Computing DIS Heavy Flavor Contributions at NNLO

Fabian Wissbrock, DESY

in collaboration with Johannes Blümlein and Francis Brown

Erice, June 2012
Contents

1. Heavy flavor contributions in DIS
2. Performing integrals using hyperlogarithms
3. Conclusions
Introduction

Deep-inelastic scattering:

\[ W_{\mu \nu} \]

kinematic quantities:
\[ Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}, \]
\[ \nu := \frac{pq}{M} \]

differential cross-section:
\[ \frac{d\sigma}{dQ^2 dx} \sim W_{\mu \nu} L^{\mu \nu} \]

- u, d and s-quarks are considered massless

Structure of the hadronic tensor:

\[ W^\gamma_{\mu \nu} (q, P, s) = \frac{1}{4\pi} \int d^4 \xi \exp(iq\xi) \langle P, s | [J^em_{\mu}(\xi), J^em_{\nu}(0)] | P, s \rangle \]

unpol. \( \left\{ \begin{aligned}
&= \frac{1}{2x} \left( g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left( P_{\mu} P_{\nu} + \frac{q_{\mu} P_{\nu} + q_{\nu} P_{\mu}}{2x} - \frac{Q^2}{4x^2} g_{\mu \nu} \right) F_2(x, Q^2) \\
\end{aligned} \right. \]

- The structure functions \( F_{2, L} \) contain light and heavy quark contributions.
Scaling violations

Comparision of the scaling violations in the massless and in the massive case:

LO contributions: massless vs. massive (PDFs from [Alekhin, Melnikov, Petriello, 2006.])
→ different slopes in $\log(Q^2)$,
→ the massive contributions at lower values of $x$ are of order 20%-35%.
Need for heavy flavor computation in DIS

- The knowledge of heavy flavor DIS contributions to $O(\alpha_s^3)$ allows to measure $\Lambda_{QCD}$ and $\alpha_s(M_Z)$ with increased precision, $|\Delta \alpha_s| < 1\%$.
- The unification of forces depends crucially on the value of $\alpha_s$ at low scales.
- The detailed understanding of parton densities is necessary to interpret the hadron induced processes at HERA, TEVATRON & LHC.

→ Important case:

hadronic Higgs Boson production $\propto \alpha_s^2 G^2(x, Q^2)$. 
Structure functions at leading twist

The structure functions factorize into Wilson coefficients and parton densities:

\[ F_i(x, Q^2) = \sum_j C_i^j \left( x, \frac{Q^2}{\mu^2} \right) \otimes f_j(x, \mu^2) \]

\[ \otimes = \text{Mellin convolution} \]

- The Wilson coefficients are process dependent and contain purely light and heavy quark contributions:
  \[ C_i^j(x, Q^2/\mu^2, m_h) = C_i^{j,\text{light}}(x, Q^2/\mu^2) + H_i^j(x, Q^2/\mu^2, Q^2/m_h^2), \ h = c, b. \]

- For \( Q^2 \gg m^2 \), for \( F_2: Q^2 \geq 10m_h^2 \), the \( H_i^j \)'s obey the asymptotic representation [Buza, Matiounine, Smith, Migneran, van Neerven, 1996.]:

\[ H_{(2, L), i}^{S, NS} \left( x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = A_{k, i}^{S, NS} \left( x, \frac{m^2}{\mu^2} \right) \otimes C_{(2, L), k}^{S, NS} \left( x, \frac{Q^2}{\mu^2} \right). \]

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- The \( C_i^{j,\text{light}} \)'s are known to NNLO. [Moch, Vermaseren, Vogt, 2005]

- This convolution becomes a simple product after a Mellin transformation:
  \[ f'(N) := \int_0^1 dx \ x^{N-1} f(x). \]
Status of the massive $O(\alpha_s^3)$ computation for $F_2$:

- Renormalization has been worked out. Fixed Mellin Moments for $F_2$ from $N = 2, \ldots, 10(14)$.  
  [Bierenbaum, Blümlein, Klein, 2009]

- Contributions $\propto N_F$ (all $N$)  
  [Ablinger, Blümlein, Klein, Schneider, Wißbrock, 2010; Blümlein, Hasselhuhn, Klein, Schneider 2012]

- First contributions $\propto T_F^2$ to $F_2$ (all $N$) and first contributions for two massive quark flavors.  
  [Ablinger, Blümlein, Klein, Schneider, Wißbrock, 2011 ]

- Contributions from ladder diagrams.  
  [Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock 2012 ]

- All contributions $\propto \ln^k(Q^2/m^2)$ to $F_2$ are completely known from renormalization.

