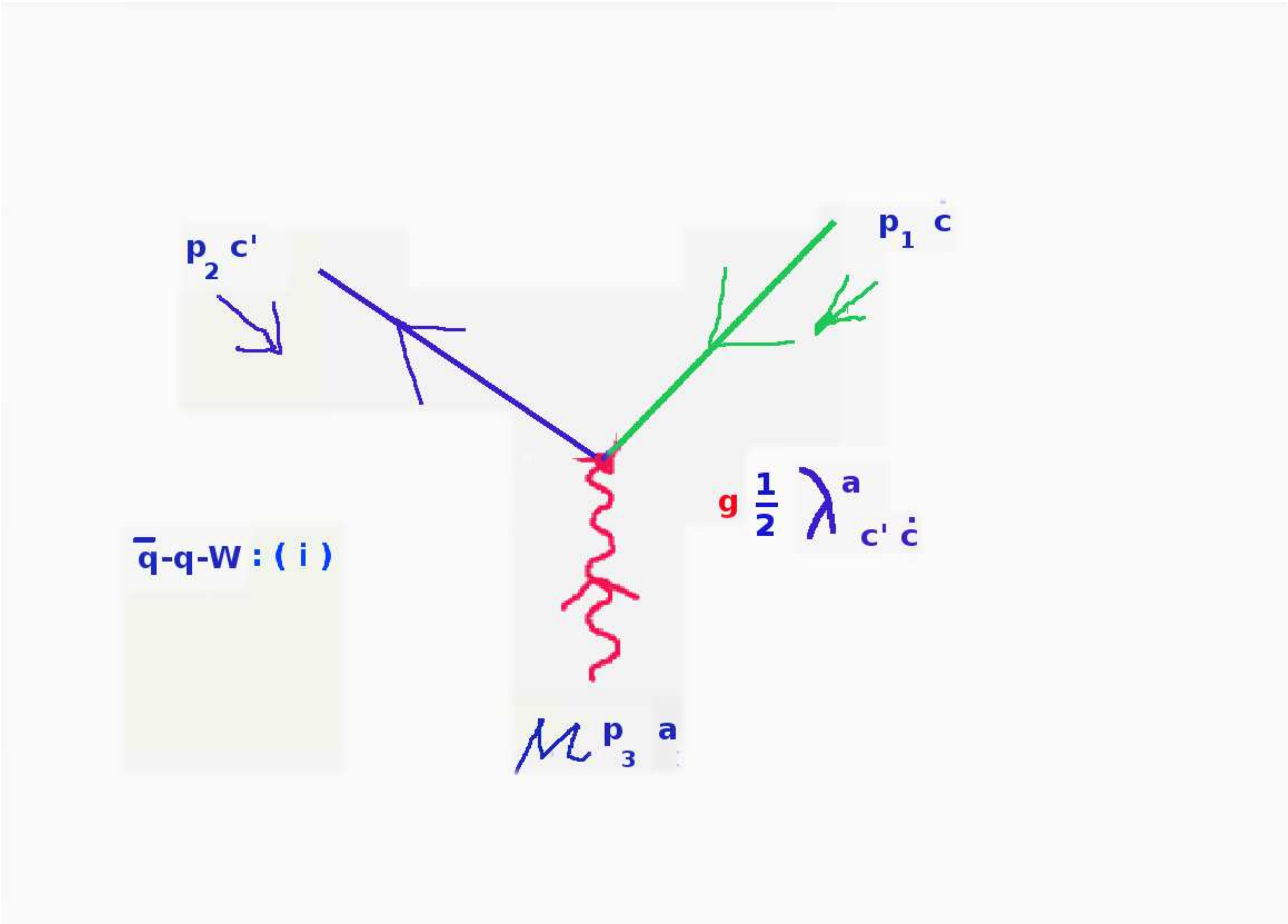

Fig. 4 : $Z_{1(\overline{q}qW)}$ $\leftrightarrow$ W-3 vertex for $\overline{q}\,q$ to gauge fields

→

the complete coupling constant renormalization equations

The minimal renormalization constants defined in eqs. 7 and 8, referring beyond the vertices shown in figures 1-4 to the two point functions for gauge connections , Z_3 , and ghost fields , $\widetilde{Z}_2$, thus need extension (also) to $q\,,\overline{q}$ fields and their two point function, Z_2 , as shown below

$$
\begin{array}{llrcl}
g & = (Z_3)^{1/2}\,(Z_3/Z_1)\,g^{(0)} & ;\; W^{\,r}_{\mu\,pert} & = & (Z_3)^{-\frac{1}{2}}\left(W^{\,r}_{\mu\,pert}\right)^{(0)} \\
g & = (Z_3)^{1/2}\left(\widetilde{Z}_2/\widetilde{Z}_1\right)g^{(0)} & ;\; (c\,,\,\overline{c})^{\,r} & = & \left(\widetilde{Z}_2\right)^{-\frac{1}{2}}\left((c\,,\,\overline{c})^{\,r}\right)^{(0)} \\
g^2 & = Z_3\,\left(Z_3/Z_{1(4W)}\right)\left(g^{(0)}\right)^2 & & &
\end{array}
$$

$$
\begin{array}{llrcl}
g & = (Z_3)^{1/2}\left(Z_2/Z_{1(\overline{q}qW)}\right)g^{(0)}\;; & q^{\,c}_{\mathcal{S}\,f} & = & (Z_2)^{-\frac{1}{2}}\left(q^{\,c}_{\mathcal{S}\,f}\right)^{(0)} \\
(9) & & \overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}} & = & (Z_2)^{-\frac{1}{2}}\left(\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}}\right)^{(0)}
\end{array}
$$

Maintaining gauge invariance through renormalization thus implies the relations

(10) $$Z_3/Z_1 = \widetilde{Z}_2/\widetilde{Z}_1 = Z_2/Z_{1(\overline{q}qW)}\;\;;\;\; Z_3/Z_{1(4W)} = (Z_3/Z_1)^2$$

It remains to include quark mass renormalization and associated rescaling. →

renormalization equations for quark masses m_f

We perform quark mass renormalization 'at zero mass' [4-1973] , with the following relations , using the $q , \overline{q}$ fields and their field renormalization constant, Z_2 , as defined in eqs. 9 and 10, through the extension

(11) $$(\mathcal{L}_m)^{(0)} = - \sum_f Z_2 Z_m m_f \quad ; \quad m_f = (Z_m)^{-1} m_f^{(0)}$$

The coefficients of the quark mass rescaling functions are known to four loops as calculated by Chetyrkin and independently by others [5-1997] . They are given in eq. 147 in Appendix 1.

Here we subsum the complete set of renormalized rescaling functions, which follows from the renormalization constants defined in eqs. 8 - 10

$$\left(\begin{array}{l} \mu^2 \partial_{\mu^2} + (\beta (g) / g) g^2 \partial_{g^2} + \\ + \gamma_m m_f \partial_{m_f} - 2 \gamma_3 (\eta \partial_\eta) - \gamma_{J \mathcal{O}} \end{array} \right) C_{J \mathcal{O}}^{T (\Pi)} (z ; \mu , g , m_\beta , \eta) = 0$$

(12) $$\left\{ \begin{array}{l} \beta (g) = - g b (g^2) \\ \gamma_m (g^2) \equiv - \chi_m \\ \gamma_{J \mathcal{O}} (g^2 , (\eta)) \\ \gamma_3 (g^2 , \eta) \end{array} \right\} = \mu^2 d / d \mu^2 \left\{ \begin{array}{c} 2 \log \left((Z_3)^{3/2} (Z_1)^{-1} \right) \\ \log \left((Z_m)^{-1} \right) \\ \log \left(Z_{\mathcal{O}} / Z_J^2 \right) \\ \log \left((Z_3)^{1/2} \right) \end{array} \right\}$$

→

We can always return to the standard form of the renormalization group equation

$$(13)\quad \left(\begin{array}{l} \mu \, \partial_{\mu} + \beta(g) \, \partial_{g} + (2 \gamma_{m}) \, m_{f} \, \partial_{m_{f}} \\ - 2 (2 \gamma_{3}) (\eta \, \partial_{\eta}) - (2 \gamma_{J \mathcal{O}}) \end{array} \right) C_{J \mathcal{O}}^{T \, (\Pi)} \left(z \, ; \, \mu \, , \, g \, , \, m_{\beta} \, , \, \eta \right) = 0$$

without changing the definition of β, transforming the lower relation in eq. 12 , which amounts to double the coefficients of the quantities γ_{m} , γ_{3} , $\gamma_{J \mathcal{O}}$, as displayed in eq. 13

$$(14)\quad \left\{ \begin{array}{l} \beta(g) = - g \, b(g^{2}) \\ 2 \gamma_{m}(g^{2}) \equiv - 2 \chi_{m} \\ 2 \gamma_{J \mathcal{O}}(g^{2}, (\eta)) \\ 2 \gamma_{3}(g^{2}, \eta) \end{array} \right\} = \mu \, d / d \mu \left\{ \begin{array}{c} \log \left((Z_{3})^{3/2} (Z_{1})^{-1} \right) \\ \log \left((Z_{m})^{-1} \right) \\ \log \left(Z_{\mathcal{O}} / Z_{J}^{2} \right) \\ \log \left((Z_{3})^{1/2} \right) \end{array} \right\}$$

→

rescaling equations in the $\overline{\text{MS}}$ scheme for coupling constant g and quark masses m_f

$\beta(g)$ and $\gamma_m(\kappa)$, $\kappa = g^2 / (16\pi^2)$

We turn to the scale (-square) evolution of the rationalized coupling constant and quark masses, introducing the variables

(15)
$$\tau = \log\left(\mu^2 / \mu_0^2\right) \;\; ; \;\; \kappa = g^2 / \left(16\pi^2\right) \;\; \rightarrow$$
$$\left\{ \widehat{\beta} = (\beta / g)\, \kappa = \widehat{\beta}(\kappa)\, ,\, \gamma_m(\kappa) \right\}$$

The renormalization group equation in the form given in eq. 12 then becomes

(16)
$$\left(\begin{array}{l} \partial_\tau + \widehat{\beta}\, \partial_\kappa + \gamma_m\, m_f\, \partial_{m_f} \\ -2\gamma_3 \left(\eta\, \partial_\eta\right) - \gamma_{J\mathcal{O}} \end{array} \right) C^{T\,(\Pi)}_{J\,\mathcal{O}} \left(z\, ;\, \mu\, ,\, g\, ,\, m_\beta\, ,\, \eta\right) = 0$$

The partial derivative terms in brackets in eq. 16 determine the sliding scale equations with respect to the quantities $\tau, \overline{\kappa}, \overline{m}_f, \overline{\eta}$ in the sense of its initial value structure. We restrict the discussion to $\tau, \overline{\kappa}(\tau), \overline{m}_f(\tau)$ here for simplicity .

(17)
$$\overset{\bullet}{\overline{\kappa}} = \widehat{\beta}(\overline{\kappa})\, ,\, \overset{\bullet}{\overline{m}}_f = \gamma_m(\overline{\kappa})\, \overline{m}_f \;\; ; \;\; \bullet = d / d\tau$$
$$\overline{\kappa}(\tau = 0) = \kappa\, ,\, \overline{m}_f(\tau = 0) = m_f$$

$\longrightarrow$

sliding coupling constant : $\dot{\overline{\kappa}} = \widehat{\beta}(\overline{\kappa})$

We associate integration variables , initial values and endpoint variables in the following way

(18) $$\theta \leftrightarrow [\tau\,,\,0]\;,\;\lambda \leftrightarrow [\overline{\kappa}\,,\,\kappa]$$

The integration of the rescaling differential equation becomes

(19) $$\int_0^{\tau} d\theta = \tau = \int_{\kappa}^{\overline{\kappa}} d\lambda \left(\widehat{\beta}(\lambda)\right)^{-1} \longrightarrow \tau = \int_{\overline{\kappa}}^{\kappa} d\lambda \left(-\widehat{\beta}(\lambda)\right)^{-1}$$

The first two coefficients in the expansion of $\widehat{\beta}$ in powers of λ necessitate a twofold subtraction for $\lambda \downarrow 0$ for the integral on the right hand side of eq. 19 to converge at the lower limit of integration

(20) $$\widehat{\beta}(\lambda) = \sum_{n=0}^{\infty} \widehat{\beta}_n \lambda^{n+2} \;\;;\;\; \widehat{\beta}_n \equiv -b_n$$

The first four coefficients $b_n\,;\,n = 0, \cdots, 3$ are known , (ref. [6-1997]) and given in eq. 146 in Appendix 1 .

We perform this subtraction splitting $\widehat{\beta}$

(21) $$\widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)}\widehat{\beta} \;\;;\;\; \left[\begin{array}{l} \widehat{\beta}_{(2)} = -\left(b_0\lambda^2 + b_1\lambda^3\right) \\ \Delta_{(2)}\widehat{\beta} = -\sum_{n=2}^{\infty} b_n \lambda^{n+2} \end{array}\right.$$

The subtraction, displayed in eq. 21 →

gives rise to the substitutions

$$\zeta_{(2)} = \frac{\Delta_{(2)} \widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 b_0^{-1} \lambda^2 (1 + O(\lambda))$$

(22) $$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \Big/ (1 + \zeta_{(2)}) = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

$$\psi_{(2)} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 (b_0)^{-2} (1 + O(\lambda))$$

ψ_2 as defined in eq. 22 has a regular power series expansion in λ, fully determined by $\widehat{\beta}$ and hence the sought subtractions involve only $\widehat{\beta}_{(2)}$, i.e. the beta function truncated to the first two terms.

We proceed in the reduction of $\left(-\widehat{\beta}\right)^{-1}$ using the substitutions in eq. 22

$$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

(23) $$\left(-\widehat{\beta}_{(2)}\right)^{-1} = b_0^{-1} \lambda^{-2} \left(1 + b_1^{(0)} \lambda\right)^{-1}; \quad b_1^{(0)} = b_1 / b_0$$

$$= b_0^{-1} \lambda^{-2} \left(1 - b_1^{(0)} \lambda\right) \left(1 - \left(b_1^{(0)} \lambda\right)^2\right)^{-1}$$

→

The last expression in eq. 23 decomposes into

(24)
$$\left(- \widehat{\beta}_{(2)} \right)^{-1} = b_0^{-1} \lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} + \phi_{(2)}$$
$$\phi_{(2)} = b_0^{-1} \left(b_1^{(0)} \right)^2 \left(1 - \left(b_1^{(0)} \lambda \right)^2 \right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$

$\phi_{(2)}$ **defined in eq. 24 , as** ψ_2 **defined in eq. 22 , has a regular power series expansion in** λ **.**
Thus we rearrange the expressions in eq. 23

$$\left(- \widehat{\beta} \right)^{-1} = \left(- \widehat{\beta}_{sing.} \right)^{-1} + \left(- \widehat{\beta}_{reg.} \right)^{-1} \; ; \; \widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)} \widehat{\beta}$$
$$\left(- \widehat{\beta}_{sing.} \right)^{-1} = b_0^{-1} \lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} \, , \; \left(- \widehat{\beta}_{reg.} \right)^{-1} = \phi_{(2)} - \psi_{(2)}$$

---

(25)
$$\phi_{(2)} = b_0^{-1} \left(b_1^{(0)} \right)^2 \left(1 - \left(b_1^{(0)} \lambda \right)^2 \right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$
$$\psi_{(2)} = \left(- \widehat{\beta}_{(2)} \right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 \left(b_0 \right)^{-2} \left(1 + O\left(\lambda \right) \right)$$
$$\zeta_{(2)} = \frac{\Delta_{(2)} \widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 \, b_0^{-1} \lambda^2 \left(1 + O\left(\lambda \right) \right)$$

→

We recall that in the decomposition $\widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)} \widehat{\beta}$, retained in eq. 25 , $\widehat{\beta}_{(2)}$ and $\Delta_{(2)} \widehat{\beta}$ denote the beta function truncated to the first wo terms and the remainder term respectively . It further follows from eq. 25

(26) $$\left(- \widehat{\beta}_{reg.} \right)^{-1} = b_0^{-1} \left((b_1 / b_0)^2 - b_2 / b_0 \right) (1 + O (\lambda))$$

We return to eq. 19 and integrate the singular part $\left(- \widehat{\beta}_{sing.} \right)^{-1}$

$$\begin{aligned} \tau &= \int_{\overline{\kappa}}^{\kappa} d \lambda \left(- \widehat{\beta} (\lambda) \right)^{-1} \\ &= \int_{\overline{\kappa}}^{\kappa} d \lambda \left\{ \left(- \widehat{\beta}_{reg.} (\lambda) \right)^{-1} + \left(- \widehat{\beta}_{sing.} (\lambda) \right)^{-1} \right\} \end{aligned}$$

$$\int_{\overline{\kappa}}^{\kappa} d \lambda \left(- \widehat{\beta}_{sing.} (\lambda) \right)^{-1} = b_0^{-1} \left[\begin{array}{c} \overline{\kappa}^{-1} - \kappa^{-1} \\ - b_0^{-1} b_1 \left(\log \left(\overline{\kappa}^{-1} \right) - \log \left(\kappa^{-1} \right) \right) \end{array} \right]$$

(27)

Eq. 19 thus allows the separation of variables $\overline{\kappa}$ and κ →

(28)
$$\tau = F(\overline{\kappa}) - G_{reg.}(\overline{\kappa}) - (F(\kappa) - G_{reg.}(\kappa))$$
$$F(\overline{\kappa}) = b_0^{-1}\,\overline{\kappa}^{-1} - (b_1/b_0^2)\log(\overline{\kappa}^{-1})$$
$$G_{reg}(\overline{\kappa}) = \int_0^{\overline{\kappa}} d\lambda \left(-\widehat{\beta}_{reg.}(\lambda)\right)^{-1}$$

the substitution $\tau \to t = \tau - \log(\mu_0^2/\Lambda^2)$ and

inverting the functional relation $t = t(\overline{\kappa}) \longleftrightarrow \overline{\kappa} = \overline{\kappa}(t)$

We rewrite eq. 28 separating sliding scale parts and associated scale μ and initial value parts associated with scale μ_0, as indicated in eqs. 15 - 17

(29)
$$\left[\begin{array}{l} \tau + b_0^{-1}\kappa^{-1} - (b_1/b_0^2)\log\left(b_0^{-1}\kappa^{-1}\right) - G_{reg.}(\kappa) + \\ \qquad + (b_1/b_0^2)\log\left(b_0^{-1}\overline{\kappa}^{-1}\right) + G_{reg.}(\overline{\kappa}) \end{array} \right] = b_0^{-1}\,\overline{\kappa}^{-1}$$

The substitutions $\kappa \to b_0\,\kappa$ and $\overline{\kappa} \to b_0\,\overline{\kappa}$ in the inverse of the arguments of the logarithm terms in eq. 29 cancels out in the difference. Next we adjust the scale parameter $\mu_0 \to \Lambda$ $\rightarrow$

setting

$$-b_0^{-1}\kappa^{-1} + (b_1/b_0^2)\log\left(b_0^{-1}\kappa^{-1}\right) + G_{reg.}(\kappa) = \log(\mu_0^2/\Lambda^2)$$

(30) $$t = \tau - \log(\mu_0^2/\Lambda^2) = \log(\Lambda^2/\mu^2)$$

$$x = b_0\overline{\kappa} \;\; ; \;\; G_{reg.}(x/b_0) = G(x) \;\; ; \;\; b_1/b_0^2 = a$$

