Quantum field theory and unification of knot theories

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Knots

Knot $\mathcal{K}$ is a mapping of S^1 into $\mathbb{R}^3$, $\quad \mathcal{K} : S^1 \to \mathbb{R}^3$

Examples:

Figure: Examples: unknot, trefoil, eight-figure knot

Knots can be very complicated. So the problem is to construct invariants of knots, which are not changed under transformations preserving topology. If one considers a projection of a knot on a 2-d plane, these transformations can be reduced to compositions of the so-called Reidemeister moves:

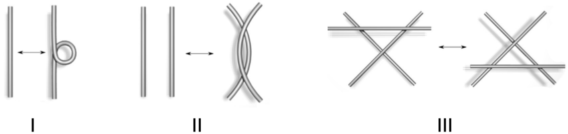

Figure: Three types of Reidemeister moves

Various knot theories

The complete theory of knot invariants is not constructed yet. Moreover, there exist several various theories, which give different invariants of knots. Doubly graded theories, i.e. 2-variable polynomials, and 1-variable polynomials are known. The central role is played by HOMFLY.

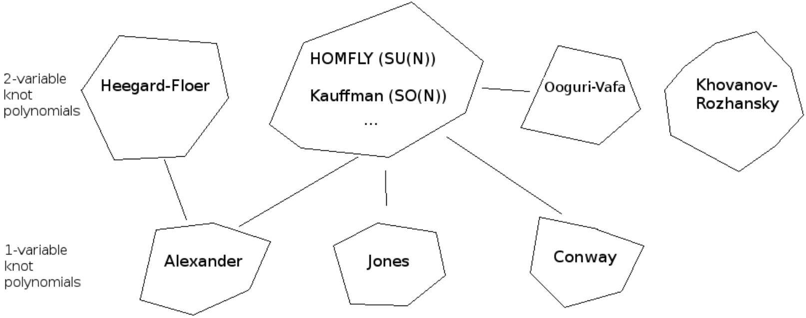

Figure: 1- and 2-graded theories

Quantum topological field theory (Chern-Simons theory)

In 1989 it was discovered by E.Witten that quantum field theory can describe polynomial invariants of knots. For this reason one needs to consider topological QFT with the Chern-Simons action:

$$S_{CS} = \frac{1}{g_{cs}} \int\limits_{\mathbb{R}^3} tr \left(AdA + \frac{2}{3} A^3 \right) \tag{1}$$

Let us consider the following vacuum expectation value

$$\langle W_R(\mathcal{K}) \rangle = \int \mathcal{D}A \; exp\{-2\pi i S_{CS}\} \; W_R(\mathcal{K}), \tag{2}$$

where $W_R(\mathcal{K}) = tr_R Pexp\{\oint_{\mathcal{K}} A\}$. If we fix gauge in an appropriate way and take $SU(N)$ as the gauge group, then

$$\langle W_R(\mathcal{K}) \rangle (g_{cs}, N) = \mathrm{HOMFLY(q,a)}, \; \mathrm{q = e^{g_{cs}}}, \; \mathrm{a = q^N}.$$

Unification

In 2005 Dunfield, Gukov, Rasmussen suggested a hypothetical theory, which can unify all other known knot theories. It is a triply graded theory, which we call the theory of superpolynomial invariants.

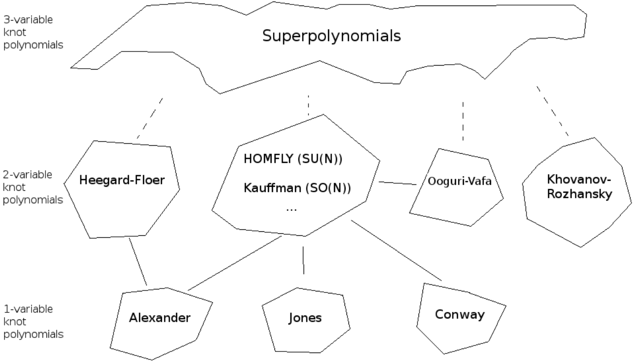

Figure: Unification of knot theories

Torus knots

Torus knots are a special kind of knots embedded in the torus in $\mathbb{R}^3$. They are defined by two co-prime numbers $[m, n]$.

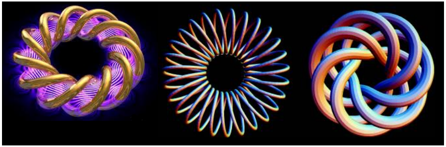

Figure: Examples of torus knots

Torus knots are naturally divided into the following series $[m, mk + r]$, $k = 1, 2, 3, ...$:

$r = 0$	$[1, n]$	$[2, 2k]$	$[3, 3k]$	$[4, 4k]$	$[5, 5k]$	$\dots$
$r = 1$		$[2, 2k + 1]$	$[3, 3k + 1]$	$[4, 4k + 1]$	$[5, 5k + 1]$	$\dots$
$r = 2$			$[3, 3k + 2]$	$[4, 4k + 2]$	$[5, 5k + 2]$	$\dots$
$r = 3$				$[4, 4k + 3]$	$[5, 5k + 3]$	$\dots$
$r = 4$					$[5, 5k + 4]$	$\dots$

Superpolynomials for torus knots

$$\mathcal{P}_{m,r}\{p\} = \sum_{Q\vdash m} c_Q^{(m,r)} M_Q\{p\} = \sum_{\substack{Q\vdash m\\ l(Q)\le r}} h_Q^{(m,r)} L_Q\{p\} \tag{3}$$

$$h_Q = q^{\nu(Q)}(1-t)^{l(Q)-1}\hat{h}_Q$$

$$\boxed{\hat{h}_Q^{(m,1)} = \begin{cases} 1 & l(Q)=1 \\ 0 & \text{otherwise} \end{cases}} \quad \boxed{\hat{h}_Q^{(m,2)} = \begin{cases} 1 & l(Q)=1 \\ 1 & l(Q)=2 \\ 0 & \text{otherwise} \end{cases}} \tag{4}$$

$$\boxed{\hat{h}_Q^{(m,3)} = \begin{cases} 1 & l(Q)=1 \\ 1+t+(q-t)[\alpha]_q & l(Q)=2,3 \\ 0 & \text{otherwise} \end{cases}} \tag{5}$$

with $\alpha = \min(Q_1 - Q_2, Q_2 - Q_3)$, Q is a Young diagram, $\nu(Q)$ is a function on Q, $\nu(Q) = \sum_i Q_i(i-1)$.

Superpolynomials for torus knots II

$$\mathcal{P}_{m,r}\{p\} = \sum_{Q\vdash m} c_Q^{(m,r)} M_Q\{p\} = \sum_{\substack{Q\vdash m \\ l(Q)\leq r}} h_Q^{(m,r)} L_Q\{p\} \qquad (6)$$

$$\boxed{\mathcal{P}_{m,mk+r}\{p\} = \sum_{Q\vdash m} c_Q^{(m,r)} q^{-k\nu(Q^T)} t^{k\nu(Q)} M_Q\{p\} = e^{k\hat{W}} \mathcal{P}_{m,r}\{p\}}$$

Thank you for your attention!