High energy scattering in QCD and in quantum gravity

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Content

1 Pomeron in hadron interactions

High energy kinematics

$$s = 4E^2 \gg (-t) = \vec{q}^{\,2} \sim m^2$$

Pomeron contribution and Pomeranchuck theorem

$$A_P(s,t) \approx is\,\gamma^2(t)\,s^{\omega(t)}\,,\ \omega(t) = \Delta + \alpha' t\,,\ \sigma_{pp} = \sigma_{p\bar{p}} \sim s^{\Delta}$$

Mandelstam cut contribution

$$A_{Mand}(s,t) = -is \int \frac{d^2k}{(2\pi)^2} \Phi^2(k, q-k) s^{\omega(-k^2)} s^{\omega(-(q-k)^2)}$$

t-channel partial wave and unitarity relations

$$A(s,t) = is \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} s^{\omega} f_{\omega}(t)\,,\ \Im_t f_{\omega}(t) \sim \sum_n \int d\Omega_n |f_{\omega}^{(n)}|^2$$

2 Gribov Pomeron calculus

Ordering of particle clusters in rapidities

$$y_r = \frac{1}{2} \ln \frac{\sqrt{k^2 + m_r^2} + k}{\sqrt{k^2 + m_r^2} - k}\,, \;\; 1 << y_r - y_{r-1} << \ln s$$

Non-relativistic Pomeron propagator

$$G_0 = \frac{1}{E + \Delta - \frac{k^2}{2m}}\,, \;\; E = -\omega\,, \;\; \alpha' = \frac{1}{2m}$$

Gribov effective action

$$S = \int dy\, d^2\rho \left(\phi^*(\partial_y - \Delta)\phi + \frac{1}{2m}\, |\partial_\mu \phi|^2) + i\lambda(\phi^* \phi^2 + \phi^* \phi^2) + ... \right)$$

Weak coupling solution and analogy with the graviton exchange

$$\Delta = 0\,, \;\; \gamma_{ii}(0) = const\,, \;\; \gamma_{ir}(0) = 0$$

3 Gluon reggeization in QCD

QCD Born amplitude at high energies $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,,\ \ \alpha_s \ln s \sim 1\,,\ \ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

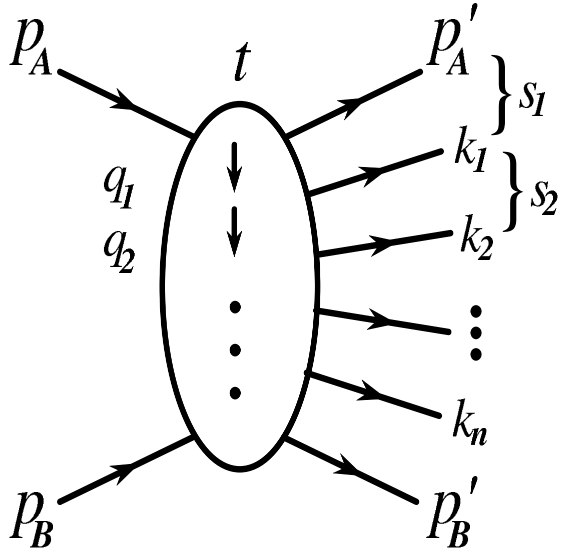
Gluon trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Bootstrap equation

$$\omega f = 1 + \Big(\omega(-|k|^2) + \omega(-|q-k|^2) + \hat{K}_8\Big)\, f\,,\ \ f = \frac{1}{\omega - \omega(-|q|^2)}$$

4 Amplitudes in multi-Regge kinematics



$$M^{BFKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q_r^2|}{\lambda^2}\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

5 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\,,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\;,\;\; \rho_r = x_r + iy_r$$

Möbius invariance and Pomeron intercept

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\; m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu\,,$$

$$E = \epsilon_m + \epsilon_{\widetilde{m}}\;,\;\; \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\;\; \Delta = \frac{g^2 N_c}{\pi^2}\ln 2 > 0$$