Further we substitute the sliding scale variable $\overline{\kappa} \longrightarrow b_0\overline{\kappa} = x$. With these substitutions eq. 29 becomes

(31) $$t + a\log(x^{-1}) + G(x) = x^{-1}$$

$$G(x) = x\sum_{n=0}^{\infty} G_n x^n$$

In order to establish the asymptotic 'ultraviolet' expansion for $t \rightarrow \infty$ and $Y = x^{-1} \rightarrow \infty$ we introduce the notation , for clarity, including a change of variables $t = \exp(L)$; $L = \log(t)$

(32) $$\log(t) = L \;\; ; \;\; x = x(L) \longrightarrow Y = Y(L) \equiv (x(L))^{-1}$$

$$\log(Y) = Z(L) \equiv \log\left((x(L))^{-1}\right)$$

$\longrightarrow$

Eq. 31 takes the form

(33)
$$L + \log \left[1 + \frac{a\, Z(L) + G(x = Y^{-1})}{\exp(L)} \right] = Z(L)$$

$$t = \exp(L) \;\; ; \;\; Y = \exp(Z) \longleftarrow Z = Z(L)$$

Eq. 33 can now be solved by iteration in the asymptotic limit $L \rightarrow \infty$

$$L = \log \left[\log \left(\Lambda^2 / \mu^2 \right) \right] \rightarrow \infty$$

$$Z(L) = lim_{n \rightarrow \infty} Z_n(L)$$

(34)
$$L + \log \left[1 + \frac{a\, Z_n(L) + G\left(x_n = Y_n^{-1} \right)}{\exp(L)} \right] = Z_{n+1}(L)$$

anchor : $Z_0(L) = L$

The first anchor takes care of the largest contribution to to Z , i.e. L . $\longrightarrow$

Thus we obtain $Z_1(L)$ and $Y_1 = \exp(Z_1)$ from rq. 34

(35)
$$Z_1(L) = L + \log\left[\, 1 + a\,L\,\exp(-L) + \frac{G\,(\exp(-L))}{\exp(L)} \,\right]$$

$$Y_1(L) = e^L + a\,L + G\,(\exp(-L)) \;\; ; \;\; t = e^L$$

Comparing withe recursive equations (eq. 34 , repeated below

$$L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \rightarrow \infty$$

$$Z(L) = lim_{n\rightarrow\infty}\, Z_n(L)$$

(36)
$$L + \log\left[\, 1 + \frac{a\,Z_n(L) + G\left(x_n = Y_n^{-1}\right)}{\exp(L)} \,\right] = Z_{n+1}(L)$$

anchor : $Z_0(L) = L$

it follows that all contributions from the function G , i.e. resulting from the subleading terms proportinal in the beta function truncated by the first to terms $b_0\,\kappa^2 + b_1\,\kappa^3$ give vanishing contributions to $Y(L) \equiv (b_0\,\overline{\kappa})^{-1}(L)$ for $L \rightarrow \infty$. This can be verified recursively. →

Thus modulo terms vanishing for $L \to \infty$ we can solve the simpler functional equation, omitting G from eqs. 31, 35 and 36

(37) $$t + a \log \left(\widetilde{Y} \right) = \widetilde{Y} \;\; ; \;\; \widetilde{Y} = x^{-1} \;\; ; \;\; Y - \widetilde{Y} \to 0 \;\; \textbf{for} \;\; t \to \infty$$

or equivalently

(38) $$L + \log \left(1 + a \widetilde{Z} e^{-L} \right) = \widetilde{Z} \;\; ; \;\; \widetilde{Y} = e^{\widetilde{Z}} \;\; ; \;\; t = e^{L}$$

The same argument which led to the elimination of the term proportional to G in eq. 36 as far as nonvanishing contributions for Y are concerned in the limit $L \to \infty$ imply that expanding the logarithm on the left hand side of eq. 38 only the first term need be retained

(39) $$\log \left(1 + a \widetilde{Z} e^{-L} \right) \sim a \widetilde{Z} e^{-L}$$

leading to the equivalent approximate functional equation simplifying eq. 38

(40) $$Z \sim \widetilde{Z} \sim L / \left(1 - a e^{-L} \right) \sim L + a e^{-L} L \;\; \longrightarrow$$
$$Y \sim \widetilde{Y} \sim e^{L} \left(1 + a e^{-L} L \right) = e^{L} + a L + [\, 0 \,] \;\; \textbf{for} \;\; L \to \infty$$

In eq. 40 the $\sim$ symbol is meant to imply modulo additive terms to Y , vanishing for $L \to \infty$, proving the remarkable fact that the potential constant in the asymptotic expansion for Y vanishes . →

We conclude this subsection comparing side by side the functional dependence of $t \equiv \log\left(\mu^2/\Lambda^2\right) = t\left(\overline{\kappa}(\mu)\right)$ and $Y \equiv (b_0\overline{\kappa})^{-1} = Y(t)$.

The first relation is given in eq. 31

$$t = x^{-1} - a\log\left(x^{-1}\right) - G(x)$$

$$G(x) = x\sum_{n=0}^{\infty} G_n x^n \; ; \; x = b_0\overline{\kappa}(\mu^2)$$

$$t = \log(\mu^2/\Lambda^2) \longrightarrow$$

(41)

$$\Lambda^2 = \mu^2\left[\exp\left(-\frac{1}{b_0\overline{\kappa}}\right)\right]\left[\frac{1}{b_0\overline{\kappa}}\right]^a \exp\left[\, G(b_0\overline{\kappa})\,\right]$$

$$a = b_1/b_0^2$$

The second relation is displayed in eq. 40 , as an asymptotic expansion for $t = e^L \rightarrow \infty$

(42)

$$Y = Y(L) \equiv (b_0\overline{\kappa})^{-1}(L) \text{ for } L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \rightarrow \infty$$

$$\sim e^L + aL + [0]$$

$\longrightarrow$

Eqs. 41 and 42 constitute the essence of this subsection, giving rise to some remarks :

1) The central scale Λ

with all subtle properties as defined, in the $\overline{\text{MS}}$ scheme, constitutes a renormalization group invariant finite mass scale, given a region of sliding scale coupling constant $\kappa = g^2 / (16 \pi^2)$ **generically or** $\overline{\kappa}$ **in the perturbative region region here, nonvanishing but appropriately small.**

This scale does *not* depend on quark masses , only on the number of quark flavors , while derivations from measurable quantities (in the asymptiotic region) suffer from systematic errors , which do depend on quark massses .

The combined analysis of the sliding scale coupling constant , as of 2009 by Bethke , is shown in Fig. 5 below.

I quote here a discussion of Λ **for** $N_{fl} = 3$ **by Bodenstein et al., ref. [8-2011] , from data in the mass scale region of the** τ **lepton**

(43) $\Lambda_{N_{fl}=3} = 382 \pm 24$ **MeV** $\longleftrightarrow \alpha_s = 4\pi \kappa (\mu = m_\tau) = 0.344 \pm 0.014$

The reconstruction of Λ **from the 'perturbatively accessible region' implies a hard breaking of scale invariance , and hence equivalently a nonvanishing trace of the co-renormalizable and renormalized energy momentum density operator. This shall be discussed in the next subsection.**

$\longrightarrow$

2) sliding scale quark masses

The rescaling equations (eqs. 15 - 17) extend to the quark mass parameters , again universally and quark mass independently rescaling m_f , keeping ratios invariant

(44) $$r_{f_1 f_2} = m_{f_1} / m_{f_2}$$

Here no detailed discussion of extracting sliding scale quark masses including heavy flavors c , b is given. For heavy flavor comparisons with initially the reaction $e^+ e^- \rightarrow f \overline{f} -$ flavored hadrons , the QCD sum rule approach is the main tool , as pioneered by Shifman, Vainshtein and Zakharov [7-1979] , together with lattice QCD .

The sliding quark masses as for 3 flavors are shown in figure 6 below , restricted to two loop approximation . References [8-2011] - [11-2011] are representative for present refinement to four loops in the rescaling functions and derived results , e.g.

(45) $$\overline{m}_b (\overline{m}_b) = \begin{cases} 4163 \pm 16 \text{ MeV} & \text{ref. [9-2010]} \\ 4171 \pm 7 \text{ MeV} & \text{ref. [10-2011]} \\ 4177 \pm 11 \text{ MeV} & \text{ref. [11-2011]} \end{cases}$$

$$\overline{m}_c (3 \text{ GeV}) = 986 \pm 13 \text{ MeV} , \text{ ref. [9-2010]}$$

$$\overline{m}_c (\overline{m}_c) = 1262 \pm 17 \text{ MeV} , \text{ ref. [11-2011]}$$

$\longrightarrow$

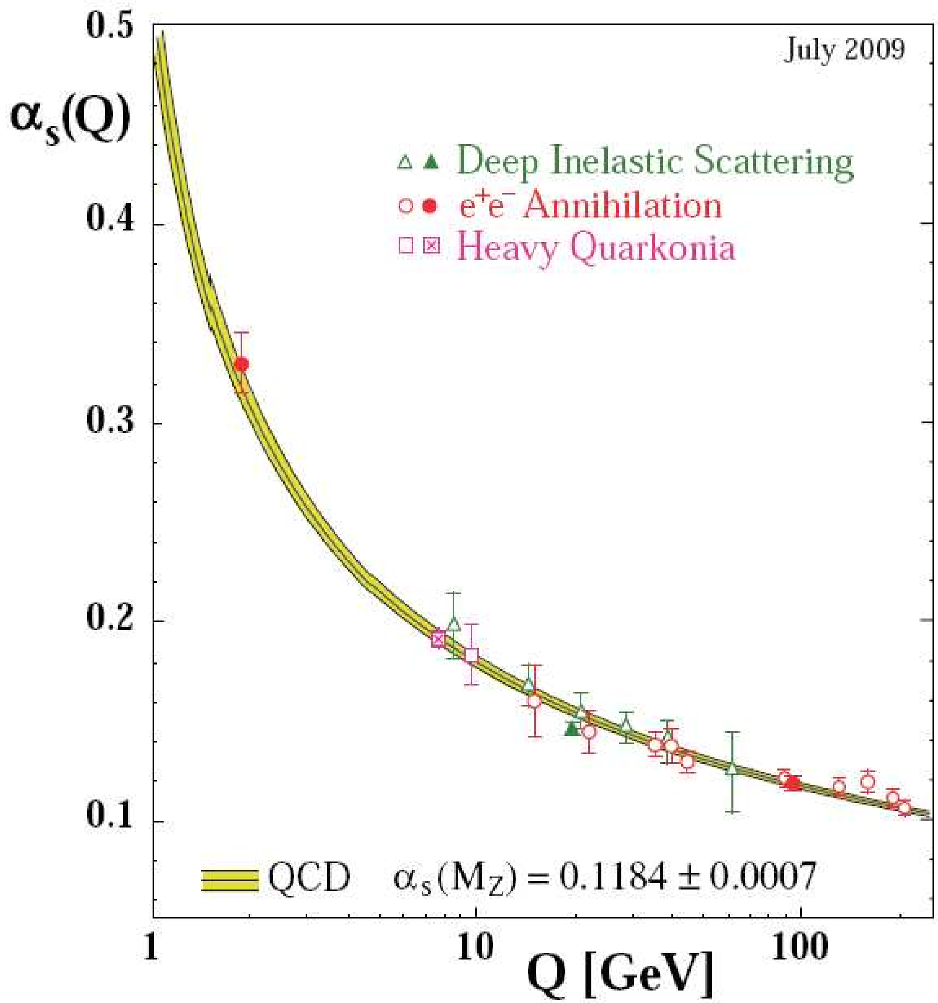

Fig. 5 : $\alpha_s(Q) = 4\pi\,\kappa(\mu = Q)$ **from ref. [A215-2009] .**

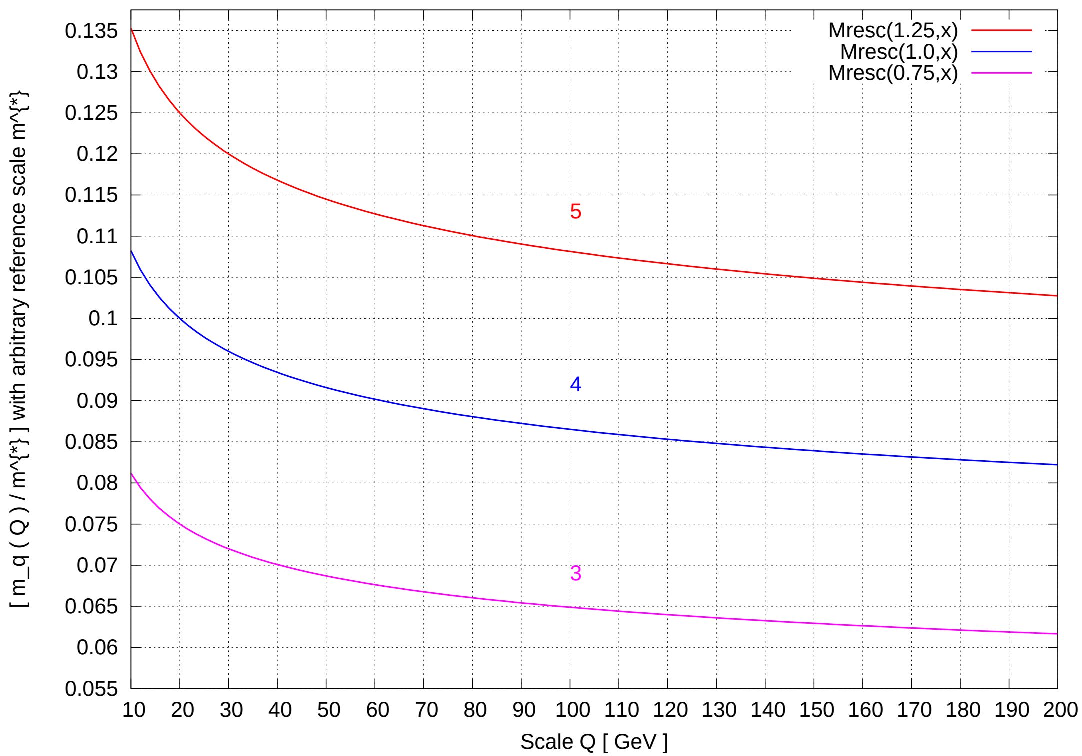

Fig. 6 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$$

from eqs. 15 - 17 , 146 - 147

$\longrightarrow$

1-1-b – Gauge boson binary bilocal and adjoint (here octet-) string operators

One goal is, to identify – not just some candidate resonance – gluonic mesons, binary and higher modes, and to relate them to the base quantities within QCD.

$$B_{[\mu_1 \nu_1],[\mu_2 \nu_2]}(x_1, x_2) =$$

(46)
$$= B^{\,r}_{[\mu_1 \nu_1]}(x_1)\, U(x_1, r; x_2, s)\, B^{\,s}_{[\mu_2 \nu_2]}(x_2)$$

$$r, s, \cdots = 1, \cdots, 8 \; ; \quad \text{no flavor but spin}$$

$B^{\,r}_{[\mu\nu]}(x)$ **denote the local color octet of field strengths.**

The quantity $U(x, r; y, s)$ **in eq. (46) denotes the octet string operator, i. e. the path ordered exponential over a straight line path** $\mathcal{C}$ **from y to x**

$$U(x, r; y, s) \quad = \quad P \exp\left(\int_y^x \Big|_{\mathcal{C}} d z^{\mu} \tfrac{1}{i} v^{\,t}_{\mu}(z) \mathcal{F}_t \right) \Bigg|_{r\,s}$$

(47)
$$= \quad P \exp\left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W^{\,t}_{\mu}(z) \left(\tfrac{1}{i} \mathcal{F}_t \right) \right) \Bigg|_{r\,s}$$

$$(\mathcal{F}_t)_{rs} = i f_{r\,t\,s} \; ; \quad (ad_t)_{rs} = \tfrac{1}{i} (\mathcal{F}_t)_{rs} = f_{r\,t\,s}$$

The path ordered exponential as a matrix function of the argument is to be performed before the matrix elements, denoted $\big|_{..}$ **in eq. 47 , are taken. The local limit becomes** →

(48) $$B_{[\mu_1\nu_1],[\mu_2\nu_2]}(x_1 = x_2 = x) = (:)\, B^{r}_{[\mu_1\nu_1]}(x)\, B^{r}_{[\mu_2\nu_2]}(x)(:)$$

no flavor but spin

1-1-c – $\overline{q}\,q$ bilinears and triplet-string operators

$$B^{q}_{[\dot{\mathcal{A}}\dot{f}_1,\mathcal{B}f_2]}(x_1,x_2) = \overline{q}^{\,\dot{c}_1}_{\mathcal{B}\dot{f}_2}(x_1)\, U(x_1,c_1;x_2,\dot{c}_2)\, q^{c}_{\mathcal{A}f_1}(x_2)$$

flavor *and* spin

(49) $$U(x,c_1;y,\dot{c}_2) = P\exp\left(\int_y^x\Big|_{\mathcal{C}} dz^{\mu}\,\frac{1}{i}\,v^{t}_{\mu}(z)\,\frac{1}{2}\lambda_t\right)\Bigg|_{c_1\dot{c}_2}$$

$$= P\exp\left(-\int_y^x\Big|_{\mathcal{C}} dz^{\mu}\,W^{t}_{\mu}(z)\,\left(\frac{1}{i}\,\frac{1}{2}\lambda_t\right)\right)\Bigg|_{c_1\dot{c}_2}$$

with the local limit

(50) $$B^{q}_{[\dot{\mathcal{B}}\dot{f}_2,\mathcal{A}f_1]}(x_1 = x_2 = x) = (:)\,\overline{q}^{\,\dot{c}}_{\mathcal{B}\dot{f}_2}(x)\, q^{c}_{\mathcal{A}f_1}(x)\,(:)$$

The symbols $(:)$ in eqs. 48 and 50 should indicate that normal ordering of regulating the local limits is required and further that such normal ordering is *not* unique , and dependent on quark masses in the case of the $\overline{q}\,q$ bilinears . →

1-1-d – Connection and curvature - forms
preparing the ensueing analysis of regularity conditions

Lets begin this (sub-)section rewriting the bilocal (formally) unitary operators forming the gauge connection dependent octet- (eq. 47) and triplet (eq. 49) QCD strings , substituting an equivalent , matrix oriented notation

$$\text{octet string}: U(x, r \;;\; y, s) = P \exp\left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t}(z) \left(\tfrac{1}{i} \mathcal{F}_t \right) \right)\Bigg|_{r\, s}$$

$$\text{triplet string}: U(x, c_1 ; y, \dot{c}_2) = P \exp\left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t}(z) \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t \right) \right)\Bigg|_{c_1 \dot{c}_2}$$

with the substitutions $\longrightarrow$

$$\left.\begin{array}{l} \text{octet string}: \; U(x, r \;;\; y, s) \rightarrow \left(U \left(x \overset{C}{\leftarrow} y \right) \right)_{rs} \\ \text{triplet string}: \; U(x, c_1 ; y, \dot{c}_2) \rightarrow \left(U \left(x \overset{C}{\leftarrow} y \right) \right)_{c_1 \dot{c}_2} \end{array} \right\} \rightarrow \begin{array}{r} \left(U(x, C, y ; \mathcal{D}) \right)_{\alpha\beta} \\ \in \mathcal{D}(\mathcal{G}) \end{array}$$

with $\mathcal{G} =$ **simple compact gauge group** ; $\mathcal{D}$: **general irreducible representation of** $\mathcal{G}$

