Computing DIS Heavy Flavor Contributions at NNLO

Fabian Wissbrock, DESY

in collaboration with Johannes Blümlein and Francis Brown

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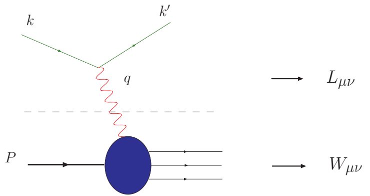
Introduction

Deep-inelastic scattering:



- kinematic quantities:
  $Q^2 := -q^2, \quad x := \frac{Q^2}{2pq},$
  $\nu := \frac{Pq}{M}$
- differential cross-section:
  $\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$

- u, d and s-quarks are considered massless

Structure of the hadronic tensor:

$$W_{\mu\nu}^{\gamma^*}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

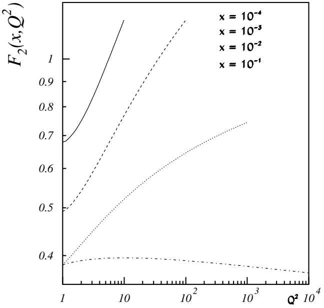
$$\text{unpol.}\left\{ = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

.

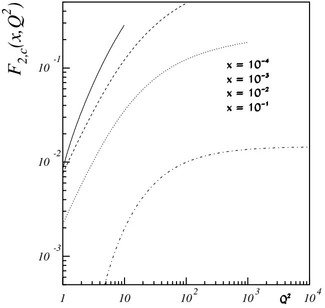
- The structure functions $F_{2,L}$ contain light and heavy quark contributions.

Scaling violations

Comparision of the scaling violations in the massless and in the massive case:



light quarks



charm contribution

LO contributions: massless vs. massive (PDFs from [Alekhin, Melnikov, Petriello, 2006.])

→ different slopes in $log(Q^2)$,

→ the massive contributions at lower values of x are of order 20%-35%.

Need for heavy flavor computation in DIS

- The knowledge of heavy flavor DIS contributions to $O(\alpha_s^3)$ allows to measure Λ_{QCD} and $\alpha_s(M_Z)$ with increased precision, $|\Delta\alpha_s| < 1\%$.
- The unification of forces depends crucially on the value of α_s at low scales.
- The detailed understanding of parton densities in necessary to interpret the hadron induced processes at HERA, TEVATRON & LHC.

$\longrightarrow$ Important case:

hadronic Higgs Boson production $\propto \alpha_s^2 G^2(x, Q^2)$.

Structure functions at leading twist

The structure functions factorize into Wilson coefficients and parton densities:

$$F_i(x,Q^2) = \sum_j \underbrace{C_i^j\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{non-perturbative}} \quad ; \otimes \; = \text{Mellin convolution} \; .$$

- The Wilson coefficients are process dependent and contain purely light and heavy quark contributions:
$C_i^j\left(x,\; Q^2/\mu^2,\; m_h\right) = C_i^{j,light}\left(x,\; Q^2/\mu^2\right) + H_i^j\left(x, Q^2/\mu^2, Q^2/m_h^2\right)\,,\;\; h = c, b\;.$

- For $Q^2 \gg m^2$, for F_2: $Q^2 \geq 10 m_h^2$, the H_i^j's obey the asymptotic representation [Buza, Matiounine, Smith, Migneron, van Neerven, 1996.]:

$$H_{(2,L),i}^{\mathrm{S,NS}}\left(x,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A_{k,i}^{\mathrm{S,NS}}\left(x,\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\mathrm{S,NS}}\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{light–flavor–Wilson coefficients}} .$$

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- The $C_i^{j,light}$'s are known to *NNLO*. [Moch, Vermaseren, Vogt, 2005]
- This convolution becomes a simple product after a Mellin transformation:
$f'(N) := \int_0^1 dx\; x^{N-1} f(x)\;.$

Status of the massive $O(\alpha_s^3)$ computation for F_2:

- Renormalization has been worked out. Fixed Mellin Moments for F_2 from $N = 2, \ldots, 10(14)$. [Bierenbaum, Blümlein, Klein, 2009]
- Contributions $\propto N_F$ (all N) [Ablinger, Blümlein, Klein, Schneider, Wißbrock, 2010; Blümlein, Hasselhuhn, Klein, Schneider 2012]
- First contributions $\propto T_F^2$ to F_2 (all N) and first contributions for two massive quark flavors. [Ablinger, Blümlein, Klein, Schneider, Wißbrock, 2011]
- Contributions from ladder diagrams. [Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock 2012]

- All contributions $\propto \ln^k(Q^2/m^2)$ to F_2 are completely known from renormalization.
- Two out of five heavy flavor Wilson Coefficients contributing to F_2 are now completely known at NNLO: $L_{qq,Q}^{PS}$, $L_{qg,Q}^{S}$.