6 Integrability of the BFKL dynamics

Holomorphic separability of BKP hamiltonian at $N_c \to \infty$ (L.)

$$H = \frac{1}{2}(h + h^*)\,,\;\; h = \sum_{k=1}^{n} \left(\ln(p_k p_{k+1}) + \frac{1}{p_k} \ln\left(\rho_{k,k+1}\rho_{k-1,k}\right)\, p_k - 2\psi(1) \right))$$

Holomorphic factorization and duality symmetry

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)\,,\;\; \rho_r \to p_r \to \rho_{r+1}$$

Monodromy matrix and Yang-Baxter equation (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}\,,\;\; L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix},$$

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)\,,\;\; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

7 Spectrum of Pomerons in QCD

Asymptotic freedom

$$\frac{\alpha_s(k^2)}{2\pi} = \frac{1}{\beta_0 \ln \frac{k^2}{\Lambda^2_{QCD}}}\,,\;\; \beta_0 = 11 - \frac{2}{3} n_f$$

Eigenfunctions of the BFKL hamiltonian

$$f_n(k) = \int_{-\infty}^{\infty} d\,\nu \left(\frac{k^2}{\Lambda^2_{QCD}}\right)^{i\,\nu} \left(e^{-2i\psi(1)}\,\frac{\Gamma\left(\frac{1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)}\right)^{\frac{12}{\beta_0\,\omega_n}}$$

Pomeron intercepts and parton distributions at HERA (KLR)

$$\omega_n \approx \frac{0.5}{1+0.95n}\,,\;\; g(x,k^2) = \sum_{n=1}^{\infty} c_n x^{-\omega_n} f_n(k)$$

Essential transverse momenta and physics BSM (KLR)

$$\bar{k}_n \sim \Lambda_{QCD}\, e^{4n}$$

8 High energy effective action in QCD

Locality in the rapidity space

$$y=\frac{1}{2}\ln\frac{\epsilon_k+|k|}{\epsilon_k-|k|}\,,\;\;|y-y_0|<\eta\,,\;\;\eta<<\ln\,s$$

Gluon and reggeized gluon fields

$$v_\mu(x)=-iT^a v_\mu^a(x)\,,\;\;A_\pm(x)=-iT^a A_\pm^a(x)\,,\,\delta A_\pm(x)=0$$

Effective action for the reggeon interactions (L., 1995)

$$S=\int d^4x\left(L_{QCD}+Tr(V_+\partial_\mu^2 A_- + V_-\partial_\mu^2 A_+)\right)\;,$$

$$V_+=-\frac{1}{g}\partial_+\,P\exp\left(-g\,\int_{-\infty}^{x^+}v_+(x')d(x')^+\right)=v_+-gv_+\frac{1}{\partial_+}v_+ + ...$$

9 Production amplitude in N=4 SUSY

Remainder factor $R = A_{2\to 4}/A_{2\to 4}^{BDS}$ (L. (2009), F.,L. (2011))

$$R\,e^{i\pi\delta} = cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\phi n}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi(\nu,n)\left(\frac{-1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)},$$

$$u_1 = \frac{ss_2}{s_{012}s_{123}}\,,\;u_2 = \frac{s_1t_3}{s_{012}t_2}\,,\;u_3 = \frac{s_3t_1}{s_{123}t_2}\,,\;|w|^2 = \frac{u_2}{u_3}\,,\;\cos\phi = \frac{1-u_1-u_2-u_3}{2\sqrt{u_2u_3}}\,,$$

$$\delta = \frac{\gamma_K}{8}\,\ln\frac{|w|^2}{|1+w|^4}\,,\;\omega_{ab} = \frac{\gamma_K}{8}\,\ln|w|^2\,,\;\Phi = 1 - a\left(\frac{E_{\nu n}^2}{2}+\frac{3}{8}\,n^2/(\nu^2+\frac{n^2}{4})^2+\zeta(2)\right)$$

$$\omega(\nu,n) = -aE_{\nu,n} - a^2(\epsilon_{\nu n}^{FL}+3\zeta(3))\,,\;E_{\nu n} = -\frac{|n|/2}{\nu^2+\frac{n^2}{4}}+2\Re\psi(1+i\nu+\frac{|n|}{2})-2\psi(1)$$

