Beyond Relativistic Quantum String Theory

or

Discreteness and Determinism in Superstrings

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Introduction

Superstring theory appears to be complicated and counter intuitive. Some physicists insist that physics at smaller distance scales will be strange and counter intuitive.

"*stranger even than quantum mechanics*" (D. Gross) .

I cannot believe this.

As Gell-Mann said: the world seems to become more and more complex, *until you suddenly reach a new understanding.* Then things become simple again.

I now want to present my theory:

Suyperstring theory is even simpler than classical mechanics!

(because there is not even chaos. . .)

If you understand what String Theory really is ...

Beyond Superstring Theory, there is something really simple . . .

it is conceptually simple, but mathematically hard . . .

The Discrete Lorentz Group

Consider a theory with all its dynamical variables defined on a lattice: $(t, x, y, z) \in \mathbb{Z}^4$. Equations could for instance take the form:

$$\phi(x, y, z, t+1) = F\Big(\phi(x \pm 1,\ y \pm 1,\ z \pm 1,\ t)\Big)$$

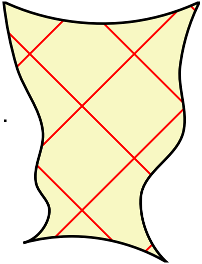
Question: is there *any* such discrete theory for which an invariance, or covariance, property can be formulated under the *discrete Lorentz transformations*? (discrete Poincaré trfs)

$X^\mu \to L^\mu{}_\nu X^\nu$, if all matrix elements $L^\mu{}_\nu$ of the $O(D-1,1)$ matrix L are integers: $L \in O(D-1,1,\mathbb{Z})$.

Examples: $L = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ or $R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

The discrete Lorentz group is *non compact*.

Bosonic String Theory



D dimensional spacetime: q^μ, $\mu = 0, 1, \cdots, D-1$.
2 dimensional world sheet: $(x,\ t)$. String: $q^\mu(x,t)$.
String action: $x^\pm = \frac{1}{\sqrt{2}}(t \pm x)$

$$S = T \int \mathrm{d}x^+ \mathrm{d}x^- \sqrt{(\partial_+ q^\mu\, \partial_- q^\mu)^2 - (\partial_+ q^\mu)^2 (\partial_- q^\nu)^2}$$

Choose $x^\pm$ such that $(\partial_+ q^\mu)^2 = (\partial_- q^\nu)^2 = 0$.

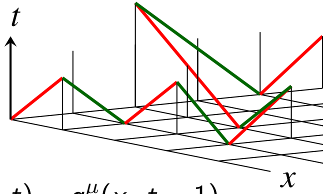
$$S \;=\; T\, \partial_+ q^\mu \partial_- q^\mu \;=\; T\left(\tfrac{1}{2}(\partial_t q^\mu)^2 - \tfrac{1}{2}(\partial_x q^\mu)^2\right)\ , \qquad T \equiv 1$$

And so: $\partial_t^2 q^\mu - \partial_x^2 q^\mu = 0$. Consequently:

$$q^\mu(x,t) = q_L^\mu(x+t) + q_R^\mu(x-t)\ ,$$

and: $(\partial_x q_L^\mu)^2 = (\partial_x q_R^\mu)^2 = 0$

Bosonic String Theory on a Lattice



World sheet: $(x,\ t)$, $x \in \mathbb{Z}$, $t \in \mathbb{Z}$
Discretized *classical* e.o.m.:

$$q^\mu(x,\ t+1) = q^\mu(x-1,\ t) + q^\mu(x+1,\ t) - q^\mu(x,\ t-1)\ .$$

Again: $q^\mu(x,t) = q^\mu_L(x+t) + q^\mu_R(x-t)$

$$\begin{aligned} a^\mu_L(x) &= q^\mu_L(x+1) - q^\mu_L(x-1)\ , \quad (a^\mu_L)^2 = 0 \\ a^\mu_R(x) &= q^\mu_R(x-1) - q^\mu_R(x+1)\ , \quad (a^\mu_R)^2 = 0 \end{aligned}$$

Space-time lattice: $q^\mu \to Q^\mu \in \mathbb{Z}$, $a^\mu \to A^\mu \in \mathbb{Z}$
Gauge condition on world sheet: $A^+_L = 1$, $A^-_L = (A^{\mathrm{tr}}_L)^2$
and $L \leftrightarrow R$; however, $A^+_L = A^0_L + A^{D-1}_L$, $A^-_L = A^0_L - A^{D-1}_L$, so if $A^0_L + A^{D-1}_L > 1$ after a Lorentz transformation, imposing this gauge will require that x^+ have a gap of length > 1.