(51)

Here $\mathcal{G} = SU3_c$ **and** $\mathcal{D}$ **is the adjoint- , triplet representation for the respective QCD** $\mathcal{D}$ **- strings.** $\rightarrow$

Further let us consider matrix valued connection 1-forms , which define the bilocal matrix valued operators $(U (x , C , y ; \mathcal{D}))_{\alpha\beta} \in \mathcal{D} (\mathcal{G})$ as given in eq. 51 . To this end the form of octet and triplet strings in eq. 51 is repeated below

$$\text{octet string}: U (x , r \; ; \; y , s) = P \exp \left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t} (z) \left(\tfrac{1}{i} \mathcal{F}_t \right) \right) \Bigg|_{r s}$$

$$\text{triplet string}: U (x , c_1 ; y , \dot{c}_2) = P \exp \left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t} (z) \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t \right) \right) \Bigg|_{c_1 \dot{c}_2}$$

(52)

The two matrices in brackets to the right of the integrand expressions in eq. 52 form an antihermitian basis of the Lie algebra representation $Lie (\mathcal{D})$ for $\mathcal{D} =$ adjoint and $\mathcal{D} =$ triplet representations of $\mathcal{G} = SU3_c$ respectively

(53) $$d_t \equiv d_t (\mathcal{D}) \leftrightarrow (d_t)_{\alpha\beta} = \begin{cases} \left(\tfrac{1}{i} \mathcal{F}_t \right)_{rs} & \text{for} \quad Lie (\mathcal{D}) = \text{adjoint} \\ \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t \right)_{c_1 \dot{c}_2} & \text{for} \quad Lie (\mathcal{D}) = \text{triplet} \end{cases}$$

$$d_t = - d_t^{\dagger} \; ; \; t = 1, \cdots, dim\, \mathcal{G} \; ; \; \alpha , \beta = 1, \cdots, dim\, \mathcal{D} \quad \text{for general } \mathcal{D}$$

$\longrightarrow$

From eqs. 52 and 53 we construct a – hopefully – consistent notation as appropriate for matrix valued $\mathcal{D}$ connections, 1-forms and strings, as well as derived 2- and higher forms. First eq. 53 is subject to the (matrix-) commutation relations

$$[\, d_r \,,\, d_s \,] \;=\; f_{rst}\, d_t \quad ; \quad \forall\, \mathcal{D}\,(\,\mathcal{G}\,) \quad ; \quad r, s, t \;=\; 1, \cdots, dim\, \mathcal{G} \qquad \longrightarrow$$

(54) $$\left(\, d_t \,(\, \mathcal{D} \;=\; \text{adjoint representation} \,)\,\right)_{sr} \;=\; (\, ad_t \,)_{sr} \;=\; f_{str} \quad : \quad \text{independent of } \mathcal{D}$$

f_{str} : totally antisymmetric, real structure constants of $Lie\,(\,\mathcal{G}\,)$

In physics the antihermitian matrix code with respect to the representations $Lie\,(\,\mathcal{D}\,)$ is (mostly) replaced by the hermitian one [a]

(55) $$\left(\, d_t \;\equiv\; \frac{1}{i}\, h_t \,\right) \,(\, Lie\,(\,\mathcal{D}\,)\,)\,\big|_{\alpha\beta} \quad ; \quad h_t \;=\; h_t^{\dagger} \quad ; \quad [\, h_r \,,\, h_s \,] \;=\; i\, f_{rst}\, h_t$$
$$\alpha, \beta \;=\; 1, \cdots, dim\, \mathcal{D}$$

Eq. 54 serves to define $\longrightarrow$

---

a A (partial) collection of historical and textbook references to the topics pertaining to 'Continous transformation groups and differential geometry' is presented at the end of references and labelled by the symbols 1H , 2H $\cdots$.

matrix valued connections built from a basis of $Lie\,(\,\mathcal{D}\,)$ representation matrices as defined in eq. 54 for general irredcible representations $\mathcal{D}\,(\,\mathcal{G}\,)$ denoted $\mathcal{W}_{\mu}\,(\,z\,,\,\mathcal{D}\,)$

(56)
$$\mathcal{W}_{\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \;=\; W_{\mu}^{\,r}\,(\,z\,)\,(\,d_{\,r}\,)_{\,\alpha\beta\;Lie\,(\,\mathcal{D}\,)} \quad ; \quad \left[\begin{array}{l} r \;=\; 1,\cdots,dim\,(\,\mathcal{G}\,) \\ \alpha\,,\,\beta \;=\; 1,\cdots,dim\,(\,\mathcal{D}\,) \end{array}\right]$$

$$\mathcal{W}_{\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \quad\longrightarrow\quad \mathcal{W}_{\mu}$$ **for compact matrix notation**

In the following it is to be understood, that $\mathcal{W}_{\mu}$ is extended to a general collection of representations $\bigcup\,\mathcal{D}$ – thought to be carried by real and spurious spin $\frac{1}{2}$ fields – care beeing taken that asymptotic freedom in the ultraviolet is not upset.

From eq. 56 we define the associated matrix valued connection 1-form displayed alongside the base definition repeated from eq. 56 in eq. 57 below

(57)
$$\begin{array}{rcll} \mathcal{W}^{\,(1)}\,(\,z\,,\,\mathcal{D}\,)\,\big|_{\,\alpha\beta} & = & d\,z^{\,\mu}\;\mathcal{W}_{\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} & \longrightarrow\quad \mathcal{W}^{\,(1)} \\ \mathcal{W}_{\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} & = & W_{\mu}^{\,r}\,(\,z\,)\,(\,d_{\,r}\,)_{\,\alpha\beta\;Lie\,(\,\mathcal{D}\,)} & \longrightarrow\quad \mathcal{W}_{\mu} \end{array}$$

and the matrix valued field-strength tensor

(58)
$$\mathcal{W}_{\mu\nu}\,(\,z,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \;=\; \left\{\begin{array}{c} \partial_{\,\mu}\,\mathcal{W}_{\nu}\,(\,z,\,\mathcal{D}\,) \;-\; \partial_{\,\nu}\,\mathcal{W}_{\mu}\,(\,z,\,\mathcal{D}\,) + \\ +\;[\,\mathcal{W}_{\mu}\,(\,z,\,\mathcal{D}\,)\,,\,\mathcal{W}_{\nu}\,(\,z,\,\mathcal{D}\,)\,] \end{array}\right\}_{\alpha\beta} \;\longrightarrow\; \mathcal{W}_{\mu\nu}$$

→

together with their associated curvature 2-form

$$\mathcal{W}^{(2)}(z,\mathcal{D})|_{\alpha\beta} = \frac{1}{2} dz^{\mu} \wedge dz^{\mu} \, \mathcal{W}_{\mu\nu}(z,\mathcal{D})|_{\alpha\beta} \longrightarrow \mathcal{W}^{(2)}$$

$$\mathcal{W}_{\mu\nu}(z,\mathcal{D})|_{\alpha\beta} = \left\{ \begin{array}{c} \partial_{\mu}\mathcal{W}_{\nu}(z,\mathcal{D}) - \partial_{\nu}\mathcal{W}_{\mu}(z,\mathcal{D}) + \\ + [\mathcal{W}_{\mu}(z,\mathcal{D}), \mathcal{W}_{\nu}(z,\mathcal{D})] \end{array} \right\}_{\alpha\beta} \longrightarrow \mathcal{W}_{\mu\nu}$$

$$= W^{\,r}_{\mu\nu}(z)\,(d_r)_{\alpha\beta\; Lie(\mathcal{D})}$$

$$W^{\,r}_{\mu\nu} = \partial_{\mu} W^{\,r}_{\nu} - \partial_{\nu} W^{\,r}_{\mu} + f_{rst}\, W^{\,s}_{\mu} W^{\,t}_{\nu} \;; \quad \textbf{independent of } \mathcal{D}$$

(59)

Two remarks are in place here

1) In order to distinguish field strength from potentials (connections) the following equivalent but different notations for the field strength shall be used

(60) $$\mathcal{W}_{\mu\nu} \equiv \mathcal{B}_{\mu\nu} \;;\; \mathcal{W}^{(2)} \equiv \mathcal{B}^{(2)} \;;\; W^{\,r}_{\mu\nu} \equiv B^{\,r}_{\mu\nu}$$

2) From the last relation in eq. 59 it may appear redundant to extend connections and curvatures to matrix valued form with respect to a wide collection of irreducible representations $\mathcal{D}(\mathcal{G})$. This however is tantamount to neglecting nontrivial global regularity conditions in the infrared . →

We end this subsection (1-1-d) displaying the bilocal (parallel transport-) operators defined in eq. 51 using the shorthand notation in eq. 59

$$(U (x , C , y ; \mathcal{D}))_{\alpha\beta} \quad = \quad P \exp \left(- \int_y^x \Big|_{\mathcal{C}} \mathcal{W}^{(1)} (z , \mathcal{D}) \right) \Bigg|_{\alpha\beta}$$

(61) $\downarrow \qquad \qquad \downarrow$

$$U (x , C , y) \quad = \quad P \exp \left(- \int_y^x \Big|_{\mathcal{C}} \mathcal{W}^{(1)} \right) \quad [\textbf{for} \quad (\bigcup \mathcal{D})]$$

1-1-e – The U1- or singlet axial current anomaly

The U1-axial central anomaly involves the local chiral current projections from $B^{\,q}_{\,[\,\dot{\mathcal{B}}\,\dot{f}\,2\,,\,\mathcal{A}\,f\,1\,]}(x)$ **in eq. 50**

$$\left(j^{\,\pm}_{\,\mu}\right)_{\,\dot{f}\,2\,f\,1}(x) \;=\; B^{\,q}_{\,[\,\dot{\mathcal{B}}\,\dot{f}\,2\,,\,\mathcal{A}\,f\,1\,]}(x)\,\left(\gamma_{\,\mu}\,\tfrac{1}{2}\,(\,\P\,\pm\,\gamma_{\,5}\,)\right)_{\mathcal{B}\,\dot{\mathcal{A}}}$$

(62)

$$=\;(:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}\,2}\,\gamma^{\,\pm}_{\,\mu}\,q^{\,c}_{\,f\,1}\,(\,x\,)\,(:)$$

$$\gamma_{\,5} \;=\; \gamma_{\,5\,R} \;=\; \tfrac{1}{i}\,\gamma_{\,0}\,\gamma_{\,1}\,\gamma_{\,2}\,\gamma_{\,3}\;\;;\;\;\gamma^{\,\pm}_{\,\mu} \;=\; \gamma_{\,\mu}\,\tfrac{1}{2}\,(\,\P\,\pm\,\gamma_{\,5}\,)$$

The equations of motion for the fermion fields are *and superficially imply* **(upon** $f_{\,1}\;\leftrightarrow\;f_{\,2}$ **)**

$$\begin{array}{lll} \partial\!\!\!/\,q^{\,c}_{\,f\,2} & = & \tfrac{1}{i}\,\left(\,\mathcal{V}\!\!\!/^{\,c\,\dot{c}'}\,+\,\delta^{\,c\,\dot{c}'}\,m_{\,f\,2}\right)\,q^{\,c'}_{\,f\,2} \\ \overline{q}^{\,\dot{c}}_{\,\dot{f}\,1}\,\overleftarrow{\partial\!\!\!/} & = & \overline{q}^{\,\dot{c}'}_{\,\dot{f}\,1}\,\tfrac{1}{i}\,\left(-\,\mathcal{V}\!\!\!/^{\,c'\,\dot{c}}\,-\,\delta^{\,c'\,\dot{c}}\,m_{\,f\,1}\right) \end{array}\quad ;\;\;\textbf{no sums over}\;\dot{f}_{\,1}\,,\,f_{\,2}\;\rightarrow$$

$$\partial^{\,\mu}\,\left(j^{\,\pm}_{\,\mu}\right)_{\,\dot{f}\,1\,f\,2} \;=\; \tfrac{1}{2\,i}\,\left(\,(\,m_{\,f\,2}\,-\,m_{\,f\,1}\,)\,S_{\,\dot{f}\,1\,f\,2}\,\mp\,(\,m_{\,f\,2}\,+\,m_{\,f\,1}\,)\,P_{\,\dot{f}\,1\,f\,2}\right)$$

$$S_{\,\dot{f}\,1\,f\,2} \;=\; (:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}\,1}\,q^{\,c}_{\,f\,2}\,(:)\;\;,\;\;P_{\,\dot{f}\,1\,f\,2} \;=\; (:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}\,1}\,\gamma_{\,5}\;\,q^{\,c}_{\,f\,2}\,(:)$$

(63)

In eq. 63 $m_{\,f}$ **denotes the** *real* , *nonnegative* **quark mass for flavor f.** $\rightarrow$

From eq. 63 the relations for vector and axial vector currents ***superficially*** **follow**

(64)
$$
\begin{aligned}
(j_\mu)_{\dot{f}_1 f_2} &= \left(j_\mu^+\right)_{\dot{f}_1 f_2} + \left(j_\mu^-\right)_{\dot{f}_1 f_2} \\
(j_\mu^5)_{\dot{f}_1 f_2} &= \left(j_\mu^+\right)_{\dot{f}_1 f_2} - \left(j_\mu^-\right)_{\dot{f}_1 f_2} \\
\partial^\mu (j_\mu)_{\dot{f}_1 f_2} &= \frac{1}{i} \left(m_{f_2} - m_{f_1}\right) S_{\dot{f}_1 f_2} \\
\partial^\mu (j_\mu^5)_{\dot{f}_1 f_2} &= \left(m_{f_2} + m_{f_1}\right) i P_{\dot{f}_1 f_2}
\end{aligned}
$$

As it follows from the original derivation by Adler and Bell and Jackiw [12-RR-1969] in QED, the vecor current Ward identities in eq. 64 can be implemented also in QCD , leaving the axial current ones reduced to the flavor non-singlet case , leaving the U1 axial current divergent anomalous

$$
\partial^\mu (j_\mu)_{\dot{f}_1 f_2} = \frac{1}{i} \left(m_{f_2} - m_{f_1}\right) S_{\dot{f}_1 f_2} \quad \surd
$$

(65)
$$
\left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}^{NS}_{\dot{f}_1 f_2} = \left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}_{\dot{f}_1 f_2} - \frac{1}{N_{fl}} \delta_{\dot{f}_1 f_2} \sum_f \left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}_{\dot{f} f}
$$

$\longrightarrow$

and similarly

(66) $$\left\{ \begin{array}{c} j^{\,5}_{\mu} \\ P \end{array} \right\}^{S}_{\dot{f}_1 f_2} = \sum_f \left\{ \begin{array}{c} j^{\,5}_{\mu} \\ P \end{array} \right\}_{\dot{f} f}$$

1-1-f – Quark masses and splittings : m_f and $\Delta m_f = m_f - \langle m \rangle$

In the subtitle above $\langle m \rangle$ stands for the mean quark mass

(67) $$\langle m \rangle = \frac{1}{N_{fl}} \sum_f m_f$$

The identities for vector currents in eqs. 64 and 65 can be extended separating the conributions proportional to Δm_f and $\langle m \rangle$

(68) $$\begin{aligned} \partial^{\mu} \left(j_{\mu} \right)_{\dot{f}_1 f_2} &= \frac{1}{i} \left(\Delta m_{f_2} - \Delta m_{f_1} \right) S_{\dot{f}_1 f_2} \quad \surd \\ \partial^{\mu} \left(j^{\,5}_{\mu} \right)^{NS}_{\dot{f}_1 f_2} &= \quad \left(\Delta m_{f_2} + \Delta m_{f_1} \right) i P^{NS}_{\dot{f}_1 f_2} \quad \surd \\ \partial^{\mu} \left(j^{\,5}_{\mu} \right)^{S}_{\dot{f}_1 f_2} &= 2 \langle m \rangle \, i P^{S} \quad \not\surd \; [\longrightarrow + \delta_5] \end{aligned}$$

[a] $$\delta_5 = \left(2 N_{fl} \right) \frac{1}{32\pi^2} B^{\,r}_{\mu\nu} \widetilde{B}^{\mu\nu r} \Big|_{\to ren.gr.inv} \; ; \; \widetilde{B}^{\,r}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} B^{\sigma\tau r}$$

→

[a] δ_5 was – as far as I know – introduced by Murray Gell-Mann in lectures $\sim$ 1970 in Hawaii .

We shall return to the question of how the local operator $ch_2 \left(B \right) \equiv \frac{1}{32 \pi^{\,2}} \, (:) \, B^{\,r}_{\mu \, \nu} \, \widetilde{B}^{\,\mu \, \nu \, r} \, (:)$ **is to be normalized and rendered renormalization group invariant [13-1991] . Here we just assume this to have been achieved and denote the U1-axial anomaly, the first of the central two, in its general form (eq. 68)**

(69)

$$\left\{ \partial^{\,\mu} \; \left(j^{\,5}_{\,\mu} \right)^{\,S} \; = \; 2 \, \left\langle \, m \, \right\rangle \, i \, P^{\,S} \; + \; \delta_{\,5} \right\} \left(\, x \, \right)$$

$$\delta_{\,5} \; = \; \left(\, 2 \, N_{\,fl} \, \right) \; \frac{1}{32 \pi^{\,2}} \; (:) \, B^{\,r}_{\mu \, \nu} \, \widetilde{B}^{\,\mu \, \nu \, r} \, (:) \, \Big|_{\,\rightarrow ren.gr.inv}$$

From here it is conceptually clear how the scale- (or trace-) anomaly arises but strictly within QCD . The renormalizability of a field theory in the limit of uncurved space-time gives rise to a local , symmetric and *conserved* **energy momentum tensor , implying exact Poincaré invariance**

(70)

$$\left\{ \, \vartheta_{\,\mu \, \nu} \; = \; \vartheta_{\,\nu \, \mu} \, \right\} \, \left(\, x \, \right)$$

$$\partial^{\,\nu} \, \vartheta_{\,\mu \, \nu} \; = \; 0$$

In connection with the normal ordering questions it is important to admit in the precise form of the energy momentum tensor a nontrivial vacuum expected value , which $\longrightarrow$

in view of exact Poincaré invariance must be of the form

$$\langle \Omega \,|\, \vartheta_{\mu\nu}(x) \,|\, \Omega \rangle = \tfrac{1}{4}\, \eta_{\mu\nu}\, \tau$$

$$\left\{ \begin{array}{c} \eta_{\mu\nu} = diag\,(1, -1, -1, -1) \\ \tau \end{array} \right\}$$ independent of x $\longrightarrow$

(71)

$$\Delta \vartheta_{\mu\nu}(x) = \vartheta_{\mu\nu}(x) - \langle \Omega \,|\, \vartheta_{\mu\nu}(x) \,|\, \Omega \rangle \times \left\{ \begin{array}{l} \widehat{¶} \\ \text{or} \quad |\, \Omega \rangle \langle \Omega \,| \end{array} \right.$$

with $\partial^{\nu} \Delta \vartheta_{\mu\nu}(x) = 0 \;\; ; \;\; \langle \Omega \,|\, \Delta \vartheta_{\mu\nu}(x) \,|\, \Omega \rangle = 0$

In eq. 71 $\widehat{¶}$ denotes the unit operator in the entire Hilbert space of states , while $P_{\Omega} = |\, \Omega \rangle \langle \Omega \,|$ stands for the projector on the ground state .

