# THE CASIMIR EFFECT IN MINIMAL LENGTH THEORIES

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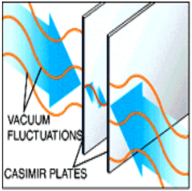
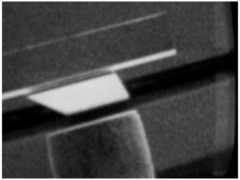
Talk based on: A.M.F., O.Panella, Phys. Rev. D 85, 045030 (2012) [arXiv:1112.2924]




# THE CASIMIR EFFECT

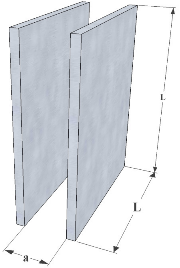


PhysRevLett.88.041804 (2002)

G. Bressi, G. Carugno, R. Onofrio, G. Ruoso

## *Casimir effect*

Interaction of a pair of neutral, parallel conducting planes due to the disturbance of the vacuum of the Electromagnetic field




$$\Delta E = \langle 0|\hat{H}(a) - \hat{H}|0\rangle$$
$$\hat{H} = \frac{1}{8\pi}\int d^3x \left[\left(\partial_0 \hat{\boldsymbol{A}}\right)^2 - \hat{\boldsymbol{A}}\nabla^2\hat{\boldsymbol{A}}\right]$$

Vacuum energy of the quantized electromagnetic field in free space (no plates)

$$E_0 =< 0|\hat{H}|0 >= \frac{c}{2}\int \frac{L^2 d^2\boldsymbol{q}}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty}\frac{a\,dp_3}{(2\pi\hbar)}\sqrt{\boldsymbol{q}^2 + p_3^2}$$

Vacuum energy in the presence of plates at a distance $a$

$$E =< 0|\hat{H}(a)|0 >= \frac{c}{2}\int \frac{L^2 d^2\boldsymbol{q}}{(2\pi\hbar)^2}\left[|\boldsymbol{q}| + 2\sum_{n=1}^{\infty}\sqrt{\left(\boldsymbol{q}^2 + \frac{n^2\pi^2\hbar^2}{a^2}\right)}\right]$$

where $\mathbf{q} = (q_1, q_2)$

The expression for the energy shift, which turns out to be finite, is known as the Casimir energy, and reads:

$$\mathcal{E} = \frac{c}{(2\pi)^2\hbar^2}\int d^2\boldsymbol{q}\left[\frac{1}{2}|\boldsymbol{q}| + \sum_{n=1}^{\infty}\sqrt{\left(\boldsymbol{q}^2 + \frac{n^2\pi^2\hbar^2}{a^2}\right)} - \int_0^{\infty} dn\sqrt{\boldsymbol{q}^2 + \frac{n^2\pi^2\hbar^2}{a^2}}\right]$$

if we define

$$G(n) = \int_{-\infty}^{+\infty} d^2\boldsymbol{q}\sqrt{\boldsymbol{q}^2 + \frac{n^2\pi^2\hbar^2}{a^2}}\,\underbrace{f\left(\sqrt{\boldsymbol{q}^2 + \frac{n^2\pi^2\hbar^2}{a^2}}\right)}_{\text{cutoff function}}$$

the difference is evaluated by the Euler-MacLaurin formula

$$\sum_{n=0}^{N} G(n) - \int_0^N dn\, G(n) =$$

$$-B_1\left[f(N) + f(0)\right] + \sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(N) - f^{(2k-1)}(0)\right] + R_p$$

$B_n$ are Bernoulli's numbers and $R_p$ is the error term for the approximation for a given $p$ (arbitrary integer)

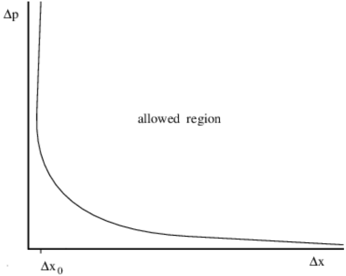
## *Casimir Energy*

After subtraction, the regularization is removed leaving the finite result

$$\mathcal{E} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3} \rightarrow \mathcal{F} = -\frac{\partial \mathcal{E}}{\partial a} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}$$

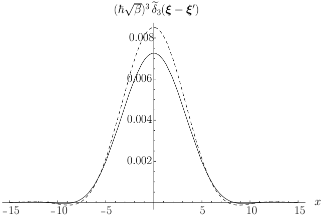
## *Applications*

- Quantum Field Theory
  - QCD
  - Kaluza-Klein field theory
- Gravitation and Cosmology
  - Non trivial topology→theory of structure formation of the Universe due to topological defects.
  - Inflation process
- Mathematical Physics: regularization and renormalization techniques
  - zeta function
  - heat kernel expansion