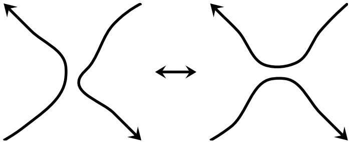
Invariance under the Discrete Lorentz (Poincaré) Group

The original discrete bosonic string equations are invariant under the transformations

$$Q^\mu \to L^\mu{}_\nu Q^\nu \ ; \qquad A^\mu \to L^\mu{}_\nu A^\nu \ ,$$

if all matrix elements $L^\mu{}_\nu$ of the $O(D-1,1)$ matrix L are integers: $L \in O(D-1,1,\mathbb{Z})$.

One can write down a *classical and unique* interaction among these classical strings: if two strings hit the same spacetime point Q^μ, two arms are exchanged:



This generates closed, interacting, oriented strings.
We still have exact invariance under $O(D-1,1,\mathbb{Z})$.
The "string constant" ϱ, determining the strength of the interaction, is fixed.

Quantization

This lattice theory can be quantized *without changing the field equations*

This means that only the states the system is in are promoted to being elements of Hilbert space. The evolution equations are still classical.

Attend the closing lecture ...

Fermions

Also the fermionic system can be handled the same way. The superstring has, besides the bosonic variables q^a, also $D-1$ fermionic fields ψ^a_A with $\psi^a_A = \psi^{a\dagger}_A$, $A = 1, 2$ (or, $A = L, R$)

$$\text{One finds that} \quad \psi^\mu_A(x,t) = \begin{pmatrix} \psi^\mu_L(x+t) \\ \psi^\mu_R(x-t) \end{pmatrix} \ .$$

The lattice theory now has Boolean degrees of freedom, $\sigma^a(x,t) = \pm 1$, $a = 1, \cdots, D-2$, obeying the equations:

$$\sigma^a(x,\ t+1) = \sigma^a(x-1,\ t)\ \sigma^a(x-1,\ t)\ \sigma^a(x,\ t-1)\ .$$

This also splits up into left- and right-movers:

$$\sigma^a(x,t) = \sigma^a_L(x+t)\ \sigma^a_R(x-t)\ .$$

Constraints

In Superstring Theory, the fermions also obey gauge conditions and constraints, which should determine $\psi_A^{\pm}$ in terms of $a^{\mathrm{tr}}_{L,R}$, and so also $\sigma^0_{L,R}$ and $\sigma^{D-1}_{L,R}$ should be determined by the transverse $\sigma^a_{L,R}$

Also a theory of Boolean variables on a $1+1$ dimensional lattice, can be quantized without modifying the evolution equations. The system is merely put in quantum states.
The Boolean variables then turn into fermion fields.

This works flawlessly in the transverse sector.

But what happens to the constraints is still difficult to characterize in detail.

Details of the lattice

Our lattice starts off being rectangular.

But that isn't the entire story. Suppose we fix $A^+ = 1$ and $A^- = \sum (A^{\rm tr})^2$. Since $A^\pm = A^0 \pm A^{D-1}$ where all A^μ are integers, A^- must be odd.

$\rightarrow$ *Restriction on the allowed lattice sites*

Perhaps we have to split all integer coordinate variables Q^a into *even* and *odd* sites. This could be what the Boolean variables σ^a actually stand for: $Q^a = \begin{cases} \text{even} & \text{if} \quad \sigma^a = 1 \ ; \\ \text{odd} & \text{if} \quad \sigma^a = -1 \ . \end{cases}$

Conjecture: the quantization procedure to be discussed in the closing lecture, will only allow for a consistent expression of the constraints if either $D = 10$, in which case no variables σ^a are needed, or in $D = 26$, in which case all transverse coordinates Q^a have to be split into even and odd variables, to be specified by the Boolean variables $\sigma^a(x, t)$.