- Two out of five heavy flavor Wilson Coefficients contributing to $F_2$ are now completely known at NNLO: $L_{qq, Q}^{PS}, L_{qg, Q}^S$. 
Massive operator matrix elements

\[
A_{ij}^{S,NS} \left( N, nf + 1, \frac{m^2}{\mu^2} \right) = \langle j \mid O_i^{S,NS} \mid j \rangle = \delta_{ij} + \sum_{k=1}^{\infty} \alpha_s^k A_{ij}^{(k),S,NS}
\]

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- The massive OMEs are obtained from massive Feynmandiagrams with additional Feynman rules for the operator insertion.

E.g. singlet heavy quark operator:

\[
O_Q^{\mu_1 \cdots \mu_N}(z) = \frac{1}{2} i^{N-1} S[\bar{q}(z) \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_N} q(z)] - \text{TraceTerms}.
\]

\[\Delta: \text{light–like momentum, } \Delta^2 = 0.\]

\[\implies \text{additional vertices with 2 and more gluons at higher orders.}\]
Example: Diagram 4

We use the $\alpha$-representation to obtain:

\begin{align*}
I_4(N) &= \int \cdots \int d\alpha_1 \, d\alpha_2 \, d\alpha_3 \, d\alpha_4 \, d\alpha_5 \, d\alpha_6 \, d\alpha_7 \, d\alpha_8 \frac{\sum_{j=0}^{N} T_{4a}^{N-j} T_{4b}^j}{U_G^{2+\varepsilon/2} M_G^{2-3/2 \varepsilon}} \\
T_{4a} &= \alpha_5 \alpha_7 \alpha_4 + \alpha_2 \alpha_3 \alpha_5 + \alpha_2 \alpha_5 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_2 \alpha_5 \alpha_8 + \alpha_8 \alpha_5 \alpha_4 + \alpha_5 \alpha_7 \alpha_8 + \alpha_2 \alpha_3 \alpha_8 \\
&\quad + \alpha_7 \alpha_2 \alpha_8 + \alpha_6 \alpha_2 \alpha_8 + \alpha_3 \alpha_7 \alpha_2 + \alpha_2 \alpha_3 \alpha_6 + \alpha_4 \alpha_2 \alpha_8 + \alpha_2 \alpha_6 \alpha_4 + \alpha_4 \alpha_7 \alpha_2 \\
T_{4b} &= +\alpha_2 \alpha_5 \alpha_4 + \alpha_4 \alpha_2 \alpha_8 + \alpha_4 \alpha_7 \alpha_2 + \alpha_2 \alpha_5 \alpha_8 + \alpha_2 \alpha_3 \alpha_5 + \alpha_7 \alpha_2 \alpha_8 + \alpha_3 \alpha_7 \alpha_2 + \alpha_8 \alpha_5 \alpha_4 \\
&\quad + \alpha_5 \alpha_7 \alpha_4 + \alpha_4 \alpha_1 \alpha_8 + \alpha_1 \alpha_7 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_5 \alpha_7 \alpha_8 + \alpha_8 \alpha_1 \alpha_7 + \alpha_1 \alpha_3 \alpha_7 \\
U_G &= \alpha_2 \alpha_5 \alpha_4 + \alpha_2 \alpha_3 \alpha_5 + \alpha_1 \alpha_3 \alpha_5 + \alpha_5 \alpha_7 \alpha_4 + \alpha_1 \alpha_6 \alpha_4 + \alpha_1 \alpha_3 \alpha_6 + \alpha_2 \alpha_3 \alpha_6 + \alpha_2 \alpha_6 \alpha_4 \\
&\quad + \alpha_5 \alpha_6 \alpha_4 + \alpha_1 \alpha_5 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_1 \alpha_3 \alpha_7 + \alpha_1 \alpha_7 \alpha_4 + \alpha_3 \alpha_7 \alpha_2 + \alpha_4 \alpha_7 \alpha_2 + \alpha_3 \alpha_5 \alpha_6 \\
&\quad + \alpha_2 \alpha_3 \alpha_8 + \alpha_2 \alpha_5 \alpha_8 + \alpha_5 \alpha_7 \alpha_8 + \alpha_8 \alpha_5 \alpha_4 + \alpha_8 \alpha_5 \alpha_6 + \alpha_5 \alpha_3 \alpha_8 + \alpha_1 \alpha_8 \alpha_5 + \alpha_1 \alpha_8 \alpha_6 \\
&\quad + \alpha_6 \alpha_2 \alpha_8 + \alpha_1 \alpha_8 \alpha_3 + \alpha_4 \alpha_1 \alpha_8 + \alpha_4 \alpha_2 \alpha_8 + \alpha_7 \alpha_2 \alpha_8 + \alpha_8 \alpha_1 \alpha_7 \\
M_G &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7.
\end{align*}