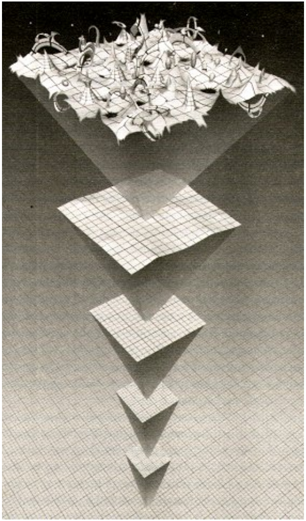



PhysRevD.52.1108 (1995)

A.Kempf, G.Mangano, R.B.Mann




PhysRevD.76.045012 (2007)
O.Panella



MINIMAL LENGTH THEORIES

## *GUP theories*

1-dim model based on Generalized Heisenberg Uncertainty Principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1+\beta\,(\Delta p)^2+\gamma\right], \qquad \beta,\gamma>0 \to \Delta x_0 = \hbar\sqrt{\beta}$$

$$[\hat{x},\hat{p}] = i\hbar\left(1+\beta\hat{p}^2+\ldots\right)$$

generalization to $n$ dimensions

$$[\hat{x}_i,\,\hat{p}_j] = i\hbar\left[f\left(\hat{p}^2\right)\delta_{ij}+g\left(\hat{p}^2\right)\hat{p}_i\hat{p}_j\right] \qquad i,j=1,\ldots,n$$

| Model I (rotationally invariant) | Model II (direct product) |
|---|---|
| $f\left(\hat{p}^2\right)=\frac{\beta\hat{p}^2}{\sqrt{1+2\beta\hat{p}^2}-1}$, $\quad g\left(\hat{p}^2\right)=\beta$ | $f\left(p^2\right)=1+\beta p^2$, $\quad g\left(p^2\right)=0$ |

## *Maximally Localized states*

$< x|\psi > \rightarrow$No physical interpretation$\rightarrow$quasi-position representation

$|\psi_x^{ML} > \rightarrow (\Delta x)_{|\psi_x^{ML}>} = \Delta x_0$ and $< \psi_x^{ML}|\hat{x}|\psi_x^{ML} > = < x >$

If ordinary commutation and uncertainty relations, the ML-states are the usual position eigenstates for which the uncertainty in position vanish.

The specific form of ML-states depends on:

- Number of dimensions
- Model considered

In literature there are at least two procedures:

- KMM (Kempf,Mangano and Mann)
- DGS (Detournay, Gabriel and Spindel)

different subset of states to which the minimization procedure is applied

# *KMM and DGS Models*

The general ML-S around the average position **r** in the momentum representation

$$\psi_{\boldsymbol{r}}^{ML} = \frac{1}{\left(\sqrt{2\pi\hbar}\right)^3}\,\Omega\left(p\right)\exp\left\{-\frac{i}{\hbar}\cdot\left[\boldsymbol{\kappa}\left(p\right)\cdot\boldsymbol{r} - \hbar\,\omega\left(p\right)t\right]\right\}$$

the functions $\Omega$, $\underline{\kappa}$ and $\omega$ change for different models. Ex: Model I

$$\kappa_i\left(p\right) = \left(\frac{\sqrt{1+2\beta p^2}-1}{\beta p^2}\right)p_i \qquad \omega\left(p\right) = \frac{pc}{\hbar}\left(\frac{\sqrt{1+2\beta p^2}-1}{\beta p^2}\right)$$

| KMM | DGS |
|---|---|
| $\Omega\left(p\right) = \left(\frac{\sqrt{1+2\beta p^2}-1}{\beta p^2}\right)^{\frac{\alpha}{2}}$ | $\Omega\left(p\right) = \frac{\sqrt{2}}{\pi}\,\frac{\sqrt{\beta p^2}}{\left(\sqrt{1+2\,\beta\,p^2}-1\right)}\,\sin\left(\frac{\pi\left(\sqrt{1+2\,\beta\,p^2}-1\right)\sqrt{2}}{2\sqrt{\beta\,p^2}}\right)$ |
| $\int\frac{d^n p}{\sqrt{1+2\beta p^2}}\left(\frac{\sqrt{1+2\beta p^2}-1}{\beta p^2}\right)^{n+\alpha}\|p\rangle\langle p\| = 1$ | $\int\frac{d^n p}{\sqrt{1+2\beta p^2}}\left(\frac{\sqrt{1+2\beta p^2}-1}{\beta p^2}\right)^{n}\|p\rangle\langle p\| = 1$ |

$\alpha = 1+\sqrt{1+n/2}$, $p = |\boldsymbol{p}|$ and $p^2 = \boldsymbol{p}\cdot\boldsymbol{p} = \sum_i^n\left(p_i\right)^2$.