The lattice does not go away!

Our quantization procedure does not send the lattice length a to zero, but uses the lattice for the quantization:

$$q^a \to Q^a \ , \quad p^a \to P^a \qquad \text{implies} \qquad [q^a,\, p^b] = \tfrac{i}{2\pi}\delta^{ab}.$$

This will imply that the typical lattice distance, a, is given by

$$a = 2\pi\sqrt{\alpha'}.$$

The string constant ρ is not freely adjustable.

We do expect that discrete Lorentz/Poincaré invariance turns into continuous Lorentz/Poincaré invariance upon quantization (this can easily be seen for the *rotations* in the transverse sector).

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II Closing Lecture

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Real numbers and integers

Imagine that, in contrast to appearances, the real world, at its most fundamental level, were *not* based on real numbers at all. We here consider systems where *only* integers describe what happens at a deeper level. Can one understand *why* our world *appears* to be based on real numbers?

A *mapping* exists of

deterministic physics of a set of $2N$ integers Q_i, P_i	onto	quantum physics on N real observables q_i with N associated momenta p_i

Canonical Variables. Why is this important? **Black Holes** show that degrees of freedom are denumerable.
In finite black holes they are finite . . .

Notation

$$\epsilon \equiv e^{2\pi} \approx 535.49\cdots \ ; \qquad \epsilon^{ipx} = e^{2\pi ipx} \ ; \qquad \epsilon^{iZ} = 1 \quad \text{if} \quad Z \in \mathbb{Z} \ .$$

All periods are 1 instead of 2π, to simplify normalization.

$$2\pi[x,\, p] = i \ , \qquad \langle x|\epsilon^{ipa}|p\rangle = \langle x + a\,|\,p\rangle \ . \quad h\ (\text{not } \hbar) = 1 \ .$$

cap. Latin letters, $N,\ P,\ Q,\ X,\ \cdots,$ indicate integers,
l.c. Latin letters, $p,\ q,\ x,\ \cdots,$ indicate real numbers,
l.c. Greek letters, $\alpha,\ \eta,\ \xi,\ \lambda,\ \cdots,$ are fractional numbers

$$-\tfrac{1}{2} < \eta \le \tfrac{1}{2} \ .$$

$$[P,Q] = 0 \ , \quad \langle Q|\,\eta_Q\rangle = \epsilon^{iQ\,\eta_Q} \ , \qquad \langle P|\,\eta_P\rangle = \epsilon^{iP\,\eta_P} \ ,$$

$$\langle Q_1|\,Q_2\rangle = \delta_{Q_1 Q_2} \ , \qquad \langle \eta_Q^1|\,\eta_Q^2\rangle = \delta(\eta_Q^1 - \eta_Q^2) \ ,$$

and on the real line: $\langle x|p\rangle = \epsilon^{ipx}$, $\qquad \langle x_1|x_2\rangle = \delta(x_1 - x_2)$,

If $N > 1$ insert δ_{ij} $\qquad \langle p_1|p_2\rangle = \delta(p_1 - p_2)$.

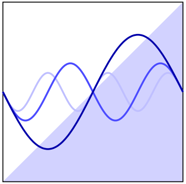
Operators

Even if we have a cellular automaton processing the numbers P_i and Q_i deterministically, we can still introduce *operators*.
Fourier transform the function η on the interval $-\frac{1}{2} < \eta < \frac{1}{2}$:

$$\eta = \sum_N \epsilon^{iN\eta} \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta \, \mathrm{d}\eta \, \epsilon^{-iN\eta} = \sum_{N\neq 0} \frac{i(-1)^N}{2\pi N} \epsilon^{iN\eta} \ ,$$