- The integral above is a projective integral: one $\alpha$-parameter may be set 1, the others have to be integrated from 0 to $\infty$.
- This integral converges in $D = 4$ dimensions.
How can one perform integrals algebraically?

One has to construct primitives.

→ We have to know the complete target space!

In the present case, the target space can be constructed in terms of hyperlogarithmic functions $L(\vec{w}, z): \mathbb{C}\setminus\Sigma \to \mathbb{C}$, where

- $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables.
- Let $\vec{w}$ be a word, where each letter corresponds to one point $\sigma_i$.

$L(\vec{w}, z)$ is uniquely defined by

1. $L(\emptyset, z) = 1$, and $L(\{0, \ldots, 0\}, z) = \frac{1}{n!} \log^n(z)$ for $n \geq 1$,

2. $L(\{\sigma_i, \vec{w}\}, z) = \int_0^z dx \frac{1}{x-\sigma_i} L(\vec{w}, z)$ otherwise.

[e.g. $L(\{\sigma_i\}, z) = \log(z - \sigma_i) - \log(\sigma_i)$]

The weight of $L(\vec{w}, z)$ is given by the number of letters in $\vec{w}$. 

[Brown, 2008; 2009]
Evaluate the primitive at the respective integration borders.

For Example: \( L \left( \{-x - 1, 0\}, y \right) \xrightarrow{y \to \infty}? \)

1. Compute \( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \):

\[
\frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) = \left( \frac{1}{y + x + 1} - \frac{1}{x + 1} \right) L \left( \{0\}, y \right) + \frac{L \left( \{-x - 1\}, y \right)}{x + 1}
\]

2. The asymptotic expansion of this derivative is

\[
\text{asympt} \left( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \right) = -\frac{L \left( \{-1\}, x \right)}{x + 1}
\]

3. Construct the primitive of this:

\[
\int \text{asympt} \left( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \right) = L \left( \{-1, -1\}, x \right)
\]

4. We have to add the respective integration constant, which is given by

\[
\text{asympt} \left( L \left( \{-1, 0\}, y \right) \right) = L \left( \{0, 0\}, y \right) - \zeta_2.
\]

5. Thus:

\[
\text{asympt} \left( L \left( \{-x - 1, 0\}, y \right) \right) = L \left( \{-1, -1\}, x \right) + L \left( \{0, 0\}, y \right) - \zeta_2.
\]

- This works if the denominators factor into linear expressions in the integration variable at every integration step.
- The Fubini algorithm determines a priori, if and for which integration order this is possible.