Next-to-leading correction to ω (F.,L. (2011))

$$\epsilon_{\nu n}^{FL} = -\frac{\Re}{2}\left(\psi''(1+i\nu+\frac{|n|}{2})-\frac{2i\nu\psi'(1+i\nu+\frac{|n|}{2})}{\nu^2+\frac{n^2}{4}}\right)-\zeta(2)\,E_{\nu n}-\frac{1}{4}\,\frac{|n|\,\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}$$

10 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - \Delta\, \nu^2 \,, \;\; \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\, t \,, \;\; t = E^2/R^2 \,, \;\; \alpha' = \frac{R^2}{2}\, \Delta$$

Large coupling asymptotics for γ and Δ (KLOV, BPST)

$$\gamma = 1 - \sqrt{1 + (j-2)/\Delta} \,, \;\; \Delta = \frac{1}{\sqrt{\lambda}} \,, \;\; \lambda = g^2 N_c$$

Exact expression for the slope of γ at $j = 2$ (KLOV, Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

11 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2\to n} = -s^2 \Gamma^{\mu'\nu'}_{\mu\nu} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} ... \Gamma^{\rho'\sigma'}_{\rho\sigma}$$

Graviton-graviton-reggeon vertex

$$\Gamma^{\mu'\nu'}_{\mu\nu} = \frac{\kappa}{4} \left(\Gamma_{\mu\mu'}\Gamma_{\nu\nu'} + \Gamma_{\mu\nu'}\Gamma_{\nu\mu'}\right)$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p^A_{\mu'} p^B_\mu + p^{A'}_\mu p^B_{\mu'}}{p^A p^B} + \frac{q^2}{2} \frac{p^B_\mu p^B_{\mu'}}{(p^A p^B)^2}$$

Reggeon–reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} \left(C_\rho C_\sigma - N_\rho N_\sigma\right) \, , \; N = \sqrt{q_1^2 q_2^2} \left(\frac{p^A}{kp^A} - \frac{p^B}{kp^B}\right)$$

12 Graviton trajectory in supergravity

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2k}{k^2 (q-k)^2} f(k,q)\,, \;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2} (k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5 \gamma_\nu \partial_\rho \psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left(\ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{|\Lambda|^2}{|q|^2} \right)$$

13 High energy action in gravity

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta << \ln\, s$$

Reggeized graviton fields

$$\delta A^{++}(x) = \delta A^{--}(x) = 0\,, \;\; \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left(\sqrt{-g}\, R + \frac{1}{2} \left(\partial_+ j^-\, \partial_\mu^2 A^{++} + \partial_- j^+\, \partial_\mu^2 A^{--} \right) \right)$$

Hamilton-Jacobi equation for effective currents $j^\pm = 2x^\pm - \omega^\pm$

$$g^{\mu\nu} \partial_\mu \omega^\pm \partial_\nu \omega^\pm = 0\,, \;\; \partial_\pm j^\mp = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_\pm} h_{\pm\pm} \right)^2 + ...$$

14 Effective currents for shock waves

Aichelburg - Sexl metric

$$(ds)^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu + a \,\ln|\overrightarrow{x}|\delta(x^-)\,(dx^-)^2\,,\; a = \frac{8}{\sqrt{2}}\, G\,\mu\,,\; z = a\,\frac{x^-}{|\overrightarrow{x}|^2}$$

Effective current for the shock wave

$$j^+ = -a\,\mu \left(\ln|\overrightarrow{x}| + \ln f(z) - \frac{1}{4}\,\frac{z}{f^2(z)}\right)\,,\; f(z) = \frac{1}{2} + \frac{\sqrt{1+2z}}{2}$$

Perturbative expansion

$$j^+ = -a \ln|\overrightarrow{x}| + \frac{a^2}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2 - \frac{a^3}{\partial_-}\,\frac{x_\mu}{2|\overrightarrow{x}|}\,\frac{\partial_\mu}{\partial_-}\left(\frac{x_\sigma}{2|\overrightarrow{x}|}\right)^2 + \ldots$$