Massive operator matrix elements

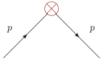
$$A_{ij}^{\mathrm{S,NS}}\left(N, nf+1, \frac{m^2}{\mu^2}\right) = \langle j \mid O_i^{\mathrm{S,NS}} \mid j\rangle = \delta_{ij} + \sum_{k=1}^{\infty} \alpha_s^k A_{ij}^{(k),\mathrm{S,NS}}$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

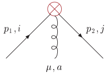
- The massive OMEs are obtained from massive Feynmandiagrams with additional Feynman rules for the operator insertion.

E.g. singlet heavy quark operator:

$$O_Q^{\mu_1,\dots,\mu_N}(z) = \frac{1}{2} i^{N-1} S[\overline{q}(z)\gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} q(z)] - \mathrm{TraceTerms}\ .$$



$$\not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\ ,$$



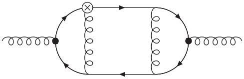
$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\ ,$$

$$\gamma_+ = 1\ , \quad \gamma_- = \gamma_5\ .$$

Δ: light–like momentum, $\Delta^2 = 0$.

$\Longrightarrow$ additional vertices with 2 and more gluons at higher orders.

Example: Diagram 4



- We use the α-representation to obtain:

$$
\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\, d\alpha_5\, d\alpha_6\, d\alpha_7\, d\alpha_8\, \frac{\sum_{j=0}^{N} T_{4a}^{N-j} T_{4b}^{j}}{U_G^{2+\varepsilon/2} M_G^{2-3/2\,\varepsilon}} \\
T_{4a} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U_G &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
M_G &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 .
\end{aligned}
$$

- The integral above is a projective integral: one α-parameter may be set 1, the others have to be integrated from 0 to ∞.
- This integral converges in $D = 4$ dimensions.

Hyperlogarithms

- How can one perform integrals algebraically?
  1. One has to construct primitives.
     $\rightarrow$ We have to know the complete target space!

In the present case, the target space can be constructed in terms of

hyperlogarithmic functions $L(\vec{w}, z) : \mathbb{C}\backslash\Sigma \rightarrow \mathbb{C}$, where

- $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables.
- Let $\vec{w}$ be a word, where each letter corresponds to one point σ_i.

$L\left(\vec{w}, z\right)$ is uniquely defined by

1. $L\left(\emptyset, z\right) = 1$, and $L(\{\underbrace{0, \ldots, 0}_{\text{n times}}\}, z) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$,
2. $L\left(\{\sigma_i, \vec{w}\}, z\right) = \int_0^z dx \frac{1}{x-\sigma_i} L\left(\vec{w}, z\right)$ otherwise.

[Brown, 2008; 2009]

- e.g. $L\left(\{\sigma_i\}, z\right) = \log(z - \sigma_i) - \log(\sigma_i)$
- The weight of $L\left(\vec{w}, z\right)$ is given by the number of letters in $\vec{w}$.

2. Evaluate the primitive at the respective integration borders.

For Example: $L(\{-x-1,0\},y) \underset{y\to\infty}{\simeq} ?$

1. Compute $\frac{\partial}{\partial x} L(\{-x-1,0\},y)$:

$$\frac{\partial}{\partial x} L(\{-x-1,0\},y) = \left(\frac{1}{y+x+1} - \frac{1}{x+1}\right) L(\{0\},y) + \frac{L(\{-x-1\},y)}{x+1}$$

2. The asymptotic expansion of this derivative is

$$asympt\left(\frac{\partial}{\partial x} L(\{-x-1,0\},y)\right) = -\frac{L(\{-1\},x)}{x+1}$$

3. Construct the primitive of this:

$$\int asympt\left(\frac{\partial}{\partial x} L(\{-x-1,0\},y)\right) = L(\{-1,-1\},x)$$

4. We have to add the respective integration constant , which is given by

$$asympt\left(L(\{-1,0\},y)\right) = L(\{0,0\},y) - \zeta_2 \ .$$

5. Thus:

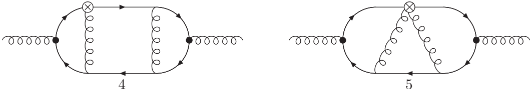
$$asympt\left(L(\{-x-1,0\},y)\right) = L(\{-1,-1\},x) + L(\{0,0\},y) - \zeta_2 \ .$$

- This works if the denominators factor into linear expressions in the integration variable at every integration step.
- The *Fubini algorithm* determines a priori, if and for which integration order this is possible.