Furthermore from the two local , *conserved* tensors in eq. 71 only $\Delta \vartheta_{\mu\nu}(x)$ with vanishing vacuum expected value is acceptable as representing the conserved 4 momentum *operators* in the integral form

(72) $$\widehat{P}_{\mu} = \int_t d^3 x\, \Delta \vartheta_{\mu\,0}(t, \vec{x})$$

$\longrightarrow$

All these arguments ***notwithstanding*** **to subtract any eventual vacuum expected values of local operators , often put forward as mathematical prerequisites , it is wise** ***not to do so*** **in the presence of spontaneous parameters , the dynamical origin of spontaneous symmetry breaking, e.g. chiral symmetries in the limit or neighbourhood of some** $m_f \to 0$ **.**

Using the (classical) equations of motion pertaining to the Lagrangean in eqs. 1 - 6

$$(D_\nu B^{\mu\nu})^r = j^{\mu\, r} (\overline{q} , q) \;\; ; \;\; B \to B_{pert}$$

$$(D_\varrho B^{\mu\,\nu})^r = \partial_\varrho B^{\mu\,\nu\, r} + f_{rst} W_\varrho^s B^{\mu\,\nu\, t}$$

$$j_\mu^r (\overline{q} , q) = g\, \overline{q}_{\dot{\mathcal{A}}\, \dot{f}}^{\dot{c}} (\gamma_\mu)_{\dot{\mathcal{A}}\, \mathcal{B}} \tfrac{1}{2} (\lambda^r)_{c\dot{c}'} q_{\mathcal{A}\, f}^{c'}$$

(73)

$$i\, (\gamma^\mu D_\mu q)_{\mathcal{A}\, f}^c = m_f\, q_{\mathcal{A}\, f}^c \quad \textbf{and} \quad q \to \overline{q}$$

$$(D_\mu q)_{\mathcal{A}\, f}^c = \left[\partial_\mu \delta_{c\dot{c}'} + \tfrac{1}{i} W_\mu^t \tfrac{1}{2} \left(\lambda^t \right)_{c\dot{c}'} \right] q_{\mathcal{A}\, f}^{c'}$$

$$W_\mu^r \equiv - v_\mu^r = g \left(W_\mu^r \right)_{pert} \equiv - g \left(v_\mu^r \right)_{pert}$$

the associated form of the energy momentum becomes $\longrightarrow$

(74)
$$\vartheta^{(cl)}_{\mu\,\nu} = \left[\begin{array}{c} \frac{1}{4\,g^{\,2}} \left[B^{\,t}_{\mu\,\varrho}\, B^{\,\varrho\,t}_{\ \ \nu} - \frac{1}{4}\,\eta_{\,\mu\,\nu}\, B^{\,t}_{\sigma\,\varrho}\, B^{\,\varrho\,\sigma\,t} \right] + \\ + \frac{1}{2} \left[\overline{q}_{\,\dot{f}}\, \gamma_{\,\mu}\, \frac{i}{2}\, \overleftrightarrow{D}_{\nu}\; q_{\,f} + \mu \leftrightarrow \nu \right] \end{array} \right]$$

and using once more the fermion part of the equations of motion the trace of the classical energy momentum tensor becomes

(75)
$$\vartheta^{\,\mu\,(cl)}_{\ \ \mu} = \sum_{f} m_{\,f}\, S_{\,\dot{f}\,f}$$
$$S_{\,\dot{f}_{1}\,f_{2}} = (:)\, \overline{q}^{\,\dot{c}}_{\,\dot{f}_{1}}\, q^{\,c}_{\,f_{2}}\, (:)$$

1-1-g – The scale- or trace- anomaly

From the classical soft fermionic contribution to the trace of the energy momentum tensor there is a clear conjecture , also by Murray Gell-Mann , of the anomalous contribution , which subsequently became the scale- or trace- anomaly within QCD

(76)
$$\vartheta^{\,\mu}_{\ \ \mu} = \sum_{f} m_{\,f}\, S_{\,\dot{f}\,f} + \delta_{\,0}$$
$$\delta_{\,0} = -\, \left(-2\,\beta\,(\,g\,)\,/\,g^{\,3} \right) \left[\frac{1}{4}\,(:)\, B^{\,t}_{\mu\,\nu}\, B^{\,\mu\,\nu\,t}\,(:) \right]_{\,\rightarrow\, ren.gr.inv}$$

$\longrightarrow$

1-1-h – The two central anomalies alongside : scale- or trace- and U1-axial anomaly

We collect the two anomalous identities in eqs. 76 and 58

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_f m_f S_{\dot{f}\,f} + \delta_0 \right\}(x)$$
$$\left\{ \partial^{\mu} \left(j^5_{\mu} \right)^S = 2 \langle m \rangle \, i P^S + \delta_5 \right\}(x)$$

(77)

$$\delta_0 = - \left(-2\beta(g)/g^3 \right) \left[\tfrac{1}{4} (:) B^t_{\mu\nu} B^{\mu\nu\, t} (:) \right]_{\rightarrow ren.gr.inv}$$
$$\delta_5 = \left(2 N_{fl} \right) \frac{1}{8\pi^2} \left[\tfrac{1}{4} (:) B^t_{\mu\nu} \widetilde{B}^{\mu\nu\, t} (:) \right]_{\rightarrow ren.gr.inv}$$

$$-\beta/g^3 = \frac{1}{16\pi^2} b_0 + O(X) \; ; \; X = g^2/(16\pi^2$$

$\beta(g)$: **Callan-Symanzik rescaling function in QCD**

The qualification 'central' for the anomalies in eq. 77 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson ***renormalized*** **Lagrangean density x dependent**

(78)

$$\mathcal{L}_{g.b.} = - \frac{1}{g^2} \frac{1}{4} (:) B^t_{\mu\nu} B^{\mu\nu\, t} (:) + \vartheta \frac{1}{8\pi^2} \frac{1}{4} (:) B^t_{\mu\nu} \widetilde{B}^{\mu\nu\, t} \longrightarrow$$
$$g^2 \rightarrow g^2(x) \; ; \; \vartheta \rightarrow \vartheta(x)$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions $\longrightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(79) $$\frac{1}{4}\,(:)\,B^{\,t}_{\mu\,\nu}\,B^{\,\mu\,\nu\,t}\,(:) \quad , \quad \frac{1}{4}\,(:)\,B^{\,t}_{\mu\,\nu}\,\widetilde{B}^{\,\mu\,\nu\,t}\,(:)$$

We will use the following definitions relative to the rescaling function β

$$-\beta\,/\,g \;=\; X\,B\,(\,X\,) \;\;;\;\; B\,(\,X\,) \;=\; b_{\,0}\,A\,(\,X\,)$$

$$B\,(\,X\,) \sim \sum_{n=0}^{\infty} b_{\,n}\,X^{\,n} \quad , \quad A\,(\,X\,) \sim \sum_{n=0}^{\infty} a_{\,n}\,X^{\,n}$$

$$\kappa \;=\; g^{\,2}\,/\,(\,16\,\pi^{\,2}\,) \;\; \text{generic} \;\; \longrightarrow \;\; X\,,\,Y$$

(80) $$b_{\,0} \;=\; \frac{1}{3}\,\big(\,33 \,-\, 2\,N_{\,fl}\,\big) \quad , \quad a_{\,0} \;=\; 1 \quad , \quad a_{\,n} \;=\; b_{\,n}\,/\,b_{\,0}$$

$$b_{\,1} \;=\; \frac{2}{3}\,\big(\,153 \,-\, 19\,N_{\,fl}\,\big)$$

$$b_{\,2} \;=\; \frac{1}{54}\,\Big(\,77139 \,-\, 15099\,N_{\,fl} \,+\, 325\,N_{\,fl}^{\,2}\,\Big)$$

$$b_{\,3} \sim 29243 \,-\, 6946.3\,N_{\,fl} \,+\, 405.089\,N_{\,fl}^{\,2} \,+\, 1.49931\,N_{\,fl}^{\,3}$$

References in conjunction with this section (’1-1’) are presented in five (partial) collections :

1 : (R) directly related to the two central anomalies

2 : (rBsquare) establishing the one renormalization group invariant quantity of dimension $[\,M^{\,4}\,]$

3 : (r-sp-1) a recent paper by Guido Altarelli and references cited therein

4 : (r-A2x) a selection of papers and textbooks for the entire realm of QCD

5 : (r-condx) : Condensation phenomena and field theory realizations

$\longrightarrow$

2 – The scale Λ , the trace of the enrgy momentum density tensor and an essential complication to establish field equations , consistent with the trace anomaly

We repeat here eq. 77 from subsection 1-1-c , subsumming th sructure of the two 'central' anomalies

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_f m_f S_{\dot{f} f} + \delta_0 \right\} (x)$$
$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \langle m \rangle \, i P^{S} + \delta_5 \right\} (x)$$

(81)
$$\delta_0 = - \left(-2 \beta(g) / g^3 \right) \left[\tfrac{1}{4} (:) B^{t}_{\mu\nu} B^{\mu\nu t} (:) \right]_{\rightarrow ren.gr.inv}$$
$$\delta_5 = \left(2 N_{fl} \right) \frac{1}{8\pi^2} \left[\tfrac{1}{4} (:) B^{t}_{\mu\nu} \widetilde{B}^{\mu\nu t} (:) \right]_{\rightarrow ren.gr.inv}$$

$$- \beta / g^3 = \frac{1}{16\pi^2} b_0 + O(X) \; ; \; X = g^2 / (16\pi^2)$$

The predicate 'central' for the anomalies in eq. 81 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson *naively renormalized* **Lagrangean density x dependent**

(82)
$$\mathcal{L}_{g.b.} = - \frac{1}{g^2} \frac{1}{4} (:) B^{t}_{\mu\nu} B^{\mu\nu t} (:) + \vartheta \frac{1}{8\pi^2} \frac{1}{4} (:) B^{t}_{\mu\nu} \widetilde{B}^{\mu\nu t} \longrightarrow$$
$$g^2 \rightarrow g^2(x) \; ; \; \vartheta \rightarrow \vartheta(x)$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions $\longrightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(83) $$\frac{1}{4}\,(:)\,B^{\,t}_{\,\mu\,\nu}\,B^{\,\mu\,\nu\,t}\,(:) \quad , \quad \frac{1}{4}\,(:)\,B^{\,t}_{\,\mu\,\nu}\,\widetilde{B}^{\,\mu\,\nu\,t}\,(:)$$

We use the following definitions relative to the rescaling function β

$$-\,\beta\,/\,g \;=\; X\,B\,(\,X\,) \;\;;\;\; B\,(\,X\,) \;=\; b_{\,0}\,A\,(\,X\,)$$

(84) $$B\,(\,X\,) \,\sim\, \sum_{\,n=0}^{\,\infty}\, b_{\,n}\,X^{\,n} \quad , \quad A\,(\,X\,) \,\sim\, \sum_{\,n=0}^{\,\infty}\, a_{\,n}\,X^{\,n}$$

$$\kappa \;=\; g^{\,2}\,/\,(\,16\,\pi^{\,2}\,) \;\; \text{generic} \;\; \longrightarrow \;\; X\,,\,Y$$

2-2 – Consequences of the trace anomaly for the canonical structure behind compatible covariant field equations for gauge fields

The issue of consistency of equations of motion for gauge fields and the trace anomaly as shown in eq. 81 was taken up anew by the author of this outline in ref. [14-a-2012] . The associated canonical structure needs precise specifications in their own right. In addition it was endowed with *hypothetical* assigned properties, influencing the phase structure of QCD , in particular for the neighbourhood of vanishing chemical potentials and a range of (ratios of) quark masses of the light (u,d,s -) flavors including the realistic case, discussed in recent work in collaboration with Sonia Kabana , ref. [15-a-2010] . The latter work is also a reappraisal of questions left open in our earlier work, ref. [16-a-2000] . $\longrightarrow$

2-2-2 – Sketch of how to construct the local, gauge invariant and conserved energy momentum density tensor, compatible with the trace anomaly

We begin studying the problem of parallel transport and the equations determining a gauge in which the time component of the connection vanishes, for complete connections with respect to a simple local compact structure group $\mathcal{G}$

(85)
$$\left(\mathcal{W}_{\mu} \left(\mathcal{D} \right) \right)_{\alpha\beta} \left(x \right) = \mathcal{W}_{\mu}^{\;r} \left(x \right) \left(d_{r} \right)_{\alpha\beta}$$
$$d_{r} = - d_{r}^{\dagger} = \frac{1}{i} J_{r} \in Lie \left(\mathcal{D} \right) \;\; ; \;\; \left[d_{p} , d_{q} \right] = f_{pqr} \, d_{r}$$
$$r , p , q = 1, \cdots , dim\, \mathcal{G} \;\; ; \;\; \alpha , \beta = 1, \cdots , dim\, \mathcal{D}$$

The quantities defined in eq. 85 follow the notation in the notefile [17-a-2011], recapitulated below

(86)

$\left(\mathcal{W}_{\mu} \left(\mathcal{D} \right) \right)_{\alpha\beta} \left(x \right)$: **local operator** $\times$ $\mathcal{D} -$ **representation valued connection over flat space time** x

$d_{r} \in Lie \left(\mathcal{D} \right)$: **basis of antihermitian matrices forming an irreducible representation of the Lie algebra of** $\mathcal{G}$

$\mathcal{W}_{\mu}^{\;r} \left(x \right)$: $4 \times dim\, \mathcal{G}$ **components of hermitian local connection fields**

A connection is called complete, if all regularity- , differentiability- and integrability conditions extend to the complete ring of representations $\mathcal{R} \mid_{\mathcal{G}}$, pertaining to $\mathcal{G}$ [18-a-1968, 17-a-2011] . $\longrightarrow$

The matrix-operator valued 1-form associated with the gauge connection $(\mathcal{W}_{\mu}(\mathcal{D}))_{\alpha\beta}(x)$ in eq. 86 is denoted

(87) $$(\mathcal{W}^{(1)}(\mathcal{D}))_{\alpha\beta} = (\mathcal{W}_{\mu}(\mathcal{D}))_{\alpha\beta}(x)\, d x^{\mu} \longrightarrow \mathcal{W}^{(1)}(\mathcal{D})$$

parallel transport along a general and special set of curves $\mathcal{C}$

We define the parallel transport operators, along an oriented curve $\mathcal{C}$ from base point y to target point x denoted $U\left(x \overset{C}{\leftarrow} y\right)$

$$U\left(x \overset{C}{\leftarrow} y\right) = P \exp\left(-\int_{C} \mathcal{W}^{(1)}(\overline{x})\right)$$

$$\rightarrow (U(x; C, \{\mathcal{W}^{(1)}\}, y))_{\alpha\beta} \in \mathcal{D}(\mathcal{G})$$

---

(88)

$$C = C\{\overline{x}\} \quad :$$

$\overline{x}(\tau) \quad ; \quad 1 \geq \tau \geq 0 \; ; \; \tau$: path parameter

$$\overline{x}(\tau = 1) = x \leftarrow x_{1} \; ; \; \overline{x}(\tau = 0) = y \leftarrow x_{0}$$

The substitutions $y \leftarrow x_{1} ; x \leftarrow x_{0}$ are relevant if the generic base- and target- points are specified in a particular way. →

A remark concerning the operator definitions inherent in general gauges as is appropriate here.

- – the linear space on which gauge boson local operators act

  The use of Lorentz-covariant gauges and the BRST algebra of associated ghost fields , [19-a-1975, 20-a-1975] , demonstrate that the linear space on which all local fields associated with gauge fields act cannot be endowed with a positive definite norm , nor can formally associated ghost fields inherit a selfadjoint or hermitian structure. At most a Hilbert space definition necessitates a projection on gauge invariant quantities, which for local fields are composite ones. This is the price to pay for the consistent use of Poincarë covriant Feynman rules and integrals in loops, behaving consistently in the perturbatively accessible region.

$$U \left(x \overset{C}{\leftarrow} y \right) = P \exp \left(- \int_C \mathcal{W}^{(1)} (\overline{x}) \right)$$

defined in eq. 88

→

satisfies a first order differential equation whence the tangent vectors along the curve $\mathcal{C}$ are introduced

(89)
$$\underline{\mathbf{v}}^{\,\mu} \left(\partial_{x\,\mu} + \mathcal{W}_{\mu} (x) \right) U (x) \Big|_{x = \overline{x}(\tau)} = 0 \;\; ; \;\; \underline{\mathbf{v}}^{\,\mu} = \dot{\overline{x}}^{\,\mu}$$
$$U (x) \longleftarrow U \left(x ; \mathcal{C} , \{ \mathcal{W}^{(1)} (\mathcal{D}) \} , y \right)$$
$$U \left(y ; \mathcal{C} , \{ \mathcal{W}^{(1)} (\mathcal{D}) \} , y \right) = ¶$$

with the initial condition as specified in the last line of eq. 89 .