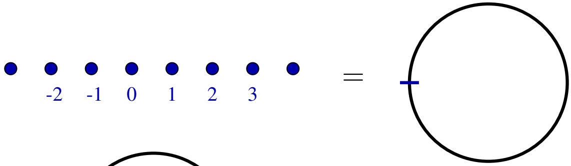
$$\epsilon^{iN\eta}|Q\rangle = |Q-N\rangle \ , \qquad \text{so}$$

$$\langle Q_1|\eta|Q_2\rangle = \frac{i}{2\pi}(1-\delta_{Q_1 Q_2})\frac{(-1)^{Q_1-Q_2}}{Q_1-Q_2} \ .$$

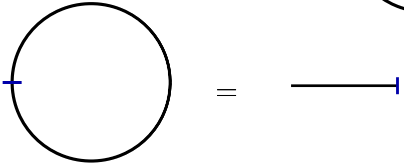


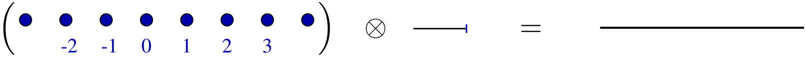
$$Q_1|[\eta_Q, \, Q]|Q_2\rangle = \tfrac{i}{2\pi}\left(\delta_{Q_1 Q_2} - (-1)^{Q_2-Q_1}\right) = \tfrac{i}{2\pi}(\mathbb{I} - |\psi\rangle\langle\psi|) \ .$$

$|\psi\rangle$ is an *edge state* : $\qquad \eta|\psi\rangle = \delta(\eta - \frac{1}{2})$.



(Fourier duality)





Make real number operators $-\infty < q < \infty$ as follows: $q = Q + \eta_P$

There is a unitary transformation of states from one basis to another:

$$\langle Q,\, \eta_P | \psi \rangle \;=\; \langle q | \psi \rangle \;.$$

Then transform $\langle Q,\, \eta_P | \psi \rangle = \sum_{P=-\infty}^{\infty} \epsilon^{-iP\eta_P} \langle Q,\, P | \psi \rangle \quad = \; \langle q | \psi \rangle$

$q = Q + \eta_P$ *Then obtain the states* $|p\rangle$ *from:* $\langle q|p\rangle = \epsilon^{ipq}$.

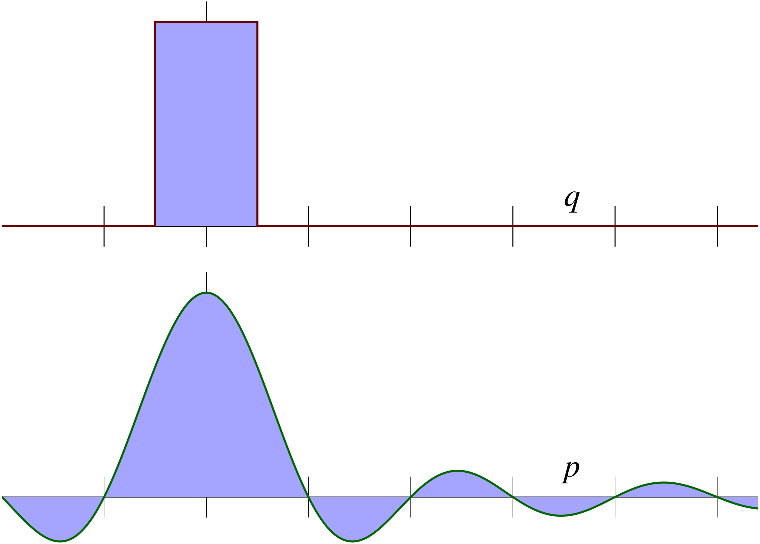
Since $-\frac{1}{2} < \eta \leq \frac{1}{2}$, we find the state $|0,0\rangle$ in q space; it is the wave function $\langle q|0,0\rangle$:

You get the other $|Q,P\rangle$ states simply as follows:

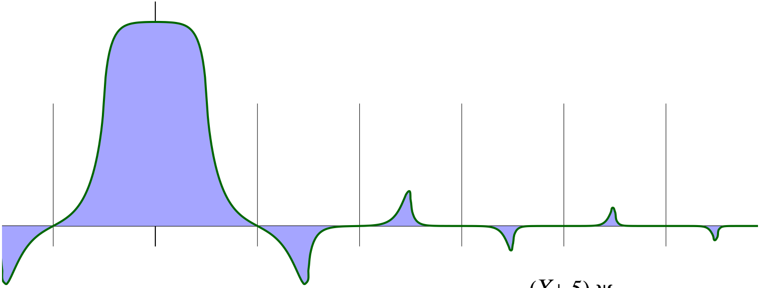
$$\langle q|Q,P\rangle = \epsilon^{iPq}\langle q-Q|0,0\rangle \ .$$

But then, in momentum space:

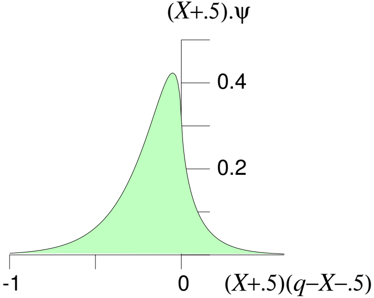
$$\langle p|Q,P\rangle = \int \mathrm{d}q \, \langle p|q\rangle\langle q|Q,P\rangle \ , \qquad \text{and this gives ...}$$



$$\langle q|0,0\rangle = \theta\left(\tfrac{1}{2} - |q|\right) \; ; \qquad \langle p|0,0\rangle = \frac{\sin \pi p}{\pi p} \; .$$



Asymptotic wave form:



$$\langle p|P,Q\rangle = \langle q|-Q,P\rangle.$$

These more convergent expressions are obtained by requiring that the operators q and p are made *periodic* in η_Q and η_P. This is done by rotating over a phase angle $\phi(\eta_Q,\,\eta_P)$. It obeys

$$\phi(\eta,\xi+1)=\phi(\eta,\xi)+\eta; \qquad \phi(\eta+1,\xi)=\phi(\eta,\xi)\ .$$

Using some mathematics, this phase angle can be obtained from an elliptic integral. We get:

$$\begin{aligned} q &= \frac{-i}{2\pi}\frac{\partial}{\partial\eta_Q}+\left(\frac{\partial}{\partial\eta_Q}\phi(\eta_P,\eta_Q)\right) = Q+\frac{\partial}{\partial\eta_Q}\phi(\eta_P,\eta_Q)\ ;\\ p &= \frac{-i}{2\pi}\frac{\partial}{\partial\eta_P}-\left(\frac{\partial}{\partial\eta_P}\phi(\eta_Q,\eta_P)\right) = P-\frac{\partial}{\partial\eta_P}\phi(\eta_Q,\eta_P)\ . \end{aligned}$$

Since $\phi(\eta,\xi)=\eta\,\xi-\phi(\xi,\eta)$, this obeys $[q,p]=i/2\pi$.

Matrix elements

In Hilbert space $\{|Q,\ P\rangle\}$, we have

$$
\begin{aligned}
q = Q + a_Q \quad &, \quad p = P + a_P \ , \\
\langle Q_1, P_1 | a_Q | Q_2, P_2 \rangle &= \frac{(-1)^{P+Q+1}\, iP}{2\pi(P^2+Q^2)} \\
\langle Q_1, P_1 | a_P | Q_2, P_2 \rangle &= \frac{(-1)^{P+Q}\, iQ}{2\pi(P^2+Q^2)} \ .
\end{aligned}
$$

From these:

$$
\langle Q_1, P_1 | \, [q,p] \, | Q_2, P_2 \rangle = \frac{i}{2\pi} (-1)^{Q_1-Q_2+P_1-P_2} (\delta_{Q_1\, Q_2} \delta_{P_1\, P_2} - 1) \ .
$$

$$
= \frac{i}{2\pi}(1 - |\psi_{\rm edge}\rangle\langle\psi_{\rm edge}|), \qquad \text{with} \qquad (\langle Q, P | \psi_{\rm edge} \rangle = (-1)^{Q+P}
$$