[Stembridge, 1998; Brown, 2008; 2009]

- Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$

e.g.:



N	Diag 4	Diag 5_a	Diag 5_b
0	$2 - 2\zeta_3$	$2\zeta_3$	$2\zeta_3$
1	$-2 + 2\zeta_3$	$-\frac{5}{2} - \zeta_3$	$-2 - 2\zeta_3$
2	$\frac{29}{12} - \frac{83}{36}\zeta_3$	$\frac{133}{72} + \frac{41}{8}\zeta_3$	$\frac{71}{24} + \frac{5}{2}\zeta_3$
3	$-\frac{17}{6} + \frac{47}{18}\zeta_3$	$-\frac{1735}{432} - \frac{35}{36}\zeta_3$	$-\frac{905}{216} - \frac{5}{2}\zeta_3$
...	...	...	...
19	$-\frac{5825158236879253094413489658569181}{2503562235895708381108915200000} - \frac{10489980717474386425 3}{5419237599135 3600}\zeta_3$	$-\frac{1280902668906280290626432157835 49}{1335233192477711136591421440 00} + \frac{2383887939492174 97}{3010687555075 20}\zeta_3$	$-\frac{25411690357579738541105025 7769}{2528850743328998364756480 0000} - \frac{1968329}{635040}\zeta_3$

[Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock 2012.]

General N-representation

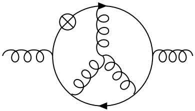
- Due to the power function the integrals do not fit directly into the framework of the algorithm for general values of N.
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed by the mapping

$$p(\alpha_i)^N \to \frac{1}{1 - x\, p(\alpha_i)} = \sum_{N=0}^{\infty} x^N p(\alpha_i)^N \quad ,$$

which defines a formal power series in x.

- Performing the α-parameter integrations then leads to an expression which contains Hyperlogarithms $L_w(x)$ in the variable x.
- Finally the Nth coefficient of this expression in x has to be extracted symbolically. This has been done with the package HarmonicSums by J.Ablinger. [Ablinger, Blümlein, Schneider 2012.]

- For example, Benz-topology :



$$
\begin{aligned}
I(x) \quad = \quad & \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big) \\
& -3L(\{-1,0,0,1\},x) + 2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) \\
& +3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x) + (-3+2x)L(\{1,0,0,1\},x) \\
& +2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x) \\
& -2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) \\
& +2L(\{1,1,0,0,1\},x) + 2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) \\
& +L(\{1,1,1,1,1\},x)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3} \\
& -\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3-(-1)^N S_{-3}-\frac{N}{6(1+N)}S_1^3+\frac{1}{24}S_1^4 \\
& -\frac{(7+22N+10N^2)}{2(1+N)^2(2+N)}S_2-\frac{19}{8}S_2^2-\frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2+\frac{9}{4}S_2-\frac{(-9+4N)}{3(1+N)}S_3 \\
& -\frac{1}{4}S_4-2(-1)^N S_{-2,1}+\frac{(-1+6N)}{(1+N)}S_{2,1}+\frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1 \\
& +4\zeta_3 S_1-\frac{(-2+7N)}{2(1+N)}S_2S_1+\frac{13}{3}S_3S_1-7S_{2,1}S_1-7S_{3,1}+10S_{2,1,1}\Bigg\}
\end{aligned}
$$

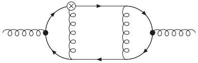
- Harmonic Sums and there generalizations are defined by

$$
S_{a_1,\ldots,a_m}(N)=\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}}\frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}}\frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}}\cdots\frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} \quad a_i\in\mathbb{Z}\setminus\{0\},
$$

[Vermaseren 1998, Blümlein, Kurth, 1998]

$$
S_{a_1,\ldots,a_m}(x_1,\cdots,x_m)(N)=\sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}}\frac{(x_1^{n_1}}{n_1^{|a_1|}}\frac{(x_2^{n_2}}{n_2^{|a_2|}}\cdots\frac{((a_m))^{n_m}x_m^{n_m}}{n_m^{|a_m|}}\ , a_i\in\mathbb{N}_+, x_i\in\mathbb{C}\setminus\{0\}.
$$

[Moch, Uwer, Weinzierl, 2002; Ablinger Blümlein, Schneider 2012.]