[Stembridge, 1998; Brown, 2008; 2009]
Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$

e.g.:

![Diagram](image)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Diag 4</th>
<th>Diag 5$_a$</th>
<th>Diag 5$_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2 - 2\zeta_3$</td>
<td>$2\zeta_3$</td>
<td>$2\zeta_3$</td>
</tr>
<tr>
<td>1</td>
<td>$-2 + 2\zeta_3$</td>
<td>$-\frac{5}{2} - \zeta_3$</td>
<td>$-2 - 2\zeta_3$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{29}{12} - \frac{83}{36}\zeta_3$</td>
<td>$\frac{133}{72} + \frac{41}{8}\zeta_3$</td>
<td>$\frac{71}{24} + \frac{5}{2}\zeta_3$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{17}{6} + \frac{47}{18}\zeta_3$</td>
<td>$-\frac{1735}{432} - \frac{35}{36}\zeta_3$</td>
<td>$-\frac{905}{216} - \frac{5}{2}\zeta_3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>19</td>
<td>$-\frac{5825158236879253094413489658569181}{2503562235895708381108915200000}$ - $\frac{104899807174743864253}{54192375991353600}\zeta_3$</td>
<td>$-\frac{1280902666890628029062643215783549}{1335233192477711136591421440000}$ + $\frac{238388793949217497}{30168755507520}\zeta_3$</td>
<td>$-\frac{254116903575797385411050257769}{252885074332899983647564800000}$ - $\frac{1968329}{635040}\zeta_3$</td>
</tr>
</tbody>
</table>

[Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock 2012]
Due to the power function the integrals do not fit directly into the framework of the algorithm for general values of $N$.

In order to use the algorithm also on integrals with general values of $N$, a generating function is constructed by the mapping

$$p(\alpha_i)^N \to \frac{1}{1 - x \ p(\alpha_i)} = \sum_{N=0}^{\infty} x^N p(\alpha_i)^N,$$

which defines a formal power series in $x$.

Performing the $\alpha$-parameter integrations then leads to an expression which contains Hyperlogarithms $L_w(x)$ in the variable $x$.

Finally the $N$th coefficient of this expression in $x$ has to be extracted symbolically. This has been done with the package HarmonicSums by J.Ablinger. [Ablinger, Blümlein, Schneider 2012.]
For example, Benz-topology:

\[
I(x) = \frac{1}{(1+N)(2+N)x} \left\{ \zeta_3 \left[ 2L \{-1\}, x \right] - 2\left(-1 + 2x\right)L \{1\}, x \right] - 4L \{1, 1\}, x \right] \\
-3L \{-1, 0, 0, 1\}, x \right] + 2L \{-1, 0, 1, 1\}, x \right] - 2xL \{0, 0, 1, 1\}, x \right] \\
+3xL \{0, 1, 0, 1\}, x \right] - xL \{0, 1, 1, 1\}, x \right] + \left(-3 + 2x\right)L \{1, 0, 0, 1\}, x \right] \\
+2xL \{1, 0, 1, 1\}, x \right] - \left(-1 + 5x\right)L \{1, 1, 0, 1\}, x \right] + xL \{1, 1, 1, 1\}, x \right] \\
-2L \{1, 0, 0, 1\}, x \right] + 3L \{1, 0, 1, 0, 1\}, x \right] - L \{1, 0, 1, 1, 1\}, x \right] \\
+2L \{1, 1, 0, 0, 1\}, x \right] + 2L \{1, 1, 0, 1, 1\}, x \right] - 5L \{1, 1, 1, 0, 1\}, x \right] \\
+L \{1, 1, 1, 1, 1\}, x \right] \right\} \]
$I(N) = \frac{1}{(N+1)(N+2)(N+3)} \left\{ \frac{648 + 1512N + 1458N^2 + 744N^3 + 212N^4 + 32N^5 + 2N^6}{(1+N)^3(2+N)^3(3+N)^3} \right.$

\[ \left. - \frac{2 \left( -1 + (-1)^N + N + (-1)^N \right)}{(1+N)} \right\} \zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)} S_1^3 + \frac{1}{24} S_4^4 \]

\[ - \frac{(7 + 22N + 10N^2)}{2(1+N)^2(2+N)} S_2 - \frac{19}{8} S_2^2 - \frac{1 + 4N + 2N^2}{2(1+N)^2(2+N)} S_1^2 + \frac{9}{4} S_2 - \frac{(-9 + 4N)}{3(1+N)} S_3 \]

\[ - \frac{1}{4} S_4 - 2(-1)^N S_{-2,1} + \frac{(-1 + 6N)}{(1+N)} S_{2,1} + \frac{54 + 207N + 246N^2 + 130N^3 + 32N^4 + 3N^5}{(1+N)^3(2+N)^2(3+N)^2} S_1 \]

\[ + 4\zeta_3 S_1 - \frac{(-2 + 7N)}{2(1+N)} S_2 S_1 + \frac{13}{3} S_3 S_1 - 7S_{2,1} S_1 - 7S_{3,1} + 10S_{2,1,1} \right\} \]