In line with the $\mathcal{D}$ projection we specify the $\mathcal{D} -$ dependent trace normalization

(90)
$$- tr_{\mathcal{D}} \; d^{\,p} d^{\,q} = \delta^{\,pq} \, T_2 (\mathcal{D})$$
$$- \sum_r (d^{\,r})^2 = C_2 (\mathcal{D}) \, ¶_{\,dim \, \mathcal{D} \times dim \, \mathcal{D}} \quad \rightarrow$$
$$T_2 (\mathcal{D}) = \frac{dim (\mathcal{D})}{dim (\mathcal{G})} \, C_2 (\mathcal{D}) \;\; ; \; \textbf{for} \; \left. \begin{array}{l} \mathcal{G} = SU3 \\ \mathcal{D} = 3 \textbf{ or } \overline{3} \end{array} \right\} :$$

$$dim \, \mathcal{D} = 3 \, , \; dim \, \mathcal{G} = 8 \, , \; T_2 (\mathcal{D}) = \tfrac{1}{2} \; C_2 (\mathcal{D}) = \tfrac{4}{3}$$

In eq. 90 $C_2 (\mathcal{D})$ denotes the eigenvalue of the second order Casimir operator projected on the irreducible representation $\mathcal{D}$ $\longrightarrow$

2-2-a – Bare Lagrangean density and equations of motion in unconstrained gauges

The bare local Lagrangean density is formed by the bare field strengths for its gauge part , and the quark-antiquark matter field part, using the relations in eq. 90

$$\mathcal{L}(x) = \frac{1}{4\,T_2(\mathcal{D})\,g^2}\, tr_{\mathcal{D}}\left[\mathcal{B}_{\mu\,\nu}\,\mathcal{B}^{\mu\nu}\right](\mathcal{D})(x) + \mathcal{L}_{\{q\}}(x)$$

(91)

$$\mathcal{L}_{\{q\}}(x) = \sum_{q-fl} \overline{q}^{\,\dot{c}'} \left\{ \frac{i}{2}\gamma^{\mu} \left[\left(\overrightarrow{D}_{\mu}^{\,(3)} \right)_{c'\dot{c}} - \left(\overleftarrow{D}_{\mu}^{\,(\overline{3})} \right)_{\dot{c}c'} \right] - m_q\,\delta_{c'\dot{c}} \right\} q^{c}(x)$$

→

The Euler-Lagrange equations for gauge fields – not including the color currents of the quark-antiquark matter fields – generating an extremum of the bare action take the form

(92)
$$\partial_{\varrho} \left\{ \frac{\delta \mathcal{L}}{\delta B^{r}_{\mu\nu}} \frac{\delta B^{r}_{\mu\nu}}{\delta \left(\partial_{\varrho} W^{s}_{\sigma} \right)} \right\} - \frac{\delta \mathcal{L}}{\delta B^{r}_{\mu\nu}} \frac{\delta B^{r}_{\mu\nu}}{\delta W^{s}_{\sigma}} = 0$$

$$\frac{\delta \mathcal{L}}{\delta B^{r}_{\mu\nu}} = \frac{1}{2 g^{2}} B^{\nu\mu\, r} \; ; \; \mathcal{L} = \mathcal{L}_{gauge} \; (\textbf{bare})$$

Including the contribution of quark-antiquark fields eq. 92 becomes

(93)
$$\partial_{\varrho} \left\{ \frac{\delta \mathcal{L}}{\delta B^{r}_{\mu\nu}} \frac{\delta B^{r}_{\mu\nu}}{\delta \left(\partial_{\varrho} W^{s}_{\sigma} \right)} \right\} - \frac{\delta \mathcal{L}}{\delta B^{r}_{\mu\nu}} \frac{\delta B^{r}_{\mu\nu}}{\delta W^{s}_{\sigma}} = \frac{\delta \mathcal{L}_{\{q\}}}{\delta W^{s}_{\sigma}}$$

$$\frac{\delta \mathcal{L}_{\{q\}}}{\delta W^{s}_{\sigma}} = \sum_{q-fl} \overline{q}^{\,\dot{c}'} \left\{ \gamma^{\sigma} \left(\tfrac{1}{2} \lambda^{s} \right)_{c' \dot{c}} \right\} q^{c} = \left(j^{\sigma\, s} \right)_{\{q\}}$$

→

Eq. 93 becomes

$$(D_{\varrho} (ad))_{sr} \left\{ \frac{1}{g^{2}} B^{\sigma \varrho r} \right\} = (j^{\sigma s})_{\{q\}}$$

(94)

$$(D_{\varrho} (ad))_{sr} = \partial_{\varrho} \delta_{sr} + W_{\varrho}^{t} (ad_{t})_{sr} \quad ; \quad (ad_{t})_{sr} = f_{str}$$

$$(j^{\sigma s})_{\{q\}} = \sum_{q-fl} \overline{q}^{\dot{c}'} \left\{ \gamma^{\sigma} \left(\frac{1}{2} \lambda^{s} \right)_{c' \dot{c}} \right\} q^{c}$$

It becomes obvious, that if we seek the energy momentum density tensor as a generalized Noether current, and compatible with the trace anomaly , the structural form of the bare Lagrangean quadratic in the gauge field strengths has to be essentially modified. This has been noted in refs. [21-a-1978, 22-a-1981] and is here taken up anew in the completion of QCD outside the perturbatively accessible region , following a first assessment in ref. [22-a-2012] . →

2-2-b - Energy momentum density tensor as a conserved generalized Noether current , restricted to $\overline{\mathcal{L}}$, replacing its bare form $\mathcal{L}$, in the absence of matter fields, i.e. neglecting $\mathcal{L}_{\{q\}}$

The search for consistent equations of motion for gauge fields is initiated by comparing the bare Lagrangean density as qualified in the title of this subsection . To this end we repeat the first part of eq. 91 below

(95) $$\mathcal{L}(x) = \mathcal{L}_{gauge} = \frac{1}{4\,T_2\,(\,\mathcal{D}\,)\,g^{\,2}}\; tr_{\mathcal{D}}\left[\mathcal{B}_{\mu\,\nu}\,\mathcal{B}^{\,\mu\nu}\right]\,(\,\mathcal{D}\,)\,(x)$$

The equations of motions induced by $\mathcal{L} \;\leftarrow\; \mathcal{L}_{gauge}$ are given in eqs. 91 - 94 , where on the right hand side either matter quark-antiquark contributions are included or , as relevant in this subsection , replaced by 0 .

In the following the suffix label $_{gauge}$ will be suppressed , for clarity, except when it is compromised .

$\longrightarrow$

Replacing the gauge field Lagrangean density to achieve consistency with the trace anomaly amounts to the substitution , as derived in ref. [22-a-2012]

(96)

$$\mathcal{L}_{gauge} = - \frac{1}{g^2} \mathcal{X} \rightarrow \overline{\mathcal{L}} = - \left[\frac{1}{\overline{g}^2 (\mathcal{X})} - J \right] \mathcal{X}$$

$$\mathcal{X} = \tfrac{1}{4} B^{\ r}_{\mu\nu} B^{\mu\nu\ r}$$

In eq. 96 J **stands for a suitable constant, while** $\overline{g} (\overline{l})$ **denotes the scale dependent coupling constant, as discussed also in ref. [21-a-1978] , established initially in the perturbative domain of QCD**

$$\overline{l} = \log (\overline{\mu} / \Lambda) = \tfrac{1}{4} \log \left(\overline{\mu}^{\,4} / \Lambda^4 \right) \rightarrow \tfrac{1}{4} \log \left(\mathcal{X} / \Lambda^4 \right) \rightarrow \tfrac{1}{8} \log \left((\mathcal{X} / \Lambda^4)^2 \right)$$

$$\overline{g} (\mathcal{X}) = \overline{g} \left(\overline{l} = \tfrac{1}{8} \log \left((\mathcal{X} / \mu^4)^2 \right) \right)$$

$$\frac{d}{d \overline{l}} \overline{g} = \beta (\overline{g}) = \overline{g} \, \overline{\kappa} \left[- \left(b_0 + b_1 \overline{\kappa}^{\,1} + \cdots \right) \right] ; \ \overline{\kappa} = \frac{\overline{g}^{\,2}}{16 \pi^2} ; \ \overline{\alpha}_s = 4 \pi \overline{\kappa}$$

$$\frac{d}{d \overline{l}} \overline{g}^{\,-2} = \frac{1}{8 \pi^2} b (\overline{\kappa}) ; \ b (\overline{\kappa}) = - (\overline{g} \, \overline{\kappa})^{-1} \beta (\overline{g}) = \left(b_0 + b_1 \overline{\kappa}^{\,1} + \cdots \right)$$

(97)

→

In order to substitute $\overline{\mu}^{\,4} \ \longrightarrow \ \mathcal{X}$ in accordance with eq. 97 in the entire domain of the field operator $\mathcal{X}$, the functional dependance of the sliding scale coupling constant $\overline{\kappa} \ (\ \mu \ / \ \Lambda \ )$ must be *known* beyond the perturbatively accessible region.

The variational derivatives of field strengths are

$$\frac{\delta \, B_{\mu\nu}^{\ r}}{\delta \ (\ \partial_{\varrho} \, W_{\sigma}^{\ s} \ )} = \delta^{\,rs} \ \left(\delta_{\mu}^{\ \varrho} \, \delta_{\nu}^{\ \sigma} \ - \ \delta_{\nu}^{\ \varrho} \, \delta_{\mu}^{\ \sigma} \right) \ ; \ \frac{\delta \, B_{\mu\nu}^{\ r}}{\delta \, W_{\sigma}^{\ s}} = f_{\,srt} \ \left(\delta_{\nu}^{\ \sigma} \, W_{\mu}^{\ t} \ - \ \delta_{\mu}^{\ \sigma} \, W_{\nu}^{\ t} \right)$$

(98)

and obtain the variational derivatives of of the quantity $\mathcal{X}$ defined in eq. 96

$$\frac{\delta \, \mathcal{X}}{\delta \ (\ \partial_{\varrho} \, W_{\sigma}^{\ s} \ )} = \frac{\delta \, \mathcal{X}}{\delta \, B_{\mu \ \nu}^{\ r}} \frac{\delta \, B_{\mu\nu}^{\ r}}{\delta \ (\ \partial_{\varrho} \, W_{\sigma}^{\ s} \ )} = B^{\,\varrho \ \sigma \ s}$$

(99)

$$\frac{\delta \, \mathcal{X}}{\delta \, B_{\mu \ \nu}^{\ r}} = \tfrac{1}{2} \, B^{\,\mu\nu \ r} \ ; \ \frac{\delta \, \mathcal{X}}{\delta \ (\ W_{\sigma}^{\ s} \ )} = \frac{\delta \, \mathcal{X}}{\delta \, B_{\mu \ \nu}^{\ r}} \frac{\delta \, B_{\mu\nu}^{\ r}}{\delta \ (\ W_{\sigma}^{\ s} \ )}$$

$$\mathcal{X} \ = \ \tfrac{1}{4} \, B_{\mu\nu}^{\ r} \, B^{\,\mu\nu \ r} \ = \ - \, g^{\,2} \, \mathcal{L}_{\,gauge}$$

$\longrightarrow$

The variation of the trace anomaly induced Lagrangean density $\overline{\mathcal{L}}$, defined in eq. 96 , is obtained by repeated use of the chain rule

(100)

$$\frac{\delta \overline{\mathcal{L}}}{\delta \left(\partial_{\varrho} W^{s}_{\sigma} \right)} = \left[\frac{\delta \overline{\mathcal{L}}}{\delta \mathcal{X}} \frac{\delta \mathcal{X}}{\delta B^{r}_{\mu\nu}} \right] \frac{\delta B^{r}_{\mu\nu}}{\delta \left(\partial_{\varrho} W^{s}_{\sigma} \right)}$$

$$\frac{\delta \overline{\mathcal{L}}}{\delta \left(W^{s}_{\sigma} \right)} = \left[\frac{\delta \overline{\mathcal{L}}}{\delta \mathcal{X}} \frac{\delta \mathcal{X}}{\delta B^{r}_{\mu\nu}} \right] \frac{\delta B^{r}_{\mu\nu}}{\delta \left(W^{s}_{\sigma} \right)}$$

$$\frac{\delta B^{r}_{\mu\nu}}{\delta \left(\partial_{\varrho} W^{s}_{\sigma} \right)} = \delta^{rs} \left(\delta^{\varrho}_{\mu} \delta^{\sigma}_{\nu} - \delta^{\varrho}_{\nu} \delta^{\sigma}_{\mu} \right)$$

$$\frac{\delta B^{r}_{\mu\nu}}{\delta W^{s}_{\sigma}} = f_{srt} \left(\delta^{\sigma}_{\nu} W^{t}_{\mu} - \delta^{\sigma}_{\mu} W^{t}_{\nu} \right)$$

→

We proceed to evaluate the derivatives in eq. 100 using eq, 97 .

The quantities in $[\,.\,]$ brackets in the first two relations of eq. 100 contain expressions wherein both canonically conjugate variables appear. Hence an ordering is necessary. This is straightforward in all such situations and amounts to neglecting all local terms arising from the noncommutativity of the latter.

(101)
$$\frac{\delta\,\overline{\mathcal{L}}}{\delta\,\mathcal{X}} = \overline{\mathcal{L}}\,/\,\mathcal{X} - \left(\frac{d}{d\,\bar{l}}\;\overline{g}^{\,-2}\right)\frac{\delta\,\bar{l}}{\delta\,\mathcal{X}}\;\mathcal{X}$$
$$\bar{l} = \tfrac{1}{8}\,\log\Big(\,(\,\mathcal{X}\,/\,\mu^{\,4}\,)^{\,2}\Big)$$

$$\frac{d}{d\,\bar{l}}\;\overline{g}^{\,-2} = \frac{1}{8\,\pi^{\,2}}\;b\,(\,\overline{\kappa}\,)$$
$$b\,(\overline{\kappa}) = -\,(\,\overline{g}\,\overline{\kappa}\,)^{\,-1}\,\beta\,(\,\overline{g}\,) = -\,\big(b_{\,0} + b_{\,1}\,\overline{\kappa}^{\,1} + \cdots\big)$$

We substitute the derivatives from eq. 99

(102)
$$\frac{\delta\,\mathcal{X}}{\delta\,B^{\,r}_{\,\mu\,\nu}} = \tfrac{1}{2}\,B^{\,\mu\nu\;r}$$

→

This yields the derivatives of $\mathcal{X}$, using eq. 100

$$\frac{\delta \, \mathcal{X}}{\delta \, (\, \partial_{\,\varrho} \, W_{\,\sigma}^{\,s} \,)} = B^{\,\varrho \, \sigma \, s} \quad ; \quad \frac{\delta \, \mathcal{X}}{\delta \, W_{\,\sigma}^{\,s}} = \tfrac{1}{2} \, B^{\,\mu\nu \, r} \, \frac{\delta \, B_{\,\mu\nu}^{\,r}}{\delta \, W_{\,\sigma}^{\,s}}$$

(103)

$$\begin{aligned} \frac{\delta \, \mathcal{X}}{\delta \, W_{\,\sigma}^{\,s}} &= - \, f_{\,str} \, W_{\,\varrho}^{\,t} \, B^{\,\varrho \, \sigma \, r} \\ &= \tfrac{1}{2} \, B^{\,\mu\nu \, r} \; f_{\,srt} \, \left(\, \delta_{\,\nu}^{\,\sigma} \, W_{\,\mu}^{\,t} \, - \, \delta_{\,\mu}^{\,\sigma} \, W_{\,\nu}^{\,t} \, \right) \\ &= \tfrac{1}{2} \, f_{\,srt} \; \left(\, W_{\,\mu}^{\,t} \, B^{\,\mu \, \sigma \, r} \, - \, W_{\,\nu}^{\,t} \, B^{\,\sigma \, \nu \, r} \, \right) \end{aligned}$$

Next we derive the energy momentum density tensor $\vartheta_{\,\nu}^{\;\;\mu} \, (\, x \,)$ as a local functional of the Lagrangean density $\overline{\mathcal{L}}$ and the gauge field variables $\partial_{\,\varrho} \, W_{\,\sigma}^{\,s} \longleftrightarrow B_{\,\varrho\sigma}^{\,s}$, $W_{\,\sigma}^{\,s}$, as they appear e.g. in eq. 103 by variational techniques going back to Emmy Noether [23-a-1918] .

First we consider general variations of the base variables, understood to depend continuously on a family parameter, denoted f . This relates to the equations of motion

(104)

$$W_{\,\sigma}^{\,s} = W_{\,\sigma}^{\,s} \, (\, f \, ; \, x \,) \quad ; \quad \delta \, W_{\,\sigma}^{\,s} \, (\, x \,) = \partial_{\,f} \; W_{\,\sigma}^{\,s} \, (\, f \, ; \, x \,) \, \big|_{\,f=0} \quad \rightarrow$$

$$\delta \, \partial_{\,\varrho} \, W_{\,\sigma}^{\,s} \, (\, x \,) = \partial_{\,\varrho} \, \delta \, W_{\,\sigma}^{\,s} \, (\, x \,)$$

→

and determine the conditions for the f-dependent action integral over a general 4-dimensional volume V

(105) $$S(V, f) = \int_V d^4x \, \overline{\mathcal{L}} \, (W_\sigma^{\ s}(f, x); \partial_\varrho W_\sigma^{\ s}(f; x))$$

to acquire an extremal value for the particular member of the family of base fields corresponding to $f = 0$.

This gives rise to the condition

$$\delta S = \partial_f \; S(V, f)|_{f=0} = 0 \quad \rightarrow$$

$$\delta S =$$

$$= \int_V d^4x \left(\delta W_\sigma^{\ s} \; \overline{\mathcal{L}}_{, W_\sigma^{\ s}} \Big|_{f=0} + \delta \partial_\varrho W_\sigma^{\ s} \; \overline{\mathcal{L}}_{, \partial_\varrho W_\sigma^{\ s}} \Big|_{f=0} \right) = 0$$

---

(106) $$\overline{\mathcal{L}}_{, W_\sigma^{\ s}} \Big|_{f=0} \rightarrow \overline{\mathcal{L}}_{, W_\sigma^{\ s}} = \frac{\partial \overline{\mathcal{L}}}{\partial W_\sigma^{\ s}(f = 0; x)}$$

$$\overline{\mathcal{L}}_{, \partial_\varrho W_\sigma^{\ s}} \Big|_{f=0} \rightarrow \overline{\mathcal{L}}_{, \partial_\varrho W_\sigma^{\ s}} = \frac{\partial \overline{\mathcal{L}}}{\partial (\partial_\varrho W_\sigma^{\ s})(f = 0; x)}$$

→

For simplicity of notation we neglect the symbol $|_{f=0}$ in the quantities $\overline{\mathcal{L}}_{,W^{s}_{\sigma}}$, $\overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}}$ in the following as indicated in eq. 106 δS (eq. 106) becomes using eq. 104

$$\delta S = \int_V d^4x \left(\delta W^{s}_{\sigma} \, \overline{\mathcal{L}}_{,W^{s}_{\sigma}} + (\partial_{\varrho} \delta W^{s}_{\sigma}) \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) \quad \rightarrow \tag{107}$$

$$(\partial_{\varrho} \delta W^{s}_{\sigma}) \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} = \partial_{\varrho} \left(\delta W^{s}_{\sigma} \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) - \delta W^{s}_{\sigma} \; \partial_{\varrho} \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}}$$

Substituting the second relation in eq. 107 we obtain

$$\delta S = \int_V d^4x \left[\begin{array}{l} \delta W^{s}_{\sigma} \; \left(\overline{\mathcal{L}}_{,W^{s}_{\sigma}} - \partial_{\varrho} \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) + \\ + \partial_{\varrho} \left(\delta W^{s}_{\sigma} \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) \end{array} \right] \tag{108}$$

The variational setting is incomplete without conditions on the boundary of the volme V – boundary conditions – which arise from the integration of the divergence in the last term in eq. 108

$$\int_V d^4x \; \partial_{\varrho} \left(\delta W^{s}_{\sigma} \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) = \int_{\partial V} d^3\sigma_{\varrho \, , \, \partial V} \; \left(\delta W^{s}_{\sigma} \, \overline{\mathcal{L}}_{,\partial_{\varrho} W^{s}_{\sigma}} \right) \tag{109}$$

$\longrightarrow$ **boundary conditions :** $\delta W^{s}_{\sigma}(x) = 0$ **for** $x \in \partial V$

With the boundary conditions as defined in eq. 109 satisfied, the extremum condition for the action integral (for all volumes V) and *otherwise* arbitrary variations $\delta W^{s}_{\sigma}(x)$ $\rightarrow$

is equivalent to the local Euler-Lagrange equations of motion (eq. 108)

(110) $$\left(\partial_{\varrho} \overline{\mathcal{L}}_{, \partial_{\varrho} W^{s}_{\sigma}} - \overline{\mathcal{L}}_{, W^{s}_{\sigma}} \right) (x) = 0$$

Together this sets the stage for deriving associating with every additional symmetry of the Lagrangean density a conserved Nöther 'current' , but with general spin, depending on the symmetry involved.

