Seiberg duality and the Superconformal Index

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Conformal Field Theories

- A conformal field theory is a QFT which is invariant under Lorentz transformations, rescalings $x^\mu \to \lambda x^\mu$ and special conformal transformations $x^\mu \to i(i(x^\mu) + a^\mu)$, $i(x^\mu) = x^\mu / x^2$.
- It is trivialy scale invariant, that is has zero β function. Although a scale invariant theory is not necessarily conformaly invariant, all known examples are.
- Interacting CFTs exist only in dimensions $2 - 6$.
- Until AdS/CFT almost all examples known are in $d = 2$ or $d = 4$. Few of the examples from AdS/CFT have explicit Lagrangian descriptions.

$d = 2$

- In 2 dimensions, CFTs are dense in the space of all QFTs.
- Scale invariance implies conformal invariance.
- 2 dimensional CFTs have been classified.

$d = 4$

- Most examples are $\mathcal{N} = 1$ SUSY gauge theories. Examples were very rare until the mid 90s.
- They can be obtained by setting the NSVZ β function

$$\beta(\alpha) = \frac{-\alpha^2}{2\pi - \alpha T(\text{adj.})}\left(3T(\text{adj.}) - \sum_i T(i)(1-\gamma_i)\right) \quad (1)$$

to zero by apropriate choice of γ_i. T is the quadratic Casmir.

- I will discuss aspects of classification in the rest of the talk.

Seiberg Duality

- Seiberg duality occurs when two different gauge theories flow to the same superconformal fixed point.
- Seiberg's original example was for a $G = SU(N_c)$ gauge theory with global symmetry group $F = SU(N_f) \times SU(N_f) \times U(1)_b \times U(1)_R$.
- This is dual to a theory with different matter content, gauge group $G = SU(N_f - N_c)$ and the same global symmetry group F.
- We require the theories to have superconformal IR fixed points so that $\frac{3N_c}{2} \leq N_f \leq 3N_f$.
- In addition over 80 other examples of duality are known.

Seiberg Electric Theory

The Seiberg electric theory has a $G = SU(N_c)$ gauge group and a $F = SU(N_f) \times SU(N_f)$ flavour group. It also has a $U(1)_B$ baryon number charge and a $U(1)_R$ R-charge. The theory has as matter content of two chiral scalar multiplets, Q and $\tilde{Q}$.

Field	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	$1/N_c$	r
$\tilde{Q}$	$\bar{f}$	1	$\bar{f}$	$-1/N_c$	r
V	adj.	1	1	0	1

Table: The field content of the Seiberg electric theory

Seiberg Magnetic Theory

The Seiberg magnetic theory has the same field content, but also a gauge singlet scalar multiplet.

Field	$SU(\tilde{N}_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	$1/\tilde{N}_c$	$\tilde{r}$
$\tilde{q}$	$\bar{f}$	1	f	$-1/\tilde{N}_c$	$\tilde{r}$
$\tilde{V}$	adj.	1	1	0	1
M	1	f	$\bar{f}$	0	r_m

Table: The field content of the Seiberg magnetic theory

$$\tilde{N}_c = N_f - N_c \tag{2}$$

Evidence

- Matching numbers of (gauge invariant) degrees of freedom
- Identical moduli spaces
- 't Hooft anomaly matching
- Equal superconformal indices

The superconformal index

- The superconformal index is deffined as

$$I(t,x) = \mathrm{tr}_{\ker\mathcal{H}}\left((-1)^F t^{\mathcal{R}} x^{2J_3}\right) \tag{3}$$

  where $\mathcal{H} = H - \frac{3}{2}R - 2J_3$
- This recieves contributions from only those parts of the BPS spectrum which cannot combine to form long multiplets.
- It is a topological invariant, so that we may calculate it at the superconformal fixed point and extend the result to all energies.