Harmonic Sums and their generalizations are defined by:

$$S_{a_1, \ldots, a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\text{sign}(a_1))^{n_1} \cdots (\text{sign}(a_m))^{n_m}}{n_1^{a_1} \cdots n_m^{a_m}} \quad a_i \in \mathbb{Z} \setminus \{0\} ,$$

[Vermaseren 1998, Blümlein, Kurth, 1998]

$$S_{a_1, \ldots, a_m}(x_1, \ldots, x_m)(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{x_1^{a_1} \cdots x_m^{a_m}}{n_1^{a_1} \cdots n_m^{a_m}} \quad a_i \in \mathbb{N}_+, x_j \in \mathbb{C} \setminus \{0\} .$$

[Moch, Uwer, Weinzierl, 2002; Ablinger Blümlein, Schneider 2012.]
Further example: Diagram 4

In intermediate example around 60,000 different Hyperlogarithms have been observed. Almost all of them canceled as predicted by the Fubini reduction algorithm.

\[
\hat{I}_4 = \frac{Q_1(N)}{2(1 + N)^5(2 + N)^3(3 + N)} + \frac{Q_2(N)}{(1 + N)^2(2 + N)^2(3 + N)^2} \zeta_3 + \frac{(-1)^N (65 + 101N + 56N^2 + 13N^3 + N^4)}{2(1 + N)^2(2 + N)^2(3 + N)^2} S_{-3} \\
+ \frac{(-24 - 5N + 2N^2)}{12(2 + N)^2(3 + N)^2} S_1 + \frac{1}{2(1 + N)(2 + N)(3 + N)} S_2^2 + \frac{1}{(2 + N)(3 + N)} S_1 S_2 \\
+ \frac{Q_4(N)}{4(1 + N)^3(2 + N)^2(3 + N)^2} S_1^2 - \frac{3}{2} S_5 - \frac{Q_5(N)}{6(1 + N)^2(2 + N)^2(3 + N)^2} S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-1} S_{-2} \\
+ \frac{(-1)^N (65 + 101N + 56N^2 + 13N^3 + N^4)}{(1 + N)^2(2 + N)^2(3 + N)^2} S_{-2,1} + \frac{(59 + 42N + 6N^2)}{2(1 + N)(2 + N)(3 + N)} S_4 + \frac{(5 + N)}{(1 + N)(3 + N)} \zeta_3 S_{1,2} \\
- \frac{Q_6(N)}{4(1 + N)^3(2 + N)^2(3 + N)^2} S_2 - \zeta_3 S_2 - \frac{3}{2} S_3 S_2 - 2S_{2,1} S_2 + \frac{(99 + 225N + 190N^2 + 65N^3 + 7N^4)}{2(1 + N)^2(2 + N)^2(3 + N)} S_{2,1} \\
+ \frac{Q_3(N)}{(1 + N)^4(2 + N)^4(3 + N)^4} S_1 - \frac{(11 + 5N)}{(1 + N)(2 + N)(3 + N)} \zeta_3 S_1 - \frac{Q_7(N)}{4(1 + N)^2(2 + N)^2(3 + N)^2} S_2 S_1 - S_{2,3} \\
+ \frac{Q_3(N)}{2(1 + N)(2 + N)(3 + N)} S_3 S_1 - \frac{(1 + N)(2 + N)(3 + N)}{3(3 + 2N)} \zeta_3 S_1 + \frac{(53 + 29N)}{2(1 + N)(2 + N)(3 + N)} S_{3,1} - 3S_{4,1} \\
+ S_{-2,1,-2} + \frac{2^{N+1} (-28 - 25N - 4N^2 + N^3)}{(1 + N)^2(2 + N)(3 + N)^2} S_{1,2} \left( \frac{1}{2}, 1 \right) - \frac{(-79 - 40N + N^2)}{(1 + N)(2 + N)(3 + N)} S_{2,1,1} \\
+ 5S_{2,2,1} + 6S_{3,1,1} + \frac{2^N (-28 - 25N - 4N^2 + N^3)}{(1 + N)^2(2 + N)(3 + N)^2} S_{1,1} \left( \frac{1}{2}, 1, 1 \right) \\
- \frac{(5 + N)}{(1 + N)(3 + N)} S_{1,1,2} \left( 2, \frac{1}{2}, 1 \right) - \frac{(5 + N)}{2(1 + N)(3 + N)} S_{1,1,1,1} \left( 2, \frac{1}{2}, 1, 1 \right)
\]
Conclusions

