Gauge Forces: From QCD to Quantum Gravity

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Content

1 Pomeron in hadron interactions

High energy kinematics

$$s = 4E^2 \gg (-t) = \vec{q}^{\,2} \sim m^2\,,\ \theta \ll 1$$

Pomeron and Pomeranchuck theorem

$$A_P(s,t) \approx is\,\gamma^2(t)\,s^{\omega(t)}\,,\ \omega(t) = \Delta + \alpha' t\,,\ \sigma_{pp} = \sigma_{p\bar{p}} \sim s^{\Delta}$$

"Potential" from Pomeron exchange (Gribov (1960))

$$V(s,\vec{\rho}) = \int \frac{d^2q}{2\pi s}\, e^{i\vec{\rho}\vec{q}}\, A_P(s,-\vec{q}^{\,2}) \approx i\,\gamma^2(0)\,\frac{s^{\Delta}}{\alpha' \ln s}\,\exp\left(-\frac{\vec{\rho}^{\,2}}{4\alpha' \ln s}\right)$$

Mandelstam cut contribution

$$A_{Mand}(s,t) = -is \int \frac{d^2k}{(2\pi)^2} \Phi^2(k,q-k) s^{\omega(-k^2)} s^{\omega(-(q-k)^2)}$$

2 Gribov Pomeron calculus

Ordering of particle clusters in rapidities

$$y_r = \frac{1}{2} \ln \frac{\sqrt{|k|^2 + m_r^2} + |k|}{\sqrt{|k|^2 + m_r^2} - |k|}\,,\ 1 \ll y_r - y_{r-1} \ll \ln s$$

Non-relativistic Pomeron propagator

$$G_0 = \frac{1}{E + \Delta - \frac{|k|^2}{2m}}\,,\ E = -\omega\,,\ \alpha' = \frac{1}{2m}$$

Gribov effective action for Pomeron interactions

$$S = \int dy\, d^2\rho \left(\phi^* (\partial_y - \Delta)\phi + \frac{1}{2m}\, |\vec{\partial}\phi|^2) + i\lambda(\phi^*\phi^2 + \phi\phi^{*2}) + ... \right)$$

Weak coupling solution and universality

$$\Delta = 0\,,\ \gamma_{ii}(0) = const\,,\ \gamma_{ir}(0) = 0$$

3 Elastic scattering in QCD

Born amplitude at high energies $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation (LLA)

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,,\ \alpha_s \ln s \sim 1\,,\ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

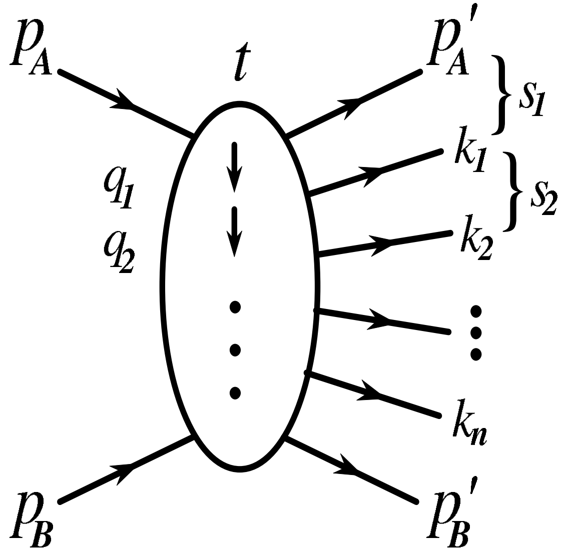
Gluon trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Bootstrap equation

$$\omega(-|q|^2) f = 1 + \left(\omega(-|k|^2) + \omega(-|q-k|^2) + \hat{K}_8\right) f$$

4 Gluon production amplitude in LLA



$$M^{BFKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$C(q_2,q_1) = -q_1^{\perp} - q_2^{\perp} + p_A\left(\frac{q_1^2}{kp_A} + \frac{kp_B}{p_Ap_B}\right) - p_B\left(\frac{q_2^2}{kp_B} + \frac{kp_A}{p_Ap_B}\right)$$

5 BFKL Pomeron

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\,,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\,,\;\; \rho_r = x_r + iy_r$$

Möbius invariance and Pomeron intercept

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\; m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu\,,$$

$$E = \epsilon_m + \epsilon_{\widetilde{m}}\;,\;\; \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)\,,\;\; \Delta = \frac{g^2 N_c}{\pi^2}\ln 2 > 0$$

6 BKP equation in multi-color QCD

Bartels-Kwiecinski-Praszalowicz equation (1979)

$$E\,\Psi = H\,\Psi\;,\;\; H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}\;,\;\; \Delta = -\frac{\alpha N_c}{2\pi}\, E_0$$

Holomorphic factorization at large N_c (L. (1988))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)\;,\;\; H = h + h^*$$

Integrability at large N_c (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},\;\; L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix},$$

$$T(u) = A(u) + D(u)\;,\;\; [T(u), h] = 0$$

7 High energy effective action in QCD

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}\,, \;\; |y - y_0| < \eta\,, \;\; \eta << \ln\, s$$

Gluon and reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x)\,, \;\; A_\pm(x) = -iT^a A_\pm^a(x)\,, \, \partial_\mp A_\pm(x) = 0$$