Calculating the index

- We can calculate the index for single particle states

$$i = -\left(\frac{1}{1-p}+\frac{1}{1-q}\right)\chi_{adj.}(g) + \sum_i \frac{t^r\chi_{R_F,i}(h)\chi_{R_G,i}(g) - t^{2-r}\chi_{\bar{R}_F,i}(h)\chi_{\bar{R}_G,i}(g)}{(1-p)(1-q)} \tag{4}$$

  Here $p = t/x$ and $q = tx$.
- This can be extended to multi particle states by the plethystic exponential

$$\mathrm{Pexp}[i;a] = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i(a^n)\right) \tag{5}$$

- Finally integrate over the gauge group

$$I = \oint \mathrm{d}\mu_G(h)\mathrm{Pexp}[i;t,x,g,h] \tag{6}$$

the Index for Seiberg dual theories

$$I_E = \frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!} \oint_{\mathbb{T}^{N_c-1}} \prod_{j=1}^{N_c-1} \frac{\mathrm{d}z_j}{2\pi i z_j} \cdot \left. \frac{\prod_{1\leq i\leq N_f} \prod_{1\leq j\leq N_c} \Gamma(t^r v^{\frac{1}{N_c}} y_i z_j, t^r v^{-\frac{1}{N_c}} \tilde{y}_i^{-1}/z_j; p, q)}{\prod_{1\leq i\leq j\leq N_c} \Gamma(z_i/z_j, z_j/z_i; p, q)} \right|_{\prod_j z_j = 1} \tag{7}$$

$$I_M = \frac{(p;p)^{\tilde{N}_c-1}(q;q)^{\tilde{N}_c-1}}{\tilde{N}_c!} \prod_{1\leq i\leq j\leq N_f} \Gamma(t^{r_m} y_i \tilde{y}_j^{-1}; p, q) \oint_{\mathbb{T}^{\tilde{N}_c-1}} \prod_{j=1}^{\tilde{N}_c-1} \frac{\mathrm{d}z_j}{2\pi i z_j} \cdot \left. \frac{\prod_{1\leq i\leq N_f} \prod_{1\leq j\leq \tilde{N}_c} \Gamma(t^r v^{-\frac{1}{\tilde{N}_c}} y_i^{-1} z_j, t^r v^{\frac{1}{\tilde{N}_c}} \tilde{y}_i/z_j; p, q)}{\prod_{1\leq i\leq j\leq N_c} \Gamma(z_i/z_j, z_j/z_i; p, q)} \right|_{\prod_j z_j = 1} \tag{8}$$

the Index for Seiberg dual theories cont.

where

$$(x;p) = \prod_{j\geq 0}(1 - xp^j) \tag{9}$$

$$\Gamma(x;p,q) = \prod_{j,k\geq 0} \frac{1 - x^{-1}p^{j+1}q^{k+1}}{1 - xp^j q^k} \tag{10}$$

Expanding the index

The proof that $I_E = I_M$ is hard analysis.
However we can get a flavour of what is going on by expanding the indices

$$I_E = 1+t^{2r}s_{(1_1)}(y)s_{(1_1)}(\tilde{y}^{-1})+t^{N_c r}\left(s_{(1_{N_c})}(y)+s_{(1_{N_c})}(\tilde{y}^{-1})\right)+\dots \tag{11}$$

$$\begin{aligned} I_M = 1 &+ t^{r_m}s_{(1_1)}(y)s_{(1_1)}(\tilde{y}^{-1}) - t^{2-r_m}s_{(1_1)}(y^{-1})s_{(1_1)}(\tilde{y}) \\ &+ t^{2\tilde{r}}s_{(1_1)}(y^{-1})s_{(1_1)}(\tilde{y}) + t^{\tilde{N}_c r}\left(s_{(1_{\tilde{N}_c})}(y^{-1})+s_{(1_{\tilde{N}_c})}(\tilde{y})\right)+\dots \end{aligned} \tag{12}$$

So we require $r_m = 2r$, $2 - r_m = 2\tilde{r}$, $N_c r = \tilde{N}_c\tilde{r}$.
Hence $r = \tilde{N}_c/N_f$, $\tilde{r} = N_c/N_f$, $r_m = 2\tilde{N}_c/N_f$.

Interpretation

- These r charges are exactly the same as those required by 't Hooft anomaly matching and a-maximisation.
- The equations arising from 't Hooft anomaly matching can also be derived by requiring the index to be 'totally elliptic'
- Matching the number of degrees of freedom and moduli spaces is equivalent to the matching of the indices in certain limits.

Conclusions

- The matching of superconformal indices apears to reproduce all the classical field theoretic tests of Seiberg duality (at least in some cases).
- There is a hope that it may be a perfect discriminant of duality.
- The index is not well defined without a conformal fixed point, despite the required mathematical identies holding.
- It does not include any information from the non BPS part of the spectrum.