In our notation, the **harmonic oscillator** hamiltonian is
$H = \pi(p^2 + q^2)$. What is $\langle Q_1, P_1|H|Q_2, P_2\rangle$?

$$\langle 0,0|q^2|0,0\rangle = \int_{-\infty}^{\infty} \mathrm{d}q \, \langle 0,0|q\rangle q^2 \langle q|0,0\rangle \ .$$

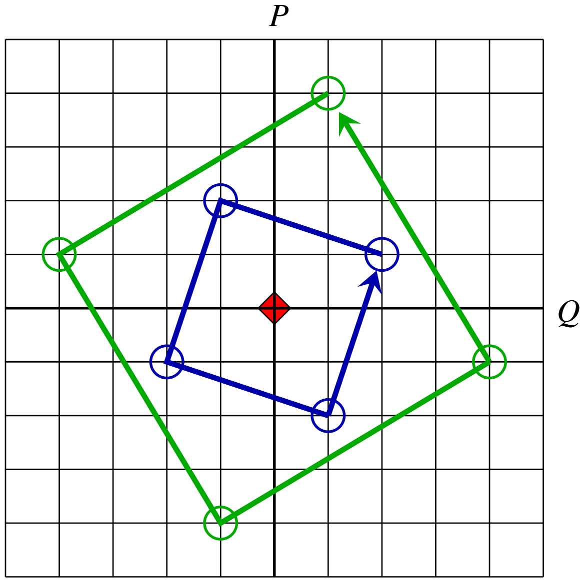
Logarithmic divergence proportional to
$(-1)^{Q_1 - Q_2 + P_1 - P_2} \quad \propto \quad \langle Q_1, P_1|\psi_{QP}\rangle\langle\psi_{QP}|Q_2, P_2\rangle$.
Only this edge state has a divergent hamiltonian.
All other elements of the hamiltonian converge rapidly.
Constraint: $\langle\psi_{QP}|\psi\rangle = 0$. We have

$$|\psi(t)\rangle = \epsilon^{-iHt}|\psi(0)\rangle \ ; \qquad H = a^\dagger a = N \ ; \qquad |\psi(N)\rangle = |\psi(0)\rangle \ .$$

At steps $t_n = \frac{1}{4}n$, we have $q \to p \to -q \to -p \to q$, or, because of our complete Q, P symmetry:

$$(Q,\, P) \to (P,\, -Q) \to (-Q,\, -P) \to (-P,\, Q) \to (Q,\, P) \ .$$

This is a *"deterministic"* system.


P
Q

Free massless bosons in $1+1$ dimensions

$$[p(x,t),\phi(y,t)]=i\delta(x-y)\ ;\qquad H=\pi\int\mathrm{d}x(p(x)^2+(\partial_x\phi)^2)\ .$$

$$\text{Discrete}\quad p_x,\phi_x\quad :\qquad \phi_{x,t}\equiv\phi(x,t)\ .$$

We have: $\phi(x,t+a)+\phi(x,t-a)=\phi(x-a,t)+\phi(x+a,t)\ .$

We would like to *map* this model one-to-one on the cellular automaton:

$$Q(x,t+a)+Q(x,t-a)=Q(x-a,t)+Q(x+a,t)\ ,$$

where Q are integers.

Applying our earlier methods directly does not work: adding real numbers does not imply that their *integer parts* are added. Leads to rounding errors: our procedure is non-linear! It *does* work if real numbers are merely *interchanged*.
This happens if you consider *left movers* and *right movers* :

$$\phi(x,t)=\phi^L(x+t)+\phi^R(x-t)\ ;\ p(x,t)=\tfrac{1}{2}a^L(x+t)+\tfrac{1}{2}a^R(x-t)\ .$$

$$a^L=p+\partial_x\phi\ ;\quad a^R=p-\partial_x\phi\ .$$

Now,
$$H=\tfrac{1}{2}(p^2+(\partial_x\phi)^2)\ =\ \tfrac{1}{4}({a^L}^2+{a^R}^2)\ ,$$

$$[a^L,\,a^R]=0\ ;\quad [a^L(x),\,a^L(y)]=\tfrac{i}{\pi}\partial_x\delta(x-y)\quad ;$$