From the scalar product of ML-s we can define the identity operator.

THE CASIMIR EFFECT IN MINIMAL LENGTH THEORIES

# *Differences from QED*

The introduction of a ML in the theory$\rightarrow$ ML-states $\rightarrow$ UV regularization.
Expansion of the field in a set of maximally localized states:

$$\hat{\mathbf{A}}(\mathbf{r},\, t) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{(2\pi)^4\,\hbar c^2}{\omega(\mathbf{p})}}\left[\hat{a}\,\epsilon < \psi^{ML}|\mathbf{p} > + \hat{a}^{\dagger}\,\epsilon^{*}\ < \mathbf{p}|\psi^{ML} >\right]$$

Boundary conditions (volume corrections)

$$\kappa_3(p_3) = \frac{n\pi\hbar}{a}$$

- Cutoff: $(\kappa_3)_{max} = \frac{\pi}{2\sqrt{\beta}} \rightarrow n_{max} = \frac{a}{2\hbar\sqrt{\beta}}$
- Analytically calculation: To compute the exact finite result we would need to compute numerically $p_3(n)$ inverting the relation between $p$ and $\kappa$

$$\left.\frac{dp_3}{d\kappa_3}\right|_{\kappa_3 = n\pi\hbar/a}$$

To obtain the first order in $\beta$ corrective term:
$\kappa_3(p_3) \underset{\beta\rightarrow 0}{\longrightarrow} p_3 = \frac{n\pi\hbar}{a} \rightarrow \left.\frac{dp_3}{d\kappa_3}\right|_{\kappa_3 = n\pi\hbar/a} \rightarrow 1$

Model II: $\kappa_i(p) = \left[\frac{1}{\sqrt{\beta}p}\arctan\left(p\sqrt{\beta}\right)\right]p_i$

Free space vacuum energy of the quantized electromagnetic field

$$E = \frac{cL^2 a}{(2\pi)^3 \hbar^3 \sqrt{\beta}} \int_{\mathbb{R}^2} d^2q \int_{-\infty}^{+\infty} dp_3 \frac{\arctan\left(p\sqrt{\beta}\right)}{\left(1+\beta p^2\right)^2}$$

change of variable from $p_3$ to $\kappa_3$: more suitable to include the b.c.

$$E = \frac{cL^2 a}{(2\pi)^3 \hbar^3 \sqrt{\beta}} \int_{\mathbb{R}^2} d^2q \int_{-(\kappa_3)_{max}}^{+(\kappa_3)_{max}} d\kappa_3 \frac{dp_3}{d\kappa_3} \frac{\arctan\left(p\sqrt{\beta}\right)}{\left(1+\beta p^2\right)^2}$$

changing again variable from $\kappa_3$ to $n$, we can define the energy shift

$$\begin{aligned}\Delta E \quad = \quad & \frac{1}{2}\frac{cL^2}{(2\pi)^2 \hbar^2 \sqrt{\beta}} \int d^2q \left\{ \sum_{n=-n_{max}}^{n_{max}} \frac{\arctan\left(\sqrt{\beta\left[q^2+p_3^2(n)\right]}\right)}{\left\{1+\beta\left[q^2+p_3^2(n)\right]\right\}^2} \left.\frac{dp_3}{d\kappa_3}\right|_{\kappa_3=\frac{\hbar\pi}{a}n} + \right. \\ & \left. - \int_{-n_{max}}^{n_{max}} dn \frac{\arctan\left(\sqrt{\beta\left[q^2+p_3^2(n)\right]}\right)}{\left\{1+\beta\left[q^2+p_3^2(n)\right]\right\}^2} \left.\frac{dp_3}{d\kappa_3}\right|_{\kappa_3=\frac{\hbar\pi}{a}n} \right\}\end{aligned}$$

exchanging sums and integrals

$$G(n) = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{+\infty} d\boldsymbol{q} \frac{\arctan\left(\sqrt{\beta\left[\boldsymbol{q}^2 + p_3(n)\right]}\right)}{\left[1 + \beta\left(\boldsymbol{q}^2 + p_3^2(n)\right)\right]^2} \left.\frac{dp_3}{d\kappa_3}\right|_{\kappa_3 = n\pi\hbar/a}$$

