Low-energy General Relativity with torsion: a systematic derivative expansion

Alexander Tumanov

St-Petersburg Academic University

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In collaboration with Dmitri Diakonov and Alexey Vladimirov

Einstein theory

$$g_{\mu\nu}, \;\; S_E = R\sqrt{g}.$$

Einstein-Cartan theory

$$e^A_\mu, \, \Omega^{AB}_\mu, \, S_{EC} = \frac{1}{4}\,\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon^{ABCD}\,\mathcal{F}^{AB}_{\mu\nu}\,e^C_\alpha\,e^D_\beta\,,$$

where

$$g_{\mu\nu} = e^A_\mu e^A_\nu, \, \mathcal{F}^{AB}_{\mu\nu} = \partial_\mu \Omega^{AB}_\nu - \partial_\nu \Omega^{AB}_\mu + \Omega^{AC}_\mu \Omega^{CB}_\nu - \Omega^{AC}_\nu \Omega^{CB}_\mu.$$

Correspondence

first approach: Ω^{AB}_μ is a dynamic field.
second approach: $\Omega^{AB}_\mu = \bar{\Omega}^{AB}_\mu\,[e]$ - saddle-point of S_{EC}.

In the absence of fermions these two approaches are equivalent and $S_{EC} \longrightarrow S_E$.

Paper

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Dmitri Diakonov, A.T., Alexey A. Vladimirov

Standard fermion action:

$$S_{\mathrm{f}} = i \int d^4x \, \det(e) \, \frac{1}{2} \left(\overline{\Psi} \, e_A^\mu \, \gamma^A \, \mathcal{D}_\mu \Psi - \overline{\mathcal{D}_\mu \Psi} \, e_A^\mu \, \gamma^A \, \Psi \right),$$

where $\mathcal{D}_\mu = \partial_\mu + \frac{1}{8}\Omega_\mu^{AB}[\gamma_A \gamma_B]$ - covariant derivative acting on spinors.

S_{f} is linear in Ω_μ^{AB}. Therefore the saddle-point value of Ω_μ^{AB} is NOT equal to $\bar{\Omega}_\mu^{AB}$. Approaches 1 and 2 give different results. In the presence of fermions torsion is non-zero:

$$T_{\mu\nu}^A = \frac{1}{2} \left((D_\mu e_\nu)^A - (D_\nu e_\mu)^A \right) \neq 0,$$

where $D_\mu^{AB} = \delta^{AB} \partial_\mu + \Omega_\mu^{AB}$ - covariant derivative acting on vectors.

Integration procedure

$$T \sim \bar{\psi}\psi,$$

$$\Omega \sim \bar{\Omega} + T,$$

$$S = S_{EC} + S_{\mathrm{f}} \longrightarrow R\sqrt{g} + TT.$$

Four fermion interaction arises in Einstein-Cartan approach.

Decomposition of torsion

$$T_{\mu\nu}{}^{\lambda} = \frac{2}{3}\,\det(e)\,a^{\kappa}\,\varepsilon_{\kappa\mu\nu\rho}\,g^{\rho\lambda} + \frac{2}{3}\,v_{\kappa}\,\delta^{\kappa}_{[\mu}\delta^{\lambda}_{\nu]} + \frac{2}{3}t_{\mu\nu}{}^{\lambda},$$

where, inversely,

$$a^{\kappa} = \frac{\varepsilon^{\kappa\alpha\beta\gamma}}{4\det(e)}\,T_{\alpha\beta,\gamma}, \quad v_{\kappa} = T_{\kappa\mu}{}^{\mu}, \quad t_{\mu\nu}{}^{\lambda} = T_{\mu\nu}{}^{\lambda} + \delta^{\lambda}_{[\mu}\,T_{\nu]\rho}{}^{\rho} - g^{\lambda\rho}\,T_{\rho[\mu,\nu]}.$$

a^{κ} - axial part, v_{κ} - vector part, $t_{\mu\nu}{}^{\lambda}$ - irreducible part.

Derivative expansion

We consider Einstein theory to be the low energy limit of some "true" theory of gravity.

It is reasonable to consider different terms in the expansion. The expansion parameter is the number of derivatives. The mass dimension of each term is equal to the number of derivatives. It must be compensated with the corresponding power of the Plank mass in the denominator. Therefore, the order of magnitude decreases with the number of derivatives. When the characteristic momenta of matter is close to the Plank mass, the expansion fails.