We apply the variations as generally given in eq. 104 to the special case relating rigid space-time translations to the canonical energy momentum tensor

(111) $$\begin{array}{c} x^{\nu} \rightarrow f a^{\nu} \quad \longleftrightarrow \quad \begin{array}{rcl} \delta_{a} W^{s}_{\sigma} & = & a^{\nu} \partial_{\nu} W^{s}_{\sigma} \\ \delta_{a} \partial_{\varrho} W^{s}_{\sigma} & = & a^{\nu} \partial_{\nu} \partial_{\varrho} W^{s}_{\sigma} \end{array} \\ \hline \delta_{a} \overline{\mathcal{L}} = a^{\nu} \left(\begin{array}{l} (\partial_{\nu} W^{s}_{\sigma}) \overline{\mathcal{L}}_{, W^{s}_{\sigma}} + \\ + (\partial_{\varrho} \partial_{\nu} W^{s}_{\sigma}) \overline{\mathcal{L}}_{, \partial_{\varrho} W^{s}_{\sigma}} \end{array} \right) = a^{\nu} \partial_{\nu} \overline{\mathcal{L}} \end{array}$$

The underlying translation symmetry reveals itself subtracting the last term in the expression for $\delta_{a} \overline{\mathcal{L}}$ in eq. 111 from the second to yield

(112) $$\begin{array}{l} a^{\nu} \left(\partial_{\varrho} \left[(\partial_{\nu} W^{s}_{\sigma}) \overline{\mathcal{L}}_{, \partial_{\varrho} W^{s}_{\sigma}} - \delta^{\varrho}_{\nu} \overline{\mathcal{L}} \right] - (\partial_{\nu} W^{s}_{\sigma}) \mathcal{E}^{\sigma s} \right) = 0 \\ \mathcal{E}^{\sigma s} (x) = \left(\partial_{\varrho} \overline{\mathcal{L}}_{, \partial_{\varrho} W^{s}_{\sigma}} - \overline{\mathcal{L}}_{, W^{s}_{\sigma}} \right) (x) \rightarrow 0 \end{array}$$

→

From eq. 112 we read off the canonical energy momentum density tensor in mixed components

(113)
$$T_{\nu}^{\ \varrho} = (\partial_{\nu} W^{\ s}_{\sigma}) \overline{\mathcal{L}}_{, \partial_{\varrho} W^{\ s}_{\sigma}} - \delta^{\varrho}_{\nu} \overline{\mathcal{L}} \ ; \ \partial_{\varrho} T_{\nu}^{\ \varrho} = 0$$
$$T_{\nu \varrho} = \eta_{\rho \tau} T_{\nu}^{\ \tau} \ ; \ T_{\nu \varrho} - T_{\varrho \nu} \neq 0$$

$T_{\nu \varrho}$ is neither symmetric nor gauge invariant. Nevertheless a symmetric and gauge invariant energy momentum density tensor can always be achieved, by also considering variations of $\overline{\mathcal{L}}$ under Lorentz transformations.

The consistensy of the gravitational coupling of $\sqrt{g} \ \overline{\mathcal{L}} (g_{\mu\nu} ; B^{\ s}_{\sigma \ \tau})$, with $g = - Det \, g_{\mu\nu}$, allows to simplify the explicit symmetrization procedure , due to Belinfante [24-a-1940] , in the limit $g_{\mu \nu} \rightarrow \eta_{\mu\nu}$, i.e. in uncurved space time , after considering a variation of the metric to first order . Here I follow my derivation in ref. [22-a-1981] , denoting the symmetric energy momentum tensor $\vartheta_{\nu \varrho}$.

$$2 \delta \left(\sqrt{g} \ \overline{\mathcal{L}} \right) = \sqrt{g} \ \vartheta_{\mu \nu} (\delta g^{\mu \nu}) \Big|_{g_{\alpha \beta} \rightarrow \eta_{\alpha \beta}}$$

---

(114)
$$\delta \left(\sqrt{g} \ \overline{\mathcal{L}} \right) = (\delta \sqrt{g}) \ \overline{\mathcal{L}} + \sqrt{g} \ \delta \overline{\mathcal{L}}$$
$$\delta \ : \ g^{\mu \nu} \rightarrow g^{\mu \nu} + \delta g^{\mu \nu} \ ; \ \left[g_{\mu\nu} \ : \ \text{base metric} \right]$$
$$g^{\mu \nu} = \left(g^{-1} \right)_{\mu \nu} \delta \sqrt{g} = - \tfrac{1}{2} \sqrt{g} \, g_{\mu \nu} (\delta g^{\mu \nu})$$

→

In eq. 114 $\overline{\mathcal{L}}$ is, for general metric, a scalar quantity and $\vartheta_{\mu\nu}$ a symmetric tensor .
The variation of $\overline{\mathcal{L}}$ becomes using eq. 101

$$\delta \overline{\mathcal{L}} = \overline{\mathcal{L}}_{,X} \delta X \quad ; \quad X = \tfrac{1}{4} B^{\,s}_{\mu\sigma} B_{\nu\tau} g^{\mu\nu} g^{\sigma\tau}$$

(115) $$\overline{\mathcal{L}}_{,X} = \frac{\delta \overline{\mathcal{L}}}{\delta \mathcal{X}} = \overline{\mathcal{L}} / \mathcal{X} - \left(\frac{d}{d\,\bar{l}} \; \overline{g}^{\,-2} \right) \frac{\delta \bar{l}}{\delta \mathcal{X}} \mathcal{X}$$

$$\bar{l} = \tfrac{1}{8} \log \left((\mathcal{X} / \mu^{4})^{2} \right)$$

Proceeding step by step we first substitute

(116) $$\frac{\delta \bar{l}}{\delta \mathcal{X}} \mathcal{X} = \tfrac{1}{4} \quad ; \quad \overline{\mathcal{L}} / \mathcal{X} = - \left(\overline{g}^{\,-2} - J \right)$$

in the second relation in eq. 115

(117) $$\delta \overline{\mathcal{L}} = \left[- \left(\overline{g}^{\,-2} - J \right) - \tfrac{1}{4} \left(\frac{d}{d\,\bar{l}} \; \overline{g}^{\,-2} \right) \right] \delta \mathcal{X}$$

→

Next we recall eqs. 96 - 97 and 101

$$\frac{d}{d\,\overline{l}}\;\overline{g}^{\,-2} \;=\; 2\,\left(\,-\,\beta\,(\,\overline{g}\,)\,/\,\overline{g}^{\,3}\,\right) \;=\; \tfrac{1}{8\,\pi^{\,2}}\,b_{\,0}\,B\,(\,\overline{\kappa}\,)$$

$$\overline{\kappa} \;=\; \frac{\overline{g}^{\,2}}{16\,\pi^{\,2}}\;;\;\overline{\alpha}_{\,s} \;=\; 4\pi\,\overline{\kappa}$$

(118)

$$B\,(\,\overline{\kappa}\,) \;=\; b\,(\,\overline{\kappa}\,)\,/\,b_{\,0} \;=\; [\,-\,\beta\,(\,\overline{g}\,)\,/\,\overline{g}\,]\;(\,b_{\,0}\,\overline{\kappa}\,)^{\,-1}$$

$$b_{\,0} \;=\; 11 \;-\; \tfrac{2}{3}\,N_{\,fl}$$

$$B\,(\,\overline{\kappa}\,) \;=\; B_{\,0} \;+\; B_{\,1}\,\overline{\kappa} \;+\; \cdots\,)$$

$$B_{\,0} \;=\; 1 \quad , \quad B_{\,n} \;=\; b_{\,n}\,/\,b_{\,0} \quad ; \quad n \;=\; 0.1.2\,\cdots$$

The differential equation in eq. 118 is to be solved for suitable initial conditions as if l , $\overline{\kappa}$ were c-numbers and the functional dependence as indicated in eq. 119 below , evaluated with the operator valued (local field valued) substitution, defined in eq. 101 . →

(119) $$\overline{\kappa} \;=\; \overline{\kappa}\,(\,l\,) \;=\; F\,(\,l\,) \;\;;\;\; l \longrightarrow \tfrac{1}{8}\,\log\Big(\,(\,\mathcal{X}\,/\,\mu^{\,4}\,)^{\,2}\,\Big)$$

In the substition, defined in eqs. 101 and 119 , local products of local operators are involved, which again requires a regularisation in the sense of normal ordering of canonical variables . We insert the relation in eq. 118 in eq. 117 and obtain

(120) $$\delta\,\overline{\mathcal{L}} \;=\; \Big[\,-\,\big(\,\overline{g}^{\,-2}\,-\,J\,\big)\,-\,\tfrac{1}{32\,\pi^{\,2}}\,b_{\,0}\,B\,(\,\overline{\kappa}\,)\,\Big]\;\delta\,\mathcal{X}$$

Next we evaluate $\delta\,\mathcal{X}$ using the first relation in eq. 101

(121) $$\begin{aligned} &\delta\,\overline{\mathcal{L}} \;=\; \overline{\mathcal{L}}_{\,,\,X}\,\delta\,X \;\;;\;\; X \;=\; \tfrac{1}{4}\,B^{\;s}_{\,\mu\sigma}\,B_{\,\nu\tau}\,g^{\,\mu\,\nu}\,g^{\,\sigma\,\tau} \;\;\longrightarrow \\ &\delta\,X \;\;=\;\; \tfrac{1}{2}\;B^{\;s}_{\,\mu\sigma}\,g^{\,\sigma\,\tau}\,B^{\;s}_{\,\nu\tau}\,\Big|_{\,g^{\,\alpha\,\beta}\,\rightarrow\,\eta^{\,\alpha\,\beta}}\;\delta\,g^{\,\mu\,\nu} \\ &\;\;=\;\; -\,\tfrac{1}{2}\;\big(\,B^{\;s}_{\,\mu\sigma}\,B^{\,\sigma\;s}_{\;\;\nu}\,\big)\;\delta\,g^{\,\mu\,\nu} \end{aligned}$$

The - sign in the last expression in eq. 121 together with a transposition of the tensor indices in $B^{\,\sigma\;s}_{\;\;\nu}$ is chosen to offset the two - signs in the expression inside [.] brackets in eq. 120 as well as to comply with the analogous expressions in QED [15-B2-1976] .

Inserting the expressions in eq. 121 in eq. 120 the final form of $\delta\,\overline{\mathcal{L}}$ becomes

(122) $$\delta\,\overline{\mathcal{L}} \;=\; \tfrac{1}{2}\,\Big[\,\big(\,\overline{g}^{\,-2}\,-\,J\,\big)\,+\,\tfrac{1}{32\,\pi^{\,2}}\,b_{\,0}\,B\,(\,\overline{\kappa}\,)\,\Big]\;\big(\,B^{\;s}_{\,\mu\sigma}\,B^{\,\sigma\;s}_{\;\;\nu}\,\big)\;\delta\,g^{\,\mu\,\nu}$$

$\longrightarrow$

Finally we complete the expression for the symmetric energy momentum tensor $\vartheta_{\mu \nu}$ in eq. 114 , which is conserved in the limit of uncurved space time

$$2 \delta \left(\sqrt{g} \, \overline{\mathcal{L}} \right) = = \sqrt{g} \, \vartheta_{\mu \nu} \left(\delta g^{\mu \nu} \right) \Big|_{g_{\alpha \beta} \rightarrow \eta_{\alpha \beta}} \quad \rightarrow$$

(123)

$$\begin{array}{l} 2 \delta \left(\sqrt{g} \, \overline{\mathcal{L}} \right) = \\ = \sqrt{g} \left[\begin{array}{l} \left[\left(\overline{g}^{-2} - J \right) + \frac{1}{32 \pi^2} b_0 B (\overline{\kappa}) \right] B^{\ \ s}_{\mu\sigma} B^{\sigma \ s}_{\ \nu} \\ - \frac{1}{4} g_{\mu\nu} \left(\overline{g}^{-2} - J \right) B^{\ \ s}_{\varrho\sigma} B^{\sigma \varrho s} \end{array} \right] \delta g^{\mu \nu} \end{array}$$

Combining the two terms proportional to $\overline{g}^{-2} - J$ in eq. 123 we obtain two characteristic contributions to $\sqrt{g} \, \vartheta_{\mu \nu} \left(\delta g^{\mu \nu} \right)$, even before the limit $g_{\mu \nu} \rightarrow \eta_{\mu \nu}$ is taken .

It follows

(124) $$\vartheta_{\mu \nu} = \left[\begin{array}{c} \left(\overline{g}^{-2} - J \right) \left[B^{\ \ s}_{\mu\sigma} B^{\sigma \ s}_{\ \nu} - \frac{1}{4} g_{\mu\nu} B^{\ \ s}_{\varrho\sigma} B^{\sigma \varrho s} \right] \\ + \frac{1}{8 \pi^2} b_0 B (\overline{\kappa}) \left[\frac{1}{4} B^{\ \ s}_{\mu\sigma} B^{\sigma \ s}_{\ \nu} \right] \end{array} \right]$$

The upper term in the outer [.] brackets in eq. 124 , proportional to $\overline{g}^{-2} - J$, is traceless , whereas the anomalous trace is contained in the lower term , proportional to $\frac{1}{8 \pi^2} b_0 B (\overline{\kappa})$. $\rightarrow$

In the limit of vanishing gravitational interactions , i.e. $g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$, the energy momentum tensor $\vartheta_{\mu\nu}$, defined in eq. 124 , is not only symmetric but conserved , provided the *modified* equations of motion can be enforced, as well as gauge invariant with respect to the local gauge group of QCD $SU3_c$

(125) $$\partial^{\mu} \vartheta_{\mu\nu}(x) = 0 \;\; ; \;\; \partial^{\mu} = \eta^{\mu\beta} \partial_{\beta} = \eta^{\mu\beta} \partial / \partial_{x^{\beta}}$$

The trace anomaly allows precisely the general form of the modified Lagrangean $\overline{\mathcal{L}}$, as introduced in eq. 96 including the a priori free constant J . It takes the form using eq. 124 , independent of J

(126) $$\overline{\mathcal{L}} = - \left[\frac{1}{\overline{g}^{2}(\mathcal{X})} - J \right] \mathcal{X} \longrightarrow$$

$$\vartheta^{\mu}{}_{\mu} = \frac{1}{8\pi^2} b_0 B(\overline{\kappa}) \left[\tfrac{1}{4} B^{s}_{\mu\sigma} B^{\sigma\mu\, s} \right] = - \frac{1}{8\pi^2} b_0 \; : \; B(\overline{\kappa}(\mathcal{X})) \, \mathcal{X} \; :$$

In the last expression in eq. 126 the need of (normal) ordering of local products of quantized fields is indicated by the : symbols , omitted before and also hereafter whenever not explicitely necessary .
The negative sign in the same expression is significant: the factor $B(\overline{\kappa})(\mathcal{X})$ is a positive operator at least for $\overline{\kappa}$ in the perturbative regime, wile $\mathcal{X}$ becomes a positive operator in the Euclidean region

(127) $$\mathcal{X} = \tfrac{1}{2} \left[\vec{B}^{s} \vec{B}^{s} - \vec{E}^{s} \vec{E}^{s} \right] \rightarrow$$
$$\rightarrow \mathcal{X}_{Euc} = \tfrac{1}{2} \left[\vec{B}^{s} \vec{B}^{s} + \vec{E}^{s} \vec{E}^{s} \right]_{Euc}$$

→

This completes the construction of the trace anomaly induced canonical variables and modified Lagrangean $\overline{\mathcal{L}}$. Consequences are discussed in the next section .

3 - Consequences arising from derivations in the previous section

We collect the reults contained in eqs. 98 - 103 , 110 - 113 , 115 - 127 in eq. 128 below

$$\vartheta_{\mu\nu} = \left[\begin{array}{c} \left(\overline{g}^{\,-2} - J \right) \left[B^{\;s}_{\mu\sigma} B^{\;\sigma\;s}_{\nu} - \frac{1}{4} g_{\mu\nu} B^{\;s}_{\varrho\sigma} B^{\;\sigma\,\varrho\,s} \right] \\ + \frac{1}{8\pi^2} b_0 B(\overline{\kappa}) \left[\frac{1}{4} B^{\;s}_{\mu\sigma} B^{\;\sigma\;s}_{\nu} \right] \end{array} \right]$$

(128) $B(\overline{\kappa}) = \leftarrow : B\left(\overline{\kappa}\left(\mathcal{X} / \Lambda^4\right)\right) : \rightarrow \; ; \; \mathcal{X} =: \frac{1}{4} B^{\;r}_{\mu\nu} B^{\mu\nu\,r} := -g^2 \mathcal{L}_{gauge}$

$$B(\overline{\kappa}) = b(\overline{\kappa}) / b_0 = \left[-\beta(\overline{g}) / \overline{g} \right] \left(b_0 \overline{\kappa} \right)^{-1} \quad ; \quad b_0 = 11 - \frac{2}{3} N_{fl}$$

$$B(\overline{\kappa}) = B_0 + B_1 \overline{\kappa} + \cdots) \; ; \; B_0 = 1 \, , \; B_n = b_n / b_0 \; ; \; n = 0.1.2 \cdots$$

The symbol $\leftarrow : B\left(\overline{\kappa}\left(\mathcal{X} / \Lambda^4\right)\right) : \rightarrow$ specifically the external arrows , shall indicate that all normal orderings which have to be applied to products of gauge field operators and functions thereof , as they appear in the various expressions in eq. 128 have to extend from the exterior leftmost to the exterior rightmost operators . →

We distinguish both the canonical energy momentum (density) tensors (eq. 113) and their symmetric equivalents according to the two choices for the gauge field Lagrangean (density) (eq. 123) , abbreviated to $\mathcal{L}_{\,gauge} \,=\, \mathcal{L}$ and $\overline{\mathcal{L}}$ respectively

(129)
$$T_{\nu}{}^{\varrho} \;=\; (\,\partial_{\nu}\;W^{\,s}_{\,\sigma}\,)\,\mathcal{L}^{\,genaral}_{\;,\,\partial_{\,\varrho}\,W^{\,s}_{\,\sigma}} \;-\; \delta^{\,\varrho}_{\,\nu}\,\mathcal{L}^{\,general}$$

$$2\,\delta\,\left(\sqrt{g}\;\;\mathcal{L}^{\,general}\right) \;=\; \sqrt{g}\;\;\vartheta_{\,\mu\,\nu}\;(\,\delta\,g^{\,\mu\,\nu}\,)\,\Big|_{\,g_{\,\alpha\,\beta}\,\rightarrow\,\eta_{\,\alpha\,\beta}} \quad\longrightarrow$$

$$T_{\nu}{}^{\varrho} \;=\; T_{\nu}{}^{\varrho}\,(\,.\,)\;;\;\vartheta_{\,\mu\,\nu} \;=\; \vartheta_{\,\mu\,\nu}\,(\,.\,)\quad;\quad \text{with}\quad .\;=\;\left\{\,\overline{\mathcal{L}}\,,\mathcal{L}\,\right\}$$

The Belinfante construction [24-a-1940] links uniquely $T^{\,\mu}_{\,\nu}\,(\,.\,)\;\leftrightarrow\;\vartheta_{\,\mu\,\nu}\,(\,.\,)$, separately for both values of $(\,.\,)$. The respective canonical energy momentum tensors are neither symmetric nor gauge invariant also for both choices of $(\,.\,)$, whereas both $\vartheta_{\,\mu\,\nu}\,(\,.\,)$ are gauge invariant.
Furthermore all energy momentum tensors are conserved , provided the equations of motion are enforced

(130)
$$\partial^{\,\mu}\,T_{\,\nu\mu}\,(\,.\,) \;=\; 0 \quad;\quad T_{\,\nu\mu}\,(\,.\,) \;=\; \eta_{\,\mu\varrho}\,T_{\,\nu}{}^{\varrho}\,(\,.\,)$$

$$\partial^{\,\mu}\,\vartheta_{\,\nu\mu}\,(\,.\,) \;=\; 0 \quad;\quad \vartheta_{\,\nu\mu}\,(\,.\,) \;=\; \vartheta_{\,\mu\nu}\,(\,.\,)$$

→

Notwithstanding the two cases

it is $\overline{\mathcal{L}}$ which needs to be chosen.