No cutoff!
In the limit $\beta \to 0$ we obtain a closed expression for $G(n)$

$$G(n) = -\frac{2\pi}{4\beta^3\left(\beta\left(\frac{\hbar n\pi}{a}\right)^2 + 1\right)} \left[\left(\arctan\left(\frac{\sqrt{\beta}\hbar\pi n}{a}\right) - \frac{\pi}{2}\right)\left(-\beta^{3/2} + \beta^{5/2}\left(\frac{\hbar\pi n}{a}\right)^2\right) + \frac{\pi\hbar n\beta^2}{a}\right]$$

to obtain the first order in $\beta$

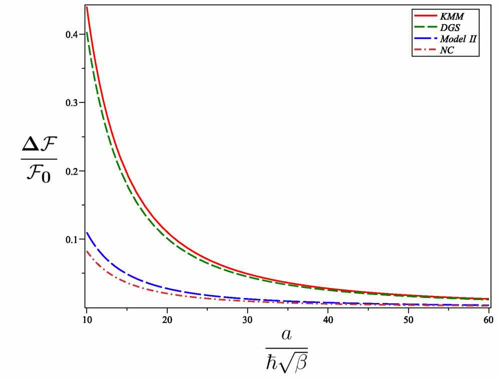
$$\frac{1}{2}G(0) + \sum_{n}^{\infty} G(n) - \int_{0}^{+\infty} dn\, G(n) = -\frac{1}{2!}B_2 G'(0) - \frac{1}{4!}B_4 G'''(0) - \frac{1}{6!}B_6 G^{\mathrm{v}}(0)$$

and the energy shift per unit area is

$$\mathcal{E} = \frac{1}{2}\frac{c}{(2\pi)^2\hbar^2}\left\{\sum_{n=0}^{n_{max}} G(n) - \int_0^{n_{max}} dn\, G(n)\right\} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}\left[1 + \pi^2\frac{2}{3}\left(\frac{\hbar\sqrt{\beta}}{a}\right)^2\right]$$

# *Results*

- Model I (KMM) $\mathcal{F} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}\left[1 + \pi^2\left(\frac{10}{3} + \frac{5\sqrt{10}}{14}\right)\left(\frac{\hbar\sqrt{\beta}}{a}\right)^2\right]$
- Model I (DGS) $\mathcal{F} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}\left[1 + \pi^2\left(\frac{20}{21} + \frac{20\pi^2}{63}\right)\left(\frac{\hbar\sqrt{\beta}}{a}\right)^2\right]$
- Model II $\mathcal{F} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}\left[1 + \pi^2\frac{10}{9}\left(\frac{\hbar\sqrt{\beta}}{a}\right)^2\right]$




comparison with volume correction in canonical NC spacetime

PhysRevD.76.025016, (2007)

R.Casadio, A.Gruppuso, B.Harms, O.Micu

# *Discussion and conclusions*

- The correction term has the same sign as QED-
  - attractive correction -
- The corrective term that scales as $a^{-5}$
- In contrast with standard QED
  - Casimir Energy does not need to be regularized -
- Upper bound on the minimal length of the theory
  plates distance $0.5\,\mu m$: $(\Delta x_0)_{\mathrm{KMM}} = \hbar\sqrt{\beta} = 29\,nm$ to $(\Delta x_0)_{\mathrm{II}} = 58\,nm$

To calculate this upper limit we impose that the correction term gives a contribution in the limit of the accuracy of the experiment

| $\Delta x_0$ | $a$ | Model I (KMM) | Model I (DGS) | Model II |
|---|---|---|---|---|
| $\hbar\sqrt{\beta}$ (m) | $3 \cdot 10^{-6}$ | $1.75 \cdot 10^{-7}$ | $1.75 \cdot 10^{-7}$ | $3.50 \cdot 10^{-7}$ |
| $\hbar\sqrt{\beta}$ (m) | $0.5 \cdot 10^{-6}$ | $2.91 \cdot 10^{-8}$ | $3.04 \cdot 10^{-8}$ | $5.84 \cdot 10^{-8}$ |

The calculation of Casimir correction for non-planar geometries could give more stringent upper bounds on the minimal length.

# *Future development*

- Numerical calculation for boundary conditions
- Nonplanar geometries

Thank you for your attention!

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- Nonplanar geometries

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