Classification of fermion actions

The most general one-derivative fermion action:

$$S_f = \int d^4x \, \det(e) \, \Psi^\dagger \gamma^\mu \left[\bar{\mathcal{D}}_\mu + a_\mu (g_1^- + g_1^+ \gamma_5) + v_\mu (g_2^- + g_2^+ \gamma_5) \right] \Psi$$

If parity is conserved, one has to put $g_1^- = g_2^+ = 0$. We denote $g_1^+ = g_a$, $g_2^- = -i g_v$.

$$S_f = \int d^4x \, \det(e) \, \bar{\Psi} \gamma^\mu \left(\bar{D}_\mu - i l_\mu \frac{1+\gamma^5}{2} - i r_\mu \frac{1-\gamma^5}{2} \right) \Psi,$$

$$l_\mu = g_v \, v_\mu + g_a \, a_\mu, \qquad r_\mu = g_v \, v_\mu - g_a \, a_\mu.$$

Additional $U(1)_{\mathrm{L}} \times U(1)_{\mathrm{R}}$ local gauge invariance

$$\Psi_{\mathrm{L}} = \psi \to e^{i\alpha(x)} \, \psi, \quad \Psi_{\mathrm{R}} = \chi \to e^{i\beta(x)} \, \chi,$$

$$l_\mu \to l_\mu + \partial_\mu \alpha, \quad r_\mu \to r_\mu + \partial_\mu \beta.$$

Classification of boson actions

No-torsion case:

Two derivatives

$$R\,\sqrt{g}.$$

Four derivatives

$$\sqrt{g}\,R^2 \quad \text{and} \quad \sqrt{g}\,R_{\mu\nu}\,R^{\mu\nu},$$

taking into account, that $\sqrt{g}\left(R^2 - 4R_{\mu\nu}\,R^{\mu\nu} + R_{\mu\nu,\lambda\rho}\,R^{\mu\nu,\lambda\rho}\right)$ is a full derivative.

Case with torsion. Two derivatives. Torsion-square terms

$$K^{A[CD],B[EF]} = \det(e)\,(T^A_{\mu\nu}\, e^{C\mu}\, e^{D\nu})(T^B_{\alpha\beta}\, e^{E\alpha}\, e^{F\beta}).$$

This object belongs to the tensor square of the following representation of the Lorentz group: $(\mathbf{2},\mathbf{2})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{2},\mathbf{4})\oplus(\mathbf{4},\mathbf{2})$. It contains five singlets corresponding to the following expressions:

$$K_1 = \det(e)\; T^A_{\mu\nu}\; T^A_{\alpha\beta}\, g^{\mu\alpha}\, g^{\nu\beta} \;=\; \det(e)\left(-\frac{8}{3}\, a^\mu a_\mu + \frac{2}{3}\, v^\mu v_\mu + \frac{4}{9}\, t^\lambda_{\mu\nu}\, t^{\mu\nu}_\lambda\right),$$

$$K_2 = \det(e)\; T^A_{\mu\nu}\; T^B_{\alpha\beta}\, e^{A\mu}\, e^{B\alpha}\, g^{\nu\beta} = \det(e)\, v^\mu v_\mu,$$

$$K_3 = \det(e)\; T^A_{\mu\nu}\; T^B_{\alpha\beta}\, e^{A\alpha} e^{B\mu}\, g^{\nu\beta} \;=\; \det(e)\left(\frac{8}{3}\, a^\mu a_\mu + \frac{1}{3}\, v^\mu v_\mu + \frac{2}{9}\, t^\lambda_{\mu\nu}\, t^{\mu\nu}_\lambda\right)$$

$$K_4 = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\; T^A_{\mu\nu}\; T^A_{\alpha\beta} \;=\; \det(e)\,\frac{8}{3}\, a^\mu v_\mu + \frac{2}{9}\,\varepsilon^{\mu\nu\alpha\beta}\, t^\lambda_{\alpha\beta}\, t_{\lambda,\mu\nu},$$

$$K_5 = \varepsilon^{\mu\alpha\rho\lambda}\; T^A_{\mu\nu}\; T^B_{\alpha\beta}\, e^A_\rho\, e^B_\lambda\, g^{\nu\beta} = \det(e)\,\frac{8}{3}\, a^\mu v_\mu - \frac{1}{9}\,\varepsilon^{\mu\nu\alpha\beta}\, t^\lambda_{\alpha\beta}\, t_{\lambda,\mu\nu}.$$

4-derivative terms quadric in curvature

$$G^{[AB][CD][EF][GH]} = \det(e)\, \mathcal{F}^{AB}_{\alpha\beta}\, \mathcal{F}^{CD}_{\gamma\delta}\, e^{E\alpha}\, e^{F\beta}\, e^{G\gamma}\, e^{H\delta}.$$

It belongs to $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$. There are 10 singlets in this representation. Two of them are full derivatives.