Our cellular automaton will be on a lattice: $(x,t)\in\mathbb{Z}$. Therefore, replace commutator by

$$[\phi(x),\,p(y)]=\tfrac{i}{2\pi}\delta_{x,y} \tag{13.1}$$

$$[a^L(x),\,a^L(y)]=\pm\tfrac{i}{2\pi}\ \text{ if }\ y=x\pm1\quad .$$

Now, we modify the hamiltonian of the continuum in such a way that its action over one time unit is a pure replacement.
Write $H = H^L + H^R$. In *momentum space* :

$$H^L = \frac{1}{2}\int_0^{1/2} \mathrm{d}k\, a^L(k)a^L(-k)\, M(k) \quad ; \quad M(\kappa) = \frac{\pi\kappa}{\sin(2\pi\kappa)} \ .$$

This hamiltonian turns $a^L(x)$ into a pure left-mover, and $a^R(x)$ into a right-mover.

Now, we can try to express $a^L(x)$ in terms of the left-movers of the cellular automaton:

$$A^L(x+t) = Q(x, t+1) - Q(x-1, t)$$

Demanding $[a^L(x),\ a^L(y)] = \pm\frac{i}{2\pi}$ if $y = x \pm 1$

Disregarding periodicity:

$$a^L(x) = A^L(x) + \eta^L_A(x-1)\ .$$

Correct procedure: in η space,

$$a^L(x) = A^L(x) + \frac{\partial}{\partial\eta^L_A(x)}\Big(\phi(\eta^L_A(x+1), \eta^L_A(x)) - \phi(\eta^L_A(x-1), \eta^L_A(x))\Big)$$

This gives the mapping.

Edge states: if two consecutive $\eta^L_A(x)$ are in the corner: $\pm\frac{1}{2}$

Fermions

Fermions are actually very easy. Suppose we have left- and right moving boolean variables: $\sigma_L(x+t)=\pm 1$ and $\sigma_R(x-t)=\pm 1$.

Introduce operators $\sigma_3(x,t)=\sigma(x,t)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$;

then: $\sigma_1(x,t)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $\sigma_2(x,t)=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

The let $\psi(x,t)=\sigma_1(x,t)\prod_{y<x}\sigma_3(y,t)$:

Jordan-Wigner transformation

$\{\psi(x,t),\,\psi(y,t)\}=\delta_{x,y}$

In this case: $\psi(x,t)=\psi^\dagger(x,t)$

Summary

– Given pairs of integers Q_i and P_i $\rightarrow$
real-valued operators q_i and p_i, with $[q_i,\, p_j] = \frac{i}{2\pi}\delta_{ij}$.

$$\langle\{Q_i,\, P_i\}|\psi\rangle \quad \leftrightarrow \quad \langle\{q_i\}|\psi\rangle \text{ or } \langle\{p_i\}|\psi\rangle .$$

Alternatively:

– Series of integers $A(x)$ $\rightarrow$
real-valued operators $a(x)$ with

$$[a(x),\, a(y)] = \pm\frac{i}{2\pi} \quad \text{if} \quad y = x \pm 1 \quad \text{else } 0\ .$$

These $A(x)$ may describe L- and R-moving string on spacetime lattice.
The $a(x)$ are the quantum degrees of freedom of string on continuum.
– Maximum momentum κ on string world sheet.
– $O(D-2,\mathbb{Z})$ symmetry on lattice $\rightarrow$ exact $O(D-2)$ on continuum.
– Lorentz symmetry $O(D-1,1)$ if anomalies in constraints cancel.
– *SUSY*: ± 1 in $D-2$ fold moving with left- and right movers.

THE END

Removing edge states

(technical details following slide ≈ 20)

The last wave functions are much better to use, for instance for constructing the form of a hamiltonian such as the harmonic oscillator, $H = \pi(q^2 + p^2)$ in P, Q variables.
How is it obtained?

