General relativity á la string: issues and new results

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Idea of embedding

String is an $2D$ surface in $4D$ ambient space. Maybe our $4D$ curved spacetime can be considered as a surface in ambient space?

Janet-Cartan theorem (1916)

An arbitrary n-dimensional Riemannian manifold can be locally isometrically embedded in N-dimensional flat space with

$$N \geq \frac{n(n+1)}{2}. \tag{1}$$

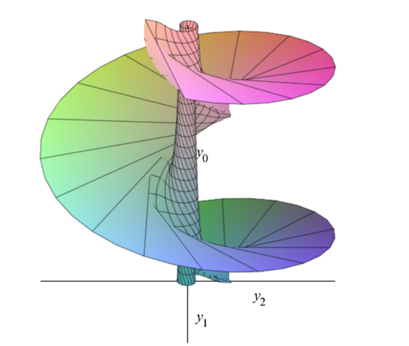
For our $4D$ manifold $N = 10$; if the manifold has symmetries, N may be smaller. Metric of this manifold can be expressed in terms of embedding function:

$$g_{\mu\nu} = \partial_\mu y^a(x) \partial_\nu y^b(x) \eta_{ab}, \tag{2}$$

where $y^a(x)$ – embedding function, η_{ab} – metric of flat ambient space.

One example of explicit embedding

New embedding for Schwarzschild black hole



$$y^0 = t,$$

$$y^1 = \sqrt{\frac{27R^3}{r}} \sin\left(\frac{t}{\sqrt{27}R} - f(r)\right),$$

$$y^2 = \sqrt{\frac{27R^3}{r}} \cos\left(\frac{t}{\sqrt{27}R} - f(r)\right),$$

$$y^3 = r\cos\theta,$$

$$y^4 = r\sin\theta\cos\phi,$$

$$y^5 = r\sin\theta\sin\phi,$$

$$f(r) = \sqrt{\frac{(r+3R)^3}{27R^2 r}}.$$

Classification of embeddings for Schwarzschild metric: S.Paston, A.S.; arXiv:1202.1204

Regge-Teitelboim equations

In 1975 Regge and Teitelboim proposed string-inspired theory in which embedding function becomes physical variable.

$$S = \frac{1}{2\varkappa}\int d^4x\sqrt{-g}R + S_m \tag{3}$$

$$\delta S = \frac{1}{2\varkappa}\int d^4x\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\delta g_{\mu\nu}. \tag{4}$$

By substitution $g_{\mu\nu} = \partial_\mu y^a(x)\partial_\nu y^b(x)\eta_{ab}$ in this formula we get

$$\delta S = \frac{1}{2\varkappa}\int d^4x\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\eta_{ab}\partial_\mu y^a(x)\partial_\nu \delta y^b(x) \tag{5}$$

and after integrating by parts we obtain Regge-Teitelboim equations:

$$\partial_\mu(\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = D_\mu((G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = 0. \tag{6}$$

Advantages

Natural appearance of flat spacetime can help to solve the following problems:

- The problem of time: there is no preferred "time slicing" of spacetime into spatial hypersurfaces. In embedding theory we can use timelike direction of flat ambient space as a natural time.
- The problem of causality: if the metric is subject to quantum fluctuations, we can't tell whether the separation between two points is spacelike, null, or timelike. Now it's possible to determine the sign of ds^2 using flat metric of ambient space.

- It's possible to reformulate embedding theory by introducing a set of fields $z^A(A = 1 \dots 6)$ living in $10D$ flat space (S.Paston, arXiv:1111.1104):

$$y^a(x^\mu) \to z^A(y^a) \tag{7}$$

Surfaces of constant value of z^A

$$z^A = \mathrm{const} \tag{8}$$

define the family of $4D$ surfaces in flat ambient space without introducing any coordinate system on the surface.
Coordinate-free theory of gravity?

Disadvantages

Higher derivatives in RT equations

It seems that RT equations $D_\mu((G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = 0$ contain higher derivatives of embedding function: there is a term proportional to $\partial G^{\mu\nu}$, $G^{\mu\nu} \propto \partial^2 g_{\mu\nu}$, $g^{\mu\nu} \propto \partial y^a$, so naive power counting shows that order of highest derivative is 4.
Solution is simple: if we look to the Gauss formula

$$R_{\mu\nu\alpha\beta} = D_\mu D_\alpha y^a D_\nu D_\beta y_a - D_\mu D_\alpha y^a D_\nu D_\beta y_a \tag{9}$$

we can see that $R_{\mu\nu\alpha\beta}$ (and $G^{\mu\nu}$) contains only $\partial^2 y^a$, not $\partial^3 y^a$.
Then using Bianchi identities $D_\mu G^{\mu\nu} = 0$ and EMT conservation $D_\mu T^{\mu\nu} = 0$ we can obtain a different form of RT equations:

$$(G^{\mu\nu} - \varkappa T^{\mu\nu}) D_\mu \partial_\nu y^a = 0, \tag{10}$$

and now it's clear that there is no higher derivatives of y^a in RT equations (Franke, Tapia 1992).

Closing of constraint algebra

In the original paper Regge and Teitelboim found that it's very difficult to prove that constraints of embedding theory are the first kind and constraint algebra is closed. Recently (Franke, Paston arXiv:0711.0576; Paston, Semenova arXiv:1003.0172) it was shown that in original paper one of the constraints was written incorrectly and after correction constraint algebra is closed.

Extra solutions

It is easy to see that RT equations are more general than the Einstein equations, so they have «extra solutions» (Deser 1976). One should remove it from theory (or try to interpret it as a source of DM). This makes it possible to consider the embedding theory as a theory of gravity which explains observed facts without any additional modification of it.

«Extra solutions» can be removed from theory in the following way:

$$\partial_\mu(\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = 0, \tag{11}$$

It looks like conservation law for some current:

$$\partial_\nu j^{a\nu} = 0, \qquad j^{a\nu} = \partial_\mu y^a \sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu}). \tag{12}$$

One can rewrite it in a form of Einstein equations:

$$G_{\mu\nu} - \varkappa T_{\mu\nu} = \frac{j^a_\nu \partial_\mu y_a}{\sqrt{-g}} = \varkappa \tau_{\mu\nu}, \tag{13}$$

$\tau^{\mu\nu}$ is additional term corresponding to «extra solutions».
Main idea: exponential increasing of $\sqrt{-g}$ during inflation leads to vanishing of «extra solutions» term $\tau^{\mu\nu}$.
It was proved with assumption of Friedmann symmetry (S.Paston, A.S.; arXiv:1106.5212)

Conclusions

- Embedding theory is a good theory of gravity on the classical level.
- In comparison to GR, it has advantages which can help to quantize it.
- It also still has several problems on quantum level, so there are many things to do.

Thank you for your attention!