Seiberg duality and the Superconformal Index

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A conformal field theory is a QFT which is invariant under Lorentz transformations, rescalings $x^\mu \rightarrow \lambda x^\mu$ and special conformal transformations $x^\mu \rightarrow i(i(x^\mu) + a^\mu)$, $i(x^\mu) = x^\mu / x^2$.

It is trivially scale invariant, that is has zero $\beta$ function. Although a scale invariant theory is not necessarily conformally invariant, all known examples are.

Interacting CFTs exist only in dimensions $2 - 6$.

Until AdS/CFT almost all examples known are in $d = 2$ or $d = 4$. Few of the examples from AdS/CFT have explicit Lagrangian descriptions.
- In 2 dimensions, CFTs are dense in the space of all QFTs.
- Scale invariance implies conformal invariance.
- 2 dimensional CFTs have been classified.
Most examples are $\mathcal{N} = 1$ SUSY gauge theories. Examples were very rare until the mid 90s.

They can be obtained by setting the NSVZ $\beta$ function

$$\beta(\alpha) = \frac{-\alpha^2}{2\pi - \alpha T(\text{adj.})} \left(3T(\text{adj.}) - \sum_i T(i)(1 - \gamma_i)\right)$$

(1)

to zero by appropriate choice of $\gamma_i$. $T$ is the quadratic Casimir.

I will discuss aspects of classification in the rest of the talk.
Seiberg Duality

- Seiberg duality occurs when two different gauge theories flow to the same superconformal fixed point.
- Seiberg’s original example was for a $G = SU(N_c)$ gauge theory with global symmetry group
  $F = SU(N_f) \times SU(N_f) \times U(1)_b \times U(1)_R$.
- This is dual to a theory with different matter content, gauge group $G = SU(N_f - N_c)$ and the same global symmetry group $F$.
- We require the theories to have superconformal IR fixed points so that $\frac{3N_c}{2} \leq N_f \leq 3N_f$.
- In addition over 80 other examples of duality are known.
The Seiberg electric theory has a $G = SU(N_c)$ gauge group and a $F = SU(N_f) \times SU(N_f)$ flavour group. It also has a $U(1)_B$ baryon number charge and a $U(1)_R$ $R$-charge. The theory has as matter content of two chiral scalar multiplets, $Q$ and $\tilde{Q}$.

<table>
<thead>
<tr>
<th>Field</th>
<th>$SU(N_c)$</th>
<th>$SU(N_f)$</th>
<th>$SU(N_f)$</th>
<th>$U(1)_B$</th>
<th>$U(1)_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>$f$</td>
<td>$f$</td>
<td>1</td>
<td>$1/N_c$</td>
<td>$r$</td>
</tr>
<tr>
<td>$\tilde{Q}$</td>
<td>$\bar{f}$</td>
<td>1</td>
<td>$\bar{f}$</td>
<td>$-1/N_c$</td>
<td>$r$</td>
</tr>
<tr>
<td>$V$</td>
<td>adj.</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table:** The field content of the Seiberg electric theory
The Seiberg magnetic theory has the same field content, but also a gauge singlet scalar multiplet.

<table>
<thead>
<tr>
<th>Field</th>
<th>$SU(\tilde{N}_c)$</th>
<th>$SU(N_f)$</th>
<th>$SU(N_f)$</th>
<th>$U(1)_B$</th>
<th>$U(1)_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$f$</td>
<td>$\bar{f}$</td>
<td>1</td>
<td>$1/\tilde{N}_c$</td>
<td>$\tilde{r}$</td>
</tr>
<tr>
<td>$\tilde{q}$</td>
<td>$\bar{f}$</td>
<td>1</td>
<td>$f$</td>
<td>$-1/\tilde{N}_c$</td>
<td>$\tilde{r}$</td>
</tr>
<tr>
<td>$\tilde{V}$</td>
<td>adj.</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$M$</td>
<td>1</td>
<td>$f$</td>
<td>$\bar{f}$</td>
<td>0</td>
<td>$r_m$</td>
</tr>
</tbody>
</table>

Table: The field content of the Seiberg magnetic theory

$$\tilde{N}_c = N_f - N_c$$
Evidence

- Matching numbers of (gauge invariant) degrees of freedom
- Identical moduli spaces
- ’t Hooft anomaly matching
- Equal superconformal indices
The superconformal index is defined as

\[ I(t, x) = \text{tr}_{\ker\mathcal{H}} \left( (-1)^F t^R x^{2J_3} \right) \quad (3) \]

where \( \mathcal{H} = H - \frac{3}{2} R - 2J_3 \)

This receives contributions from only those parts of the BPS spectrum which cannot combine to form long multiplets.