Consider the edge states. We have the states $\langle Q|\psi_Q\rangle = (-1)^Q$ *at all values of* P. Often, also the P edge states will emerge: $\langle P|\psi_P\rangle = (-1)^P$. *We should try to avoid them!*
As will be shown now, only *one* edge state is *unavoidable*: $\langle Q, P|\psi_{QP}\rangle = (-1)^{P+Q}$.

Go to η space: $|\eta_Q,\, \eta_P\rangle = \displaystyle\sum_{Q,P=-\infty}^{\infty} \epsilon^{-i(Q\,\eta_Q+P\,\eta_P)}|Q,P\rangle$.

Away from the boundaries: $Q=-\frac{i}{2\pi}\dfrac{\partial}{\partial\eta_Q}$, $P=-\frac{i}{2\pi}\dfrac{\partial}{\partial\eta_P}$.

We had introduced the operator

$$q \stackrel{?}{=} Q+\eta_P = -\frac{i}{2\pi}\frac{\partial}{\partial\eta_Q}+\eta_P \ .$$

This is not periodic in η_P ! Therefore, $p \neq P-\eta_Q$.

Let's rotate our wave functions in η space with a *quasi periodic* phase factor $\epsilon^{i\phi(\eta_Q,\eta_P)}$, which we demand to obey

$$\phi(\eta,\xi+1)=\phi(\eta,\xi)+\eta; \qquad \phi(\eta+1,\xi)=\phi(\eta,\xi) \ ;$$
$$\phi(\eta,\xi)=-\phi(-\eta,\xi)=-\phi(\eta,-\xi) \ .$$

Write $|\eta_1,\, \eta_2\rangle = \epsilon^{i\phi(\eta_1,\eta_2)}|\eta_Q,\, \eta_P\rangle$.

Now, $q = -\frac{i}{2\pi}\frac{\partial}{\partial\eta_1} + \eta_2$; $\qquad p = -\frac{i}{2\pi}\frac{\partial}{\partial\eta_2}$, $\qquad [q,p] = \frac{i}{2\pi}$,

(away from the edges).

q and p are now *fully periodic* in η_Q and η_P so the edge states do not contribute.

Mathematical expression for the phase function ϕ:

$$f(\eta_1,\,\eta_2) = \rho(\eta_1,\,\eta_2)\epsilon^{i\phi(\eta_1,\eta_2)} = \sum_{N=-\infty}^{\infty} \epsilon^{-\frac{1}{2}(N-\eta_2)^2+iN\eta_1} \ .$$

$$\begin{aligned}
\langle\eta_Q,\eta_P|q\rangle &= \delta(\eta_P-\xi)\,\epsilon^{iX\eta_Q+i\phi(\eta_Q\eta_P)}\ , \quad q = X+\xi\ ; \\
\langle\eta_Q,\eta_P|p\rangle &= \delta(\eta_Q+\kappa)\epsilon^{iK\eta_P+i\phi(\eta_Q,\eta_P)-i\eta_Q\eta_P} \\
&= \delta(\eta_Q+\kappa)\epsilon^{iK\eta_P-i\phi(\eta_P,\eta_Q)}\ , \quad p = K+\kappa\ ,
\end{aligned}$$

because of duality: $\phi(\eta_1,\eta_2)+\phi(\eta_2,\eta_1) = \eta_1\,\eta_2$

$\rightarrow$ $\qquad p \leftrightarrow q$ symmetry

Careful inspection: our waves are now smooth functions of q and p.

One surviving edge state!

However, $\phi(\eta_1, \eta_2)$ does have a singularity, because the periodicity condition demands it:

$\eta \to \eta + 1\ ,\ \xi \to \xi + 1$ does not give the same as

$\xi \to \xi + 1\ ,\ \eta \to \eta + 1,$ so that there must be a point of a full phase rotation. $\phi(\eta_1, \eta_2)$ is the phase of an analytic function with a zero in it: the *elliptic* ϑ *function*. The singularity is at $\eta_1 = \pm\,\eta_2 = \pm\,\frac{1}{2}$: the edge state $|\psi_{QP}\rangle$.

Phase contours of the function $\phi(\eta, \xi)$.