4-derivative terms linear in torsion

$$L^{AB[CD][EF][GH]} = \det(e)\, (\bar{\nabla}_\lambda \bar{R}_{\alpha\beta,\gamma\delta})\, T^{A}_{\mu\nu}\, e^{B\lambda}\, e^{C\mu}\, e^{D\nu}\, e^{E\alpha}\, e^{F\beta}\, e^{G\gamma}\, e^{H\delta}$$

There are 20 singlets, but the Bianchi identities decrease this number to four:

$$L_1 = \det(e)\, v^\lambda\, \partial_\lambda \bar{R}, \quad L_2 = \det(e)\, a^\lambda\, \partial_\lambda \bar{R},$$

$$L_3 = \det(e)\, t_{\rho,\mu\nu}\, \bar{\nabla}^{[\mu} \bar{R}^{\nu]\rho}, \quad L_4 = \varepsilon^{\lambda\rho\mu\nu}\, t^{\sigma}_{\mu\nu}\, \bar{\nabla}_\lambda \bar{R}_{\rho\sigma}\,.$$

"Standard action terms" can be expressed through K-terms

$$\frac{1}{4}\,\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon^{ABCD}\,\mathcal{F}^{AB}_{\mu\nu}\,e^{C}_{\alpha}\,e^{D}_{\beta} = \sqrt{g}\,\bar{R} + K_1 - 4K_2 + 2\,K_3 + 4\,\partial_\mu\left(\sqrt{-g}\,v^\mu\right),$$

$$\frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,\mathcal{F}^{AB}_{\mu\nu}\,e^{A}_{\alpha}\,e^{B}_{\beta} = 2K_4 - 4\partial_\mu\left(\sqrt{-g}\,a^\mu\right).$$

The general form of two-derivative boson action

$$S_b = \int d^4x\,\det(e)\frac{1}{2}\big(M^2_{aa}\,a^\mu a_\mu + 2M^2_{av}\,a^\mu v_\mu + M^2_{vv}\,v^\mu v_\mu +$$

$$+ M^2_{tt}\,t^{\lambda}_{\mu\nu}t^{\mu\nu}_{\lambda} + M^2_{\epsilon tt}\,t^{\lambda}_{\mu\nu}\,t_{\lambda,\alpha\beta}\,\frac{\varepsilon^{\mu\nu\alpha\beta}}{\det(e)}\Big).$$

The stability of vacuum

The mass matrix must be positively-defined:

$$M^2_{vv} + M^2_{aa} > 0, \qquad M^2_{vv}M^2_{aa} > M^4_{av}, \qquad M^2_{tt} > M^2_{\epsilon tt}$$

Integrating over torsion we get the 4-fermion interaction term. Note that reduced torsion part ${t_{\mu\nu}}^{\lambda}$ doesn't couple to fermions in the lowest order.

Induced 4-fermion interaction

$$\mathcal{L}^{\Psi^4} = \frac{\det(e)}{2(M_{aa}^2 M_{vv}^2 - M_{av}^4)} \Big\{ A_B A^B \left(g_1^{+\,2} M_{vv}^2 - 2 g_1^+ g_2^+ M_{av}^2 + g_2^{+\,2} M_{aa}^2 \right) +$$

$$+ V_B V^B \left(g_1^{-\,2} M_{vv}^2 - 2 g_1^- g_2^- M_{av}^2 + g_2^{-\,2} M_{aa}^2 \right) +$$

$$+ 2 A_B V^B \left(g_1^+ g_1^- M_{vv}^2 - (g_1^+ g_2^- + g_1^- g_2^+) M_{av}^2 + g_2^+ g_2^- M_{aa}^2 \right) \Big\} =$$

$$= \sqrt{-g} \left(h_{AA} A_B A^B + h_{VV} V_B V^B + 2 h_{AV} A_B V^B \right),$$

where $A^B = \bar{\Psi}\gamma^B\gamma^5\Psi$ is the axial current, and $V^B = \bar{\Psi}\gamma^B\Psi$ is the vector current.