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- Further example: Diagram 4
- In intermediate example around 60.000 different Hyperlogarithms have been observed. Almost all of them canceled as predicted by the Fubini reduction algorithm.

$$
\begin{aligned}
\hat{I}_4 \;=\;& \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\,(65+101N+56N^2+13N^3+N^4)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
&+\frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2 S_2 \\
&+\frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-2,1}S_{-2} \\
&+\frac{(-1)^N\,(65+101N+56N^2+13N^3+N^4)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3 S_1(2) \\
&-\frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3 S_2 - \frac{3}{2}S_3 S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
&+\frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3 S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2 S_1 - S_{2,3} \\
&+\frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3 S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1 S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
&+S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\,(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
&+5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\,(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
&-\frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

Conclusions

- Understanding the complete NNLO heavy flavor contributions allows a more precise determination of α_s and the parton distribution functions , which are of essential importance for the analysis of the LHC data.
- The only missing piece in the most interesting kinematic region of $Q \geq 10m^2$ are the massive Operator matrix elements .
- Presently all $log^k(Q^2/m^2)$ contributions and 2 out of 5 heavy flavor Wilson coefficients are completely known.
- Various new technologies are required for higher topologies.
- We present a method based on hyperlogarithmic functions to evaluate diagrams with operator insertions in $d = 4$ dimensions.
- The method allows a relativly fast evaluation of fixed Mellin-Moments and has been extended for general-N computations.
- It is fast, yields compact results and allows to compute diagrams, which could not be evaluated with different methods yet.
- The calculation of remaining topologies and mass-assignments are underway.

These properties allow to define different operations on the hyperlogarithmic functions:

- Construction primitives of expressions consisting of rational functions and hyperlogarithms
- Differentation with respect to z and w.r.t variables in the alphabet
- Transformations $L\left(\{f_1(x), \ldots, f_2(x)\}, z\right) \rightarrow L\left(\{g_1(z), \ldots, g_2(z)\}, x\right)$
- Series expansions around $z = 0$ and $z = \infty$.

Using these operations we may perform multiple integrals over rational functions to finite or infinite upper bounds.

- This works if the denominators factor into linear expressions in the integration variable at every integration step.
- The *Fubini algorithm* determines a priori, if and for which integration order this is possible.

[Stembridge, 1998; Brown, 2008; 2009]

Example: $L(\{-x-1,0\},y) \underset{y\to\infty}{\simeq} ?$

- Compute $\frac{\partial}{\partial x} L(\{-x-1,0\},y)$:
  - $\frac{\partial}{\partial x}\frac{\partial}{\partial y} L(\{-x-1,0\},y) = -\frac{L(\{0\},y)}{(y+1+x)^2}$

$$\frac{\partial}{\partial x} L(\{-x-1,0\},y) = \int dy \frac{\partial}{\partial x}\frac{\partial}{\partial y} L(\{-x-1,0\},y) + Const$$
$$= \left(\frac{1}{y+x+1} - \frac{1}{x+1}\right) L(\{0\},y) + \frac{L(\{-x-1\},y)}{x+1} + Const$$

  - Fix $Const$ such that $\lim_{y\to 0} \frac{\partial}{\partial x} L(\{-x-1,0\},y) = 0$, in this case $Const = 0$
- The asymptotic expansion of this derivative is

$$asympt\left(\frac{\partial}{\partial x} L(\{-x-1,0\},y)\right) = -\frac{L(\{-1\},x)}{x+1}$$

- $\int asympt\left(\frac{\partial}{\partial x} L(\{-x-1,0\},y)\right) = L(\{-1,-1\},x)$
- We have to add the respective integration constant , which is given by $asympt\left(L(\{-1,0\},y)\right) = L(\{0,0\},y) - \zeta_2$
- Thus $lim_{y\to\infty} L(\{-x-1,0\},y) = L(\{-1,-1\},x) + L(\{0,0\},y) - \zeta_2$.