It is a topological invariant, so that we may calculate it at the superconformal fixed point and extend the result to all energies.
Calculating the index

- We can calculate the index for single particle states

$$i = - \left( \frac{1}{1-p} + \frac{1}{1-q} \right) \chi_{\text{adj.}}(g)$$

$$+ \sum_i t^r \chi_R, i(h) \chi_R, i(g) - t^{2-r} \chi_{\bar{R}}, i(h) \chi_{\bar{R}}, i(g)$$

$$\frac{(1 - p)(1 - q)}{(1 - p)(1 - q)}$$

(4)

Here $p = t/x$ and $q = tx$.

- This can be extended to multi particle states by the plethystic exponential

$$\text{Pexp}[i; a] = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} i(a^n) \right)$$

(5)

- Finally integrate over the gauge group

$$I = \oint d\mu_G(h) \text{Pexp}[i; t, x, g, h]$$

(6)
the Index for Seiberg dual theories

$$I_E = \frac{(p; p)^{N_c-1}(q; q)^{N_c-1}}{N_c!} \int_{\mathbb{T}^{N_c-1}} \prod_{j=1}^{N_c-1} \frac{dz_j}{2\pi iz_j} \cdot$$

$$\prod_{1 \leq i \leq N_f} \prod_{1 \leq j \leq N_c} \Gamma(t^r v^{1/N_c} y_i z_j, t^r v^{-1/N_c} \tilde{y}_i^{-1}/z_j; p, q)$$

$$\prod_{1 \leq i \leq j \leq N_c} \Gamma(z_i/z_j, z_j/z_i; p, q)$$

$$\prod_{j} z_j = 1$$

(7)

$$I_M = \frac{(p; p)^{\tilde{N}_c-1}(q; q)^{\tilde{N}_c-1}}{\tilde{N}_c!} \prod_{1 \leq i \leq j \leq N_f} \Gamma(t^r m y_i \tilde{y}_j^{-1}; p, q) \int_{\mathbb{T}^{\tilde{N}_c-1}} \prod_{j=1}^{\tilde{N}_c-1} \frac{dz_j}{2\pi iz_j} \cdot$$

$$\prod_{1 \leq i \leq N_f} \prod_{1 \leq j \leq \tilde{N}_c} \Gamma(t^r v^{1/\tilde{N}_c} y_i^{-1} z_j, t^r v^{1/\tilde{N}_c} \tilde{y}_i / z_j; p, q)$$

$$\prod_{1 \leq i \leq j \leq \tilde{N}_c} \Gamma(z_i/z_j, z_j/z_i; p, q)$$

$$\prod_{j} z_j = 1$$

(8)
where

\[(x; p) = \prod_{j \geq 0} (1 - xp^j) \quad (9)\]

\[\Gamma(x; p, q) = \prod_{j, k \geq 0} \frac{1 - x^{-1} p^{j+1} q^{k+1}}{1 - xp^j q^k} \quad (10)\]
The proof that $I_E = I_M$ is hard analysis. However we can get a flavour of what is going on by expanding the indices

\begin{equation}
I_E = 1 + t^{2r} s_{(1)}(y) s_{(1)}(\tilde{y}^{-1}) + t^{N_c r} \left( s_{(1 N_c)}(y) + s_{(1 N_c)}(\tilde{y}^{-1}) \right) + \ldots \tag{11}
\end{equation}

\begin{equation}
I_M = 1 + t^{r_m} s_{(1)}(y) s_{(1)}(\tilde{y}^{-1}) - t^{2-r_m} s_{(1)}(y^{-1}) s_{(1)}(\tilde{y}) + t^{2 \tilde{r}} s_{(1)}(y^{-1}) s_{(1)}(\tilde{y}) + t^{\tilde{N}_c r} \left( s_{(1 \tilde{N}_c)}(y^{-1}) + s_{(1 \tilde{N}_c)}(\tilde{y}) \right) + \ldots \tag{12}
\end{equation}

So we require $r_m = 2r$, $2 - r_m = 2\tilde{r}$, $N_c r = \tilde{N}_c \tilde{r}$. Hence $r = \tilde{N}_c / N_f$, $\tilde{r} = N_c / N_f$, $r_m = 2\tilde{N}_c / N_f$. 

These $r$ charges are exactly the same as those required by ’t Hooft anomaly matching and $a$-maximisation.

The equations arising from ’t Hooft anomaly matching can also be derived by requiring the index to be ‘totally elliptic’

Matching the number of degrees of freedom and moduli spaces is equivalent to the matching of the indices in certain limits.
Conclusions

- The matching of superconformal indices appears to reproduce all the classical field theoretic tests of Seiberg duality (at least in some cases).
- There is a hope that it may be a perfect discriminant of duality.
- The index is not well defined without a conformal fixed point, despite the required mathematical identities holding.
- It does not include any information from the non BPS part of the spectrum.