Stress-energy tensor from 4-fermion interaction

We perform averaging of 4-fermion term over fermion medium with finite temperature T:

$$< (\bar{\Psi}\Gamma_1\Psi)(\bar{\Psi}\Gamma_2\Psi) >= \frac{1}{2}\int \frac{d^4p_1}{(2\pi)^4 i}\, Tr(G(p_1)\Gamma_1) \int \frac{d^4p_2}{(2\pi)^4 i}\, Tr(G(p_2)\Gamma_2) -$$

$$-\frac{1}{2}\int \frac{d^4p_1}{(2\pi)^4 i} \int \frac{d^4p_2}{(2\pi)^4 i}\, Tr(G(p_1)\Gamma_1 G(p_2)\Gamma_2).$$

We consider one species of non-interacting fermions with mass m at temperature T and chemical potential μ which corresponds to certain charge density ρ. $\omega_n = 2\pi T(n + \frac{1}{2})$ are the Matsubara frequencies.

The fermion propagator

$$G(p) = \frac{1}{m - \widehat{p}} = \frac{1}{m - (\mu + i\omega_n)\gamma^0 - p_i\gamma^i} = \frac{m + (\mu + i\omega_n)\gamma^0 + p_i\gamma^i}{m^2 + p^2 - (\mu + i\omega_n)^2}$$

Charge density

$$\rho =< j^0 >= \langle \bar{\Psi}\gamma^0\Psi \rangle = T\sum_n \int \frac{d^3p}{(2\pi)^3}\, Tr(G(p)\gamma^0) \xrightarrow{m\ll\mu,T}$$

$$\frac{\mu T^2}{3} + \frac{\mu^3}{3\pi^2} - m^2\,\frac{\mu}{2\pi^2} + \mathcal{O}(m^4).$$

Averaging of 4-fermion interaction

$$< V_B V^B >= \frac{3}{4}\rho^2 - \frac{m^2}{2}\sigma^2,$$

$$< A_B A^B >= \frac{1}{4}\rho^2 + \frac{m^2}{2}\sigma^2,$$

$$< A_B V^B >= 0,$$

where

$$\sigma = -\frac{T^2}{6} - \frac{\mu^2}{2\pi^2}.$$

4-fermion Lagrangian

$$\mathcal{L}^{4-\mathrm{ferm}} = \sqrt{-g}\left[h_{VV}\left(\frac{3}{4}\rho^2 - \frac{m^2}{2}\sigma^2\right) + h_{AA}\left(\frac{1}{4}\rho^2 + \frac{m^2}{2}\sigma^2\right)\right].$$

Stress-energy tensor

$$\Theta_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}$$

In our case we obtain in $m \longrightarrow 0$ limit:

$$\Theta^{4-\mathrm{ferm}}_{\mu\nu}\Big|_{m=0} = \left(\frac{3}{4}h_{VV} + \frac{1}{4}h_{AA}\right)\rho^2\Big|_{m=0}\left(-g_{\mu\nu} + 6\,\delta_{\mu 0}\,\delta_{\nu 0}\right)$$

Equations of state

The equation of state for ultrarelativistic fermion gas:

$$\epsilon = 3\, p.$$

4-fermion interaction contribution

If 4-fermion interaction is dominant:

$$\epsilon^{4-\mathrm{ferm}} = 5\, p^{4-\mathrm{ferm}}$$

If 4-fermion interaction is a small correction:

$$\epsilon^{4-\mathrm{ferm}} = p^{4-\mathrm{ferm}}$$

Conclusions

1. The most general one-derivative fermion lagrangian was found.
2. All two-derivative boson lagrangians are listed, and partial classification in four-derivative sector was performed.
3. The most general 4-fermion interaction term, which arises from torsion, was obtained.
4. It was averaged over fermion medium, and the equations of state were derived.
5. The contribution of the 4-fermion interaction to the stress energy tensor is of the order of $(\mu^2/M^2)T^4$. The main contribution is proportional to T^4. The four fermion interaction is not observable if torsion bosons masses are of the Plank scale; If they are of the smaller order it could have contributed during the early epochs.

Thank you for the attention!