- Understanding the complete NNLO heavy flavor contributions allows a more precise determination of $\alpha_s$ and the parton distribution functions, which are of essential importance for the analysis of the LHC data.
- The only missing piece in the most interesting kinematic region of $Q \geq 10m^2$ are the massive Operator matrix elements.
- Presently all $\log^k(Q^2/m^2)$ contributions and 2 out of 5 heavy flavor Wilson coefficients are completely known.
- Various new technologies are required for higher topologies.
- We present a method based on hyperlogarithmic functions to evaluate diagrams with operator insertions in $d=4$ dimensions.
- The method allows a relativly fast evaluation of fixed Mellin-Moments and has been extended for general-$N$ computations.
- It is fast, yields compact results and allows to compute diagrams, which could not be evaluated with different methods yet.
- The calculation of remaining topologies and mass-assignments are underway.
These properties allow to define different operations on the hyperlogarithmic functions:

- Construction primitives of expressions consisting of rational functions and hyperlogarithms
- Differentation with respect to $z$ and w.r.t variables in the alphabet
- Transformations $L(\{f_1(x), \ldots, f_2(x)\}, z) \rightarrow L(\{g_1(z), \ldots, g_2(z)\}, x)$
- Series expansions around $z = 0$ and $z = \infty$.

Using these operations we may perform multiple integrals over rational functions to finite or infinite upper bounds.

- This works if the denominators factor into linear expressions in the integration variable at every integration step.
- The *Fubini algorithm* determines a priori, if and for which integration order this is possible.  
  [Stembridge, 1998; Brown, 2008; 2009]
Example: \( L \left( \{-x - 1, 0\}, y \right) \) \( \sim \) ?

- Compute \( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \): 
  \[
  \frac{\partial}{\partial x} \frac{\partial}{\partial y} L \left( \{-x - 1, 0\}, y \right) = -\frac{L(\{0\}, y)}{(y+1+x)^2}
  \]

  \[
  \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) = \int dy \frac{\partial}{\partial x} \frac{\partial}{\partial y} L \left( \{-x - 1, 0\}, y \right) + \text{Const}
  \]

  \[
  = \left( \frac{1}{y + x + 1} - \frac{1}{x + 1} \right) L(\{0\}, y) + \frac{L(\{-x - 1\}, y)}{x + 1} + \text{Const}
  \]

- Fix \( \text{Const} \) such that \( \lim_{y \to 0} \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) = 0 \), in this case \( \text{Const} = 0 \)

- The asymptotic expansion of this derivative is
  \[
  \text{asympt} \left( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \right) = -\frac{L(\{-1\}, x)}{x + 1}
  \]

- \( \int \text{asympt} \left( \frac{\partial}{\partial x} L \left( \{-x - 1, 0\}, y \right) \right) = L \left( \{-1, -1\}, x \right) \)

- We have to add the respective integration constant, which is given by
  \( \text{asympt} \left( L \left( \{-1, 0\}, y \right) \right) = L \left( \{0, 0\}, y \right) - \zeta_2 \)

- Thus \( \lim_{y \to \infty} L \left( \{-x - 1, 0\}, y \right) = L \left( \{-1, -1\}, x \right) + L \left( \{0, 0\}, y \right) - \zeta_2. \)