Variational principle for j^+

$$j^+ = \int_{-\infty}^{x^-}\left(g^{++}(y^-, \overrightarrow{\rho}(y^-)) + (\partial_-\overrightarrow{\rho})^2\right)\,,\; \frac{\delta j^+}{\delta\overrightarrow{\rho}(y^+)} = 0$$

15 Double-logarithmic asymptotics

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2 \ln\frac{|q|^2}{\lambda^2}} \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega\,, \;\; \alpha = \frac{\kappa^2}{8\pi^2}$$

Infrared evolution equation for supergravity (BLS (2012))

$$f_\omega = 1 + b\frac{d}{d\,\omega}\frac{f_\omega}{\omega} - b\,\frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\,, \;\; b = \alpha|q|^2$$

Solution in terms of parabolic cylinder function

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln d^{(N)}(x)\,, \;\; d^{(N)}(x) = e^{\frac{x^2}{4}} D_{\frac{6-N}{2}}(x)\,, \;\; x = \frac{\omega}{\sqrt{b}}$$

Perturbative expansion of scattering amplitudes in $\xi = \alpha\,|t|\,\ln^2\frac{s}{|q|^2}$

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2 \ln\frac{|q|^2}{\lambda^2}} \left(1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} + \ldots\right)$$

16 Production amplitudes in DLA

Multi-Regge kinematics

$$s \gg s_i = (k_{i-1}+k_i)^2 \gg |k_r^\perp|^2 \,, \quad \prod_{r=1}^{n+1} s_r = s \prod_{r=1}^{n} |k_r|^2$$

One graviton production amplitude in DLA (BLS)

$$A^{(N)}_{2\to 3} = A^{Born}_{2\to 3}\, s_1^{-\alpha |q_1|^2 \ln \frac{|q_1^2|}{\mu^2}}\, s_2^{-\alpha |q_2|^2 \ln \frac{|q_2^2|}{\mu^2}}\, r^{(N)}(\rho_1,\rho_2)\,, \;\; \rho_r = \sqrt{\alpha}\ln s_r/|q_r|^2 \,,$$

$$r^{(N)} = \int_{-i\infty}^{+i\infty} \frac{dx_1 dx_2}{(2\pi i)^2}\, e^{x_1\rho_1 + x_2\rho_2}\, \frac{d^{(N)}(\sqrt{\frac{|q_2|}{|q_1|}}x_1)\, d^{(N)}(\sqrt{\frac{|q_1|}{|q_2|}}x_2)}{d^{(N)}(x_1)\, d^{(N)}(x_2)}\, \phi^{(N)}(x_1,x_2)\,,$$

$$\phi^{(N)}(x_1,x_2) = \frac{2}{6-N}\, \frac{\frac{\left(d^{(N)}\left(\sqrt{\frac{|q_2|}{|q_1|}}x_1\right)\right)'}{d^{(N)}\left(\sqrt{\frac{|q_2|}{|q_1|}}x_1\right)} - \frac{\left(d^{(N)}\left(\sqrt{\frac{|q_1|}{|q_2|}}x_2\right)\right)'}{d^{(N)}\left(\sqrt{\frac{|q_1|}{|q_2|}}x_2\right)}}{\sqrt{\frac{|q_1|}{|q_2|}}x_2 - \sqrt{\frac{|q_2|}{|q_1|}}x_1}$$

17 Discussion

1. Locality in the rapidity space and Gribov calculus.
2. BFKL Pomeron as a composite state of reggeized gluons.
3. High energy effective action in QCD.
4. Integrability of Hamiltonian for gluon composite states.
5. Pomeron-graviton duality in $N = 4$ SUSY.
6. High energy effective vertices in gravity.
7. Graviton Regge trajectory at supergravity.
8. Effective action for high energy gravity.
9. Hamilton-Jacobi equation for effective currents.
10. Double-logarithmic scattering amplitudes at supergravity.