With respect to the trace anomaly (eq. 126) and thereby with respect to dilatation transformations, for which the symmetric tensor(s) $\vartheta_{\nu\mu}(\,.\,)$ are the relevant ones, it follows

(131)
$$\begin{aligned} \vartheta^{\mu}{}_{\mu}(\overline{\mathcal{L}}) &= \frac{1}{8\pi^2}\, b_0 \,:\, B\left(\overline{\kappa}\left(\mathcal{X}/\Lambda^4\right)\right)\left[\tfrac{1}{4}\, B^{\,s}_{\mu\sigma}\, B^{\sigma\mu\, s}\right] : \\ &= -\frac{1}{8\pi^2}\, b_0 \,:\, B\left(\overline{\kappa}\left(\mathcal{X}/\Lambda^4\right)\right)\mathcal{X} :\neq 0 \end{aligned}$$

$$\vartheta^{\mu}{}_{\mu}(\mathcal{L}) = 0$$

Concerning redundancy of normal order proedures it shall be noted , that here the normal ordering is restricted to obey chain rules of (covariant) differentiation . This is exemplified by the relations in eq. 100 and as another example the relation

(132)
$$\partial_{\varrho} \,:\, B^{\,s}_{\mu\nu}(x)\, B^{\,s}_{\sigma\tau}(x) \,:=:\, \left[\begin{array}{l} \left(\left(D_{\varrho}(ad)\right)_{sr} B^{\,r}_{\mu\nu}\right) B^{\,s}_{\sigma\tau} \\ + B^{\,s}_{\mu\nu}\left(\left(D_{\varrho}(ad)\right)_{sr} B^{\,r}_{\mu\nu}\right) \end{array}\right] :$$

$$\left(D_{\varrho}(ad)\right)_{sr} = \partial_{\varrho}\,\delta_{sr} + W^{\,t}_{\varrho}\,(ad_t)_{sr} \quad ; \quad (ad_t)_{sr} = f_{str}$$

→

In the following normal ordering symbols $:\ (\ )\ :$ shall be omitted whenever possible , for shortness of notation , except if crucial to avoid misinterpretation .

3-1 - Equations of motion and canonically conjugate variables pertaining to $\overline{\mathcal{L}} + \mathcal{L}_{\{q\}}$

The conflict between the canonical forms of obviously inequivalent Lagrangean densities in the gauge field sector

(133)
$$\overline{\mathcal{L}} \longleftrightarrow \mathcal{L} \ \Big|_{gaugefields}$$

can be traced – in the perturbative sector – to maintaining Poincaré invariance and full gauge invariance , which necessitates the use of Fermi like gauges and thus a nontrivial coupling of ghost fields, which do contribute to the energy momentum tensor , as shown e.g. in ref. [20-1975] . At this stage we turn to the equations of motion for gauge fields as induced by the Lagrangean $\overline{\mathcal{L}} + \mathcal{L}_{\{q\}}$ defined in eqs. 1 - 6 and 96 - 97 , merging eqs. 103 and 120

(134)
$$\delta \overline{\mathcal{L}} = - \left[\left(\overline{g}^{\,-2} - J \right) + \frac{1}{32\,\pi^{2}}\, b_{0}\, B \left(\overline{\kappa} \right) \right] \delta \mathcal{X}$$
$$\frac{\delta \mathcal{X}}{\delta \left(\partial_{\varrho} W_{\sigma}^{\,s} \right)} = B^{\,\varrho\,\sigma\,s} \ ; \ \frac{\delta \mathcal{X}}{\delta W_{\sigma}^{\,s}} = - f_{\,str}\, W_{\varrho}^{\,t}\, B^{\,\varrho\,\sigma\,r}$$

$\longrightarrow$

It follows for the partial derivatives of $\overline{\mathcal{L}}$

(135)
$$\overline{\mathcal{L}}_{\,,\,\partial_{\varrho} W^{\,s}_{\sigma}} = + \left[\left(\overline{g}^{\,-2} - J \right) + \frac{1}{32\,\pi^{\,2}}\, b_{\,0}\, B\left(\overline{\kappa} \right) \right] B^{\,\sigma\,\varrho\,s}$$
$$\overline{\mathcal{L}}_{\,,\,W^{\,s}_{\sigma}} = - \left[\left(\overline{g}^{\,-2} - J \right) + \frac{1}{32\,\pi^{\,2}}\, b_{\,0}\, B\left(\overline{\kappa} \right) \right] f_{\,str} \times W^{\,t}_{\varrho}\, B^{\,\sigma\varrho\;r}$$

Using eq. 132

(136)
$$\left(D_{\,\varrho}\left(ad \right) \right)_{\,sr} \left\{ B^{\,r}_{\sigma\tau} \right\} = \partial_{\,\varrho} \left\{ B^{\,s}_{\sigma\tau} \right\} + f_{\,str}\; W^{\,t}_{\varrho} \left\{ B^{\,r}_{\sigma\tau} \right\}$$

the Euler-Lagrange derivative of $\overline{\mathcal{L}}$ on the left hand side of eq. 110 take the form

(137)
$$\left(\partial_{\,\varrho} \overline{\mathcal{L}}_{\,,\,\partial_{\varrho} W^{\,s}_{\sigma}} - \overline{\mathcal{L}}_{\,,\,W^{\,s}_{\sigma}} \right) \left(x \right) =$$
$$= D_{\,\varrho}\left(ad \right) \left\{ \left[\left(\overline{g}^{\,-2} - J \right) + \frac{1}{32\,\pi^{\,2}}\, b_{\,0}\, B\left(\overline{\kappa} \right) \right] B^{\,\sigma\,\varrho} \right\}^{\,s} \left(x \right)$$

The Euler-Lagrange equations including quark flavors according to $\mathcal{L}_{\,\{q\}}$ (eqs. 1 , 94 , 110) become

(138)
$$D_{\,\varrho}\left(ad \right) \left\{ \left[\left(\overline{g}^{\,-2} - J \right) + \frac{1}{32\,\pi^{\,2}}\, b_{\,0}\, B\left(\overline{\kappa} \right) \right] B^{\,\sigma\,\varrho} \right\}^{\,s} \left(x \right) =$$
$$= \left(j^{\,\sigma\;s} \right)_{\,\{q\}} \left(x \right)$$

$$\left(j^{\,\sigma\;s} \right)_{\,\{q\}} = \sum_{\,q-fl} \overline{q}^{\,\dot{c}'} \left\{ \gamma^{\,\sigma} \left(\tfrac{1}{2}\, \lambda^{\,s} \right)_{\,c'\dot{c}} \right\} q^{\,c} = \left(\mathcal{L}_{\,\{q\}} \right)_{\,,\,W^{\,s}_{\sigma}}$$

→

The quantity on the gauge field side of the 'divergence' on the right hand side of eq. 137 and on the left hand side of the first relation in eq. 138 differs in an essential way from the bare Lagrangean expressions corresponding to $\mathcal{L}_{gauge}$ (eqs. 94 , 139 below)

(139)
$$D_{\varrho} = \partial_{\varrho} + \mathcal{W}_{\varrho} \; ; \; \mathcal{W}_{\varrho} = W_{\varrho}^{r} \, ad_{r}$$
$$D^{\varrho} \left\{ \frac{1}{g^{2}} B_{\sigma\varrho} \right\}^{s} = j_{\sigma}^{s}$$

To make this more transparent lets introduce the notation

(140)
$$\overline{\mathcal{L}} \leftrightarrow \overline{G} \quad \leftrightarrow \quad \mathcal{L} \leftrightarrow G$$
$$\overline{G} = \text{local, scalar, color neutral field} \quad \leftrightarrow \quad G = g^{-2}$$

const. c-number

$$\overline{G} =: \left[\left(\overline{g}^{-2} - J \right) + \tfrac{1}{32 \pi^{2}} b_{0} B \left(\overline{\kappa} \right) \right] : (x)$$

$\longrightarrow$

The precise way , $\overline{G}$ and associated quantities, in particular the energy momentum (density) tensor pertaining to gauge fields, defined in eq. 124, are determined as local field operators , shall be (re-) specified below

$$(141) \qquad \begin{array}{l} (a) \ : \ \overline{G} \ = \ \overline{G} \ (\ l \ ) \ ; \ l \ = \ \log \ (\ \overline{\mu} \ / \ \mu \ ) \quad \rightarrow \\ (b) \ : \ l \ \rightarrow \ \frac{1}{8} \ \log \ \left(\left(\ \mathcal{X} \ (\ x \ ) \ / \ \Lambda^{\ 4} \ \right)^{\ 2} \right) \ \leftrightarrow \ e^{\ 8 \ l} \ = \ \left(\ \mathcal{X} \ (\ x \ ) \ / \ \Lambda^{\ 4} \ \right)^{\ 2} \\ \hline \mathcal{X} \ (\ x \ ) \ =: \ \frac{1}{4} \ B^{\ r}_{\ \mu\nu} \ B^{\ \mu\nu \ r} \ : \ (\ x \ ) \end{array}$$

The functional dependence $\overline{G} \ = \ \overline{G} \ (\ l \ )$ in step (a) in eq. 141 can be determined in the perturbative regime, through the renormalization group equation(s) , e.g. in the $\overline{\text{MS}}$ renormalization scheme . Through 4 loop order $\overline{\text{MS}}$ is renormalization group invariant, through the substiturions pertaining to a general sliding scale μ , $\mu \ = \ \Lambda_{QCD}$, and e.g. through the moments of deep inelastic scattering amplitudes , $\overline{\mu}^{\ 2} \ = \ Q^{\ 2}$, where Q^2 is the (positive) momentum transfer square in the deep inelastic reaction studied [29-1988, 30-1997] . In any renormalization scheme, where at least in the perturbative regime full renormalization group invariance is verified , it follows that the substitution in the second step (b) in eq. 141 can equally be performed in a renormalization group invariant manner . This is tantamount to resolve all ambiguities in the definition of the composite local field $\mathcal{X} \ (\ x \ )$ by a normalization in terms of renormalization group invariant , i.e. measurable quantities . $\longrightarrow$

Within QCD sum rules introduced by Shifman, Vainshtain and Zakharov [40-1979] , the vacuum expected value of the multiplicatively related field , denoted $\alpha_s G^2$ has been intensively studied .
I cite here a recent paper and result(s) by Stephan Narison [11-2011, 45-2011] in partcular with respect to the renormalization group invariant setting – in principle – of composite local field normalization

(142) $\alpha_s G^2 = \pi^{-1} \mathcal{X}$;

$$\begin{aligned} \langle \Omega \,|\, \alpha_s G^2 \,|\, \Omega \rangle &= (7.0 \pm 1.3)\, 10^{-2}\ \text{GeV}^4 \\ &= \pi^{-1}\,(0.22 \pm 0.04)\ \text{GeV}^4 \end{aligned}$$

However balancing of short distance ordered contributions from local operators of increasing mass dimension in a finite region of reference energies corresponding to characteristic, i.e. nonasymptotic hadronic ones, with contributions in the same energy interval from hadrons , proves to be less straightforward, than initially thought. The invoked compensations , traditionally called duality relations, may well not hold upon truncation of the ultraviolet and normal scale contributions to a few leading terms, translating into increased errors relative to initial estimates, exemplified by the errors in eq. 142 . I quote here a derivation of the ratio of up , down and strange quark masses by Cesareo Dominguez and Eduardo de Rafael [26-a-1991] in which subtle analytic methods are presented.
Having defined the local field structure of two inequivalent such fields $\mathcal{X}(x)$ and *separately* $\overline{G}(x)$ in eqs. 140 and 141, the symmetric, gauge invariant energy momentum tensor incompletely defined in eqs. 123 and 124 $\longrightarrow$

is represented as follows

$$\vartheta_{\mu\,\nu}(x) = \begin{bmatrix} :\overline{G}\,[\,B^{\;s}_{\mu\sigma}\,B^{\sigma\;s}_{\;\nu} + g_{\mu\nu}\,\mathcal{X}\,]\;: \\ -\,g_{\mu\nu}\;:\;\frac{1}{32\,\pi^{\,2}}\,b_{\,0}\,B\,(\,\overline{\kappa}\,(\,\mathcal{X}\,)\,\mathcal{X}\;: \end{bmatrix}(x)$$

(143)
$$\vartheta^{\,\mu}_{\;\;\mu}(x) = -\,\frac{1}{8\,\pi^{\,2}}\,b_{\,0}\;:\;B\,(\,\overline{\kappa}\,(\mathcal{X}\,)\,)\,\mathcal{X}\;:\,(x)$$

$$\overline{G}\,(x) = \;:\left[\left(\overline{g}^{\,-2} - J\right) + \frac{1}{32\,\pi^{\,2}}\,b_{\,0}\,B\,(\,\overline{\kappa}\,(\,\mathcal{X}\,)\,)\right]:\,(x)$$

$$B\,(\,\overline{\kappa}\,(\,\mathcal{X}\,)\,)\,(x) \rightarrow\equiv \overline{B}\,(\,\mathcal{X}\,)\,(x)$$

We complete the Euler-Lagrange equations of motion defined in eqs. 137 and 138 , which introduce a local, color neutral, hermitian scalar field , denoted $\varphi\,(\,x\,)$ below , depending implicitely on the gauge field strengths bilinear $\mathcal{X}\,(\,x\,)$ as specified in eqs. 140 - 143

$$\left(\partial_{\,\varrho}\,\overline{\mathcal{L}}_{\,,\,\partial_{\,\varrho}\,W^{\;s}_{\;\sigma}} - \overline{\mathcal{L}}_{\,,\,W^{\;s}_{\;\sigma}}\right)(x) =$$

(144)
$$= D_{\,\varrho}\,(\,ad\,)\;:\left\{\left[\begin{pmatrix} \overline{g}^{\,-2} - J + \\ +\,\frac{1}{32\,\pi^{\,2}}\,b_{\,0}\,\overline{B} \end{pmatrix}(\,\mathcal{X}\,)\right] B^{\,\sigma\,\varrho}\right\}^{s}\;:\,(x) \;\rightarrow$$

$$\varphi\,(x) = \;:\;\left(\overline{g}^{\,-2} - J + \frac{1}{32\,\pi^{\,2}}\,b_{\,0}\,\overline{B}\right)(\,\mathcal{X}\,)\;:\,(x) \equiv \overline{G}\,(x)$$

The field φ introduced in eq. 144 is *dimensionless* , which in itself is a consequence →

of the violation of dilatation invariance, or the trace anomaly .

The equations of motion for the gauge fields (eqs. 138 , 143) become , suppressing the ordering signs : ( ) :

$$D_{\varrho}\,(ad)\,\{\varphi\ B^{\sigma\varrho}\}^{s}\,(x) = (j^{\sigma s})_{\{q\}}\,(x)$$

$$(j^{\sigma s})_{\{q\}} = \sum_{q-fl} \overline{q}^{\,\dot{c}'}\left\{\gamma^{\sigma}\left(\tfrac{1}{2}\lambda^{s}\right)_{c'\dot{c}}\right\} q^{c} = \left(\mathcal{L}_{\{q\}}\right)_{,W_{\sigma}^{s}}$$

(145)
$$\left[\begin{array}{c} D_{\varrho}\,(ad) \\ \{\varphi\ B^{\sigma\varrho}\}^{s} \\ (x) \end{array}\right] = \left[\begin{array}{c} \partial_{x\,\varrho}\,\{\varphi\,(x)\,B^{\sigma\varrho s}\,(x)\}\,+ \\ +\,f_{str}\,\{W_{\varrho}^{\,t}\,(x)\,\varphi\,(x)\,B^{\sigma\varrho\, r}\,(x)\} \end{array}\right]$$

$$= \left[\begin{array}{c} \{\partial_{x\,\varrho}\,\varphi\,(x)\}\,\{B^{\sigma\varrho s}\,(x)\}\,+ \\ +\,\varphi\,(x)\,\{D_{\varrho}\,(ad)\,B^{\sigma\varrho}\}^{s}\,(x) \end{array}\right]$$

$$\varphi\,(x) = :\left(\overline{g}^{\,-2} - J + \frac{1}{32\,\pi^{2}}\,b_{0}\,\overline{B}\right)(\mathcal{X}) : (x)$$

In deriving eq. 145 we assumed that chain rules for normal partial derivatives and covariant derivatives are maintained through the ordering processes , suppressed for simplicity of notation . →

3-2 - Concluding remarks and outlook

To set the present results in perspective I quote from ref. [27-a-2010] (PM , November 2010, Singapore)

"$\cdots$ The conventional form for the bosonic Bogoliubov transformation is the one corresponding to $|\,\vartheta\,| \; < \; 1$.

Let me remark , that neither the real form nor the conventinal normalization (= 1) are representing the full structure associated with *one* bosonic pair of oscillators .

Epilogue

The embedding of chiral symmetry depends in a nontrivial way on the strength of the *gauge field strength pair-* Bose condensate as does the excitation of binary and higher gauge boson compounds ('glueballs') *and* the phase structure of QCD .

There is some way to go. I hope to come back to this theme soon. $\cdots$ "

and further $\longrightarrow$

3-11-b

”. . .

1) Perturbative accessibility of renormalization in asymptotically free theories

While the entire renormalization procedure thus (eqs. 9 - 16) comes within *perturbative accessibility* **– as explained in textbooks [21] , [22-1982] – the associated renormalization group equation serves to restore renormalization group invariant properties, in particular such definitions of operators .**

2) Infrared instability

is associated with all physical scales *not* **accessible to perturbative approximations .**

3) Quark mass dependence

We neglect in the considerations followed here the quark mass dependence of all Green functions in the deep Euclidean region on quark masses , the latter also *to be renormalized* **and thus** *not* **renormalization group independent [23-1975] . This is in line with the main short distance contributions , which are sorting out by the twist characteristic leading contributions** *modulo less dominant ones modulo powers of inverse Euclidean distance* **. These dimensional hierarchies** *also* **break down whence the region of perturbative accessibility is transgressed . For small quark masses at a generic scale of** $\sim 1\,GeV$ **the quark mass associated mixing of operators with different dimensions sets in in subtle ways governed by approximate chiral symmetry also outside the deep Euclidean region .**

. . . ”

$\longrightarrow$

4) Canonical structure

The step performed here brings in line the structure of the trace anomaly with the derivation of consistent equations of motion for the gauge fields , displayed in eq. 145 . It shows that to transcend the perturbatively accessible region the canonical structure of gauge boson associated 'p'-s and 'q'-s has to be modified accordingly .

5) Outlook

The completion of QCD as a gauge field theory in uncurved space time remains a far goal . Along the way let us keep in mind that (for all we know) the physical reality transcends much further, in particular to curved space time of unknown dimensionality .

— Thank you —

Appendix 1 - expansion coefficients of the rescaling functions $\widehat{\beta}\,,\gamma$ to four loops

$$-\beta/g = X\,B(X) \;;\quad B(X) = b_0\,A(X)$$

$$B(X) \sim \sum_{n=0}^{\infty} b_n X^n \;,\quad A(X) \sim \sum_{n=0}^{\infty} a_n X^n$$

$\kappa = g^2/(16\pi^2)$ **generic** X

(146)
$$b_0 = \tfrac{1}{3}\left(33 - 2\,N_{fl}\right) \;,\quad a_0 = 1 \;,\quad a_n = b_n/b_0$$

$$b_1 = \tfrac{2}{3}\left(153 - 19\,N_{fl}\right)$$

$$b_2 = \tfrac{1}{54}\left(77139 - 15099\,N_{fl} + 325\,N_{fl}^2\right)$$

$$b_3 \sim 29243 - 6946.3\,N_{fl} + 405.089\,N_{fl}^2 + 1.49931\,N_{fl}^3$$

$\rightarrow$

A1-2

$$-\gamma_m^0 = 4\,, \quad -\gamma_m^1 = \frac{202}{3} - n_{fl}\frac{20}{9} \qquad \Big| \; -\gamma_m^l \equiv \chi_m^l$$

$$-\gamma_m^2 = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] N_{fl} - \frac{140}{81} N_{fl}^2$$

$$-\gamma_m^3 = \left\{ \begin{array}{l} \dfrac{4603055}{162} + \dfrac{135680}{27}\zeta(3) - 8800\,\zeta(5) + \\ + \left[-\dfrac{91723}{27} - \dfrac{34192}{9}\zeta(3) + 880\,\zeta(4) + \dfrac{18400}{9}\zeta(5) \right] N_{fl} + \\ + \left[\dfrac{5242}{243} + \dfrac{800}{9}\zeta(3) - \dfrac{160}{3}\zeta(4) \right] N_{fl}^2 + \\ + \left[-\dfrac{332}{243} + \dfrac{64}{27}\zeta(3) \right] N_{fl}^3 \end{array} \right\} \tag{147}$$

→

References

[1-2001] O. Bär and U.J. Wiese, 'Can one see the number of colors?', MIT-CTP-3144, May 2001, 24 pp., Nucl.Phys. B609 (2001) 225-246, hep-ph/0105258 .

[2-1964] M. Gell-Mann and Y. Neeman, 'The eightfold way', W. A. Benjamin, Inc., New York 1964 .

[3-1975] H. Fritzsch and P. Minkowski, 'Psi Resonances, Gluons and the Zweig Rule', March 1975, 61 pp., CALT-68-492, Nuovo Cim. A30 (1975) 393 .

[4-1973] S. Weinberg, 'New approach to the renormalization group', Phys. Rev. D8 (1973) 3497 .

[A215-2009] S. Bethke, 'The 2009 Wolrd Average of $\alpha_s(M_Z)$ ', Eur.Phys.J.C64 (2009) 689-703, arXiv:0908.113 [hep-ph] .

[5-1997] K.G. Chetyrkin, 'Quark mass anomalous dimension to $O(\alpha_s^4)$ ', March 1997, 7 pp., MPI-PHT-97-019, Phys.Lett. B404 (1997) 161-165, hep-ph/9703278 , and references cited therein .

[6-1997] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, 'The Four loop beta function in quantum chromodynamics', UM-TH-97-01, NIKHEF-97-001, Jan 1997. 9pp., Phys.Lett.B400 (1997) 379 , hep-ph/9701390 , and references cited therein .

[7-1979] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, 'QCD and resonance physics. Theoretical Foundations.' , Nucl. Phys. B147 (1979) 385 , 448 .

→

References

[8-2011] S. Bodenstein, C.A. Dominguez, S.I. Eidelman, H. Spiesberger and K. Schilcher, 'Confronting electron-positron annihilation into hadrons with QCD: An Operator product expansion analysis', Oct 2011, 17 pp., UCT-TP-282-10,MZ-TH-10-40, JHEP 1201 (2012) 039, arXiv:1110.2026 [hep-ph] , and references cited therein .

[9-2010] K. Chetyrkin, J.H. Kühn, A. Maier, P. Maierhöfer, P. Marquard, M. Steinhauser and C. Sturm, 'Precise Charm- and Bottom-Quark Masses: Theoretical and Experimental Uncertainties', Theor.Math.Phys. 170 (2012) 217-228, arXiv:1010.6157v2 [hep-ph] , and references cited therein .

[10-2011] S. Bodenstein, J. Bordes, C.A. Dominguez, J. Penarrocha and K. Schilcher, 'Bottom-quark mass from finite energy QCD sum rules', Nov 2011, UCT-TP-288-11, MZ-TH-11-37, Phys.Rev. D85 (2012) 034003, arXiv:1111.5742 [hep-ph] .

[11-2011] S. Narison, 'Gluon Condensates and $m_{c,b}$ from QCD-Moments and their ratios to Order α_s^3 and $< G^4 >$ ', Phys.Lett. B706 (2012) 412-422, arXiv:1105.2922 [hep-ph] .

[12-RR-1969] S. L. Adler, 'Axial vector vertex in spinor electrodynamics', Phys.Rev.177 (1969) 2426-2438. J. Bell and R. Jackiw, 'A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the sigma model', Nuovo Cim.A60 (1969) 47-61 .

$\longrightarrow$

References

[13-1991] P. Minkowski, ’Proton spin and quark spin’, in Proceedings of the ”Workshop on Effective Field Theories of the Standard Model”, Dobogókõ, Hungary August 22 - 26, 1991 ; U. Meissner edt., World Scientific Singapore 1992 , BUTP-91-2, Aug 1991. 31pp. .

[14-a-2012] P. Minkowski, ’Canonical (anti-) commutation rules in QCD abd unbroken gauge invariance, QCD - the two central anomalies and canonical structure’, Feb 2012, arXiv:1202.2488 [hep-th] .

[15-a-2010] S. Kabana and P. Minkowski, ’On the thermal phase structure of QCD at vanishing chemical potentials’, Jan 2010, 7 pp., CERN-PH-TH-2010-002, Int.J.Mod.Phys. A26 (2011) 3035-3050, arXiv:1001.0707 [hep-ph] .

[16-a-2000] S. Kabana and P. Minkowski, ’Mapping out the QCD phase transition in multiparticle production’, Oct 2000, 53 pp., New J.Phys. 3 (2001) 4, hep-ph/0010247 .

[17-a-2011] P. Minkowski, ’QCD and the incomplete links between mass and gauge’, massgauge-kyoto2011.pdf,
(URL) http://www.mink.itp.unibe.ch/lectures.html ,

[18-a-1968] G. Segal, ’The representation ring of a compact Lie group’, Publications Mathématiques de L’IHS, Volume 34, Number 1 (1968) 113-128, DOI: 10.1007/BF02684592 , and references cited therein .

⟶

References

[19-a-1975] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. 42 (1975) 127 , Ann. Phys. 98 (1976) 287 , and in 'Renormalization Theory', G. Velo and A. S. Wightman eds., Reidel , Dordrecht 1976 .

[20-a-1975] I. V. Tyutin, Lebedev Institute preprint N39 , 1975 .

[21-a-1978] H. Pagels and E. Tomboulis, 'Vacuum of the quantum Yang-Mills theory and magnetostatics', Nuclear Physics B143 (1978) 485-502.

[22-a-1981] P. Minkowski, 'On The Ground State Expectation Value Of The Field Strength Bilinear In Gauge Theories And Constant Classical Fields', Nuclear Physics B 177 (1981) 203-217 , and references cited therein .

[22-a-2012] P. Minkowski, 'Canonical (anti-)commutation rules in QCD abd unbroken gauge invariance, QCD - the two central anomalies and canonical structure', Feb 2012, arXiv:1202.2488 [hep-th] .

[23-a-1918] E. Noether, "Invariante Variationsprobleme", Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse 1918 , 235-257 ; english translation : Emmy Noether, M. A. Tavel, 'Invariant Variation Problems', (Submitted on 8 Mar 2005), arXiv:physics/0503066v1 [physics.hist-ph] .

References

[24-a-1940] F. J. Belinfante, ’On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields’, Physica 7 (1940) 449-474 , a full derivation can be found in the textbook by Res Jost [25-a-1965] .

[25-a-1965] R. Jost, ’The general theory of quantzed fields’, American Mathematical Society, Providence, Rhode Island , 1965 .

[26-a-1991] C. Dominguez and E. de Rafael, ’Light Quark Masses in QCD from Local Duality’, Annals of Physics 174 (1987) 272 , and references cited therein .

[27-a-2010] P. Minkowski, ’Oscillations of three Majorana neutrinos and a possible connection to QCD’, URL : http://www.mink.itp.unibe.ch/publications.html [file : singafl2010.pdf] .

References

[14-B2-1975] H. Kluberg-Stern and J. B. Zuber, ’Ward Identities and Some Clues to the Renormalization of Gauge Invariant Operators’, Phys.Rev. D12 (1975) 467 ,
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions’, Phys.Rev. D12 (1975) 482 and
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 2. Gauge Invariant Operators’, Phys.Rev.D12 (1975) 3159-3180 .

[15-B2-1976] P. Minkowski, ’On the anomalous divergence of the dilatation current in gauge theories’, Bern preprint 1976 , unpublished , URL : http://www.mink.itp.unibe.ch/publications.html [file : all.pdf] .
For fairness of citation let me quote the papers published in 1977 on the trace anomaly
S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D15 (1977) 1712 ,
N. K. Nielsen. Nucl. Phys. B 120 (1977) 212 ,
S. L. Adler, A. Duncan and S. D. Joglekar, Phys. Rev. D 16 (1977) 438 .

References

[16-1972] H. Fritzsch and M. Gell-Mann, ’Current algebra: Quarks and what else?’, Proceedings of 16th International Conference on High-Energy Physics, Batavia, Illinois, 6-13 Sep 1972, published in eConf C720906V2 (1972) 135-165, also in Physics, Proceedings of the XVI International Conference on High Chicago 1972 p.135 (J. D. Jackson, A. Roberts, eds.), hep-ph/0208010 .

[17-1973] H. Fritzsch, Murray Gell-Mann and H. Leutwyler , ’Advantages of the Color Octet Gluon Picture’, CALT-68-409, 1973, Phys.Lett.B47 (1973) 365-368 .

[18-1973(4)] D.J. Gross and F. Wilczek,

’Asymptotically free gauge theories. 1’, Phys.Rev.D8 (1973) 3633-3652 ,

’Asymptotically free gauge theories. 2’, Phys.Rev.D9 (1974) 980-993 .

[19-1973(4)] D. Politzer, ’ Reliable Perturbative Results for Strong Interactions?’, Phys. Rev. Lett. 30 (1973) 1346 ,

H. Georgi and D. Politzer, ’Electroproduction scaling in an asymptotically free theory of strong interactions’, Phys. Rev. D9 (1974) 416 .

→

References

[20-1975] H. Kluberg-Stern and J. B. Zuber, 'Ward Identities and Some Clues to the Renormalization of Gauge Invariant Operators', Phys.Rev. D12 (1975) 467 ,
'Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions', Phys.Rev. D12 (1975) 482 and
'Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 2. Gauge Invariant Operators', Phys.Rev.D12 (1975) 3159-3180 .

[21] C. Itzykson and J. B. Zuber, 'Quantum field theory', McGraw-Hill, New York 1980.
M. E. Peskin, D. V. Schroeder, 'An Introduction to quantum field theory', Addison-Wesley, Reading, MA 1995 .
J. Zinn-Justin, 'Quantum field theory and critical phenomena', The International Series of Monographs on Physics, Vol. 113 , Oxford University Press, Oxford 2002 .

[22-1982] G. Altarelli, 'Partons in quantum chromodynamics', NFN-ROME-783, Jul 1981. 211pp., Phys.Rept.81 (1982) .

[23-1975] H. Fritzsch and P. Minkowski, 'Psi Resonances, Gluons and the Zweig Rule', Caltech preprint CALT-68-492, Mar 1975. 61pp., Nuovo Cim.A30 (1975) 393 .

→

References

[24-1970] C. G. Callan, 'Broken scale invariance in scalar field theory', Phys.Rev.D2 (1970) 1541-1547 .

[25-1970] K. Symanzik, 'Small distance behavior in field theory and power counting', Commun.Math.Phys.18 (1970) 227-246 .

[26-1971(2)] G. 't Hooft, 'Renormalizable Lagrangians for Massive Yang-Mills Fields', Nucl.Phys.B35 (1971) 167-188 ,
G. 't Hooft and M.J.G. Veltman, 'Regularization and Renormalization of Gauge Fields', Nucl.Phys.B44 (1972) 189-213 .

[27-1969] I. B. Khriplovich, 'Greens functions in theories with a non-abelian gauge group' , Yad. Fiz. 10 (1969) 409 , engl. transl. Sov. J. Nucl. Phys. 10 (1970) 235 .

[28-1972] G. 't Hooft, Remarks on the ultraviolet stable sign of the beta function at the Colloquium on renormalization of Yang-Mills fields and applications to particle physics, Marseille 1972 .

[29-1988] S. G. Gorishnii , A. L. Kataev and S. A. Larin, 'Next-To-Leading $O(\alpha_s^3)$ QCD Correction to $\Sigma_t\left(e^+e^- \rightarrow \text{Hadrons}\right)$: Analytical Calculation and Estimation of the Parameter Lambda (MS)', JINR-E2-88-254, Apr 1988. 18pp., Phys.Lett.B212 (1988) 238-244 .

→

References

[30-1997] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, 'The Four loop beta function in quantum chromodynamics', UM-TH-97-01, NIKHEF-97-001, Jan 1997. 9pp., Phys.Lett.B400 (1997) 379 , hep-ph/9701390 .

[31-2009] S. Bethke, 'The 2009 Wolrd Average of $\alpha_s(M_Z)$ ', arXiv:0908.1135 [hep-ph] .

[32-1969] K. Wilson, ' Nonlagrangian models of current algebra', Phys. Rev. 179 (1969) 1499 .

[33-1954] M. Gell-Mann and F. Low, 'Quantum electrodynamics at small distances', Phys. Rev. 95 (1954) 1300 .

[34-1973] S. Weinberg, 'New approach to the renormalization group', Phys. Rev. D8 (1973) 3497 .

[35-1982] O. V. Tarasov, Dubna report JINR P2-82-900 (1982) , unpublished .

[36-1994] K. G. Chetyrkin, C. A. Dominguez, D. Pirjol and K. Schilcher, 'Mass singularities in light quark correlators : the strange quark case', Phys.Rev.D51 (1995) 5090-5100 , hep-ph/9409371 .

[37-2006] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. 96 (2006) 012003 .
P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, 'Massless propagators: Applications in QCD and QED', SFB-CPP-08-06, TTP08-03, Oct 2008. 7pp., Contributed to 8th International Symposium on Radiative Corrections (RADCOR 2007): Application of Quantum Field Theory to Phenomenology, Florence, Italy, 1-6 Oct 2007, PoS RADCOR2007 (2007) 023 , arXiv:0810.4048 [hep-ph] .

$\longrightarrow$

References

[38-2008] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, 'Hadronic Z- and tau-decays in order α_s^4', Phys.Rev.Lett.101 (2008) 012002 , arXiv:0801.1821v2 [hep-ph] .

[39-2008] C.A. Dominguez, N.F. Nasrallah, R.H. Röntsch and K. Schilcher, 'Up and down quark masses from Finite Energy QCD sum rules to five loops', UCT-TP-271-08, Jun 2008. 4pp. , Phys.Rev.D79 (2009) 014009 , arXiv:0806.0467 [hep-ph] .

[40-1979] M. A. Shifman , A. I. Vainshtein and V. I. Zakharov , 'QCD and resonance physics. Theoretical Foundations.' , Nucl. Phys. B147 (1979) 385 , 448.

[41-2001] J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142,
H. Leutwyler, 'Nonlattice determinations of the light quark masses', Nucl.Phys.Proc.Suppl.94 (2001) 108, hep-ph/0011049.

[42-2008] B. Blossier et al., European Twisted Mass Collaboration, 'Light quark masses and pseudoscalar decay constants from N(f)=2 Lattice QCD with twisted mass fermions', DESY-07-148, FTUV-07-2709, IFIC-07-57, MS-TP-07-23, RM3-TH-07-11, ROM2F-2007-16, SFB-CPP-07-58, TUM-HEP-676-07, Sep 2007. 20pp., JHEP 0804 (2008) 020 , arXiv:0709.4574 [hep-lat].

[43-2008] C. Amsler et al. (Particle Data Group), Physics Letters B667 (2008) 1.

→

References

[44-2008] T. Becher and M. D. Schwartz, 'A Precise determination of alpha(s) from LEP thrust data using effective field theory', FERMILAB-PUB-08-048-T, Mar 2008, 34pp., JHEP 0807 (2008) 034 , arXiv:0803.0342 [hep-ph] .

[45-2011] S. Narison, 'Gluon Condensates and $m_b(m_b)$ from QCD-Exponential Moments at Higher Orders', May 2011. 4pp., arXiv:1105.5070 [hep-ph] .

[46-2011] S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, K. K. Szabo, 'The QCD equation of state with dynamical quarks', WUB-10-12, Jul 2010. 31pp., JHEP 1011 (2010) 077, arXiv:1007.2580 [hep-lat] .

Historical and textbook references to ’Continous transformation groups and differential geometry’

References

[1H-1893] S. Lie and G. Scheffers, ’Vorlesungen über continuierliche Gruppen, mit geometrischen und anderen Anwendungen’, Leipzig 1893 .

[2H-1890] W. Killing, ’Die Zusammensetzung der stetigen endlichen Transformationsgruppen’, Mathematische Annalen 31 (1888) 252-290; 33 (1888) 1-48; 34 (1889) 57-122; 36 (1890) 161-189 .

[3H-1894] É. Cartan, ’Sur la structure des groupes de transformations finis et continues’, Thèse, Paris 1884 , IIème édition 1933 .

[4H-1897] F. Klein and A. Sommerfeld, ’Über die Theorie des Kreisels’, 4 volumes, Teubner Verlag 1897 .

[5H-1903] L. Bianchi, ’Lezioni sulla teoria dei gruppi continui finiti di trasformazioni’, Pisa 1903 , IIa edizione 1918 .

[6H-1927] F. Peter und H. Weyl, ’Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe’ , Ann. Math. 97 (1927) 737-755 .

[7H-1933] A. Haar, ’Der Massbegriff in der Theorie der kontinuierlichen Gruppen’, Ann. Math. (2) 34 (1933) 147-169 .

[8H-1933] H. Weyl, ’The structure and representation of continous groups’, I and II, Princeton 1933 and 1934-35.

$\longrightarrow$

References

[9H-1933] L.P. Eisenhart, 'Continuous groups of tranformations', Princeton 1933.

[10H-1941-1986] L. S. Pontryagin, 'Topological Groups', Gordon and Breach Sci. Publ., New York/London/Paris 1986 , first edition in russian : 1941 .

[11H-1951-1961] G. Racah, 'Group theory and spectroscopy', CERN Yellow report 61-8, 1961, URL : http://doc.cern.ch/cernrep/1961/1961-008/1961-008.html , reprinted from lectures dlivered (by the author) at the Institute for Advanced Study in Princeton, 1951 ; and references therein quoted below.

[12H-1953] G. W. Mackey, 'Induced Representations of Locally Compact Groups II. The Frobenius Reciprocity Theorem', The Annals of Mathematics, Second Series, Vol. 58, No. 2 (Sep., 1953) 193-221 .

[13H-1962] S. Helgason, 'Differential geometry and Symmetric Spaces', Academic Press New York and London 1962 .

[14H-1963] S. Kobayashi and K. Nomizu, 'Foundations of Differential Geometry', 2 volumes, John Wiley and Sons, Inc., Hoboken, NJ 1963, Wiley Classics Library 2009 .

End of historical and textbook references to
'Continous transformation groups and differential geometry'