Effective action for the reggeon interactions (L., 1995)

$$S = \int d^4x \left(L_{QCD} + Tr(V_+ \partial_\mu^2 A_- + V_- \partial_\mu^2 A_+) \right) \,,$$

$$V_+ = -\frac{1}{g}\partial_+ \, P \exp \left(-g \int_{-\infty}^{x^+} v_+(x')d(x')^+ \right) = v_+ - g v_+ \frac{1}{\partial_+} v_+ + ...$$

8 Amplitude $A_{2\to 4}$ in $N=4$ SUSY

Remainder function $R = A_{2\to 4}/A^{BDS}_{2\to 4}$ (L. (2009), F.,L. (2011))

$$R\, e^{i\pi\delta} = cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\phi n}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi(\nu,n)\left(\frac{-1}{\sqrt{u_2 u_3}}\right)^{\omega(\nu,n)},$$

$$u_1 = \frac{s s_2}{s_{012}s_{123}}\,,\ u_2 = \frac{s_1 t_3}{s_{012}t_2}\,,\ u_3 = \frac{s_3 t_1}{s_{123}t_2}\,,\ |w|^2 = \frac{u_2}{u_3}\,,\ \cos\phi = \frac{1-u_1-u_2-u_3}{2\sqrt{u_2 u_3}}\,,$$

$$\delta = \frac{\gamma_K}{8}\,\ln\frac{|w|^2}{|1+w|^4}\,,\ \omega_{ab} = \frac{\gamma_K}{8}\,\ln|w|^2\,,\ \Phi = 1 - a\left(\frac{E_{\nu n}^2}{2} + \frac{3}{8}\,n^2/(\nu^2+\frac{n^2}{4})^2 + \zeta(2)\right)$$

$$\omega(\nu,n) = -aE_{\nu,n} - a^2(\epsilon^{FL}_{\nu n} + 3\zeta(3))\,,\ E_{\nu n} = -\frac{|n|/2}{\nu^2+\frac{n^2}{4}} + 2\Re\psi(1+i\nu+\frac{|n|}{2}) - 2\psi(1)\,,$$

Next-to-leading eigenvalue (F.,L. (2011))

$$\epsilon^{FL}_{\nu n} = -\frac{\Re}{2}\left(\psi''(1+i\nu+\frac{|n|}{2}) - \frac{2i\nu\psi'(1+i\nu+\frac{|n|}{2})}{\nu^2+\frac{n^2}{4}}\right) - \zeta(2)\,E_{\nu n} - \frac{1}{4}\,\frac{|n|\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}$$

9 BFKL equation in $N = 4$ SUSY

BFKL kernel in two loops (F., L. and C.,C. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\;M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\;\beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. 2002)

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

10 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Pomeron intercept at large $\lambda = g^2 N_c$ (KLOV, BPST, KL)

$$j = 2 - \Delta\,,\ \Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2} + O(\lambda^{-5/2})$$

Anomalous dimension slope at $j = 2$ (KLOV, Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

11 Perturbation series divergency

Perturbative expansion in Yang-Mils model $SU(2)$

$$f = \sum_{k=0}^{\infty} g^{2k}\, c_k\,,\quad \lim_{k\to\infty} c_k = k!\, a^k\, k^{\delta}\, C$$

Solution for the Higgs model with $L_{int} = G|\phi|^4/2$ (BLM (1979))

$$A_\mu^a = \frac{4\eta^a_{\mu\nu} x_\nu (\rho^2 - 1)/g}{(x^2+\rho^2)(1+x^2\rho^2)}\,,\ \hat{\phi} = \frac{i\hat{U}\, 4\sqrt{3}/g}{\sqrt{(x^2+\rho^2)(1+x^2\rho^2)}}\,,\ \rho^4 = 12\frac{G}{g^2} - 1$$

Anomalous dimension in $N=4$ SUSY at $N_c \to \infty$

$$\gamma(\omega) = \sum_{k=1}^{\infty} \lambda^k\, c_k(\omega)\,,\ \lim_{k\to\infty} c_k(\omega) = \frac{\lambda_{cr}^{-k}}{k^{3/2}}\,\frac{1}{2\sqrt{\pi}}\, a\,,$$

$$\omega = \omega_0(\lambda_{cr}) = 1 - \Delta(\lambda_{cr})\,,\ a = \sqrt{\frac{\lambda_{cr}\,\omega_0'(\lambda_{cr})}{D(\lambda_{cr})}}$$

12 High energy action in gravity

Graviton and reggeized graviton fields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,\;\; \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa}\int d^4x \left(\sqrt{-g}\,R + \frac{1}{2}\left(\partial_+ j^-\,\partial_\mu^2 A^{++} + \partial_- j^+\,\partial_\mu^2 A^{--}\right)\right)$$

Hamilton-Jacobi equation for $j^\pm = 2x^\pm - \omega^\pm$

$$g^{\mu\nu}\partial_\mu\omega^\pm\partial_\nu\omega^\pm = 0\,,\;\; \partial_\pm j^\mp = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2}\,\frac{\partial_\rho}{\partial_\pm}h_{\pm\pm}\right)^2 + \ldots$$

Hamilton-Jacobi functional

$$\omega = \min_{x^\nu,p_\mu}\int_{-\infty}^{\tau} d\tau\, L\,,\;\; L = p_\mu x^\mu - \frac{e(\tau)}{2}\,g^{\mu\nu}p_\mu p_\nu$$

13 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2\to n} = -s^2 \Gamma^{\mu'\nu'}_{\mu\nu} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} \ldots \Gamma^{\rho'\sigma'}_{\rho\sigma}$$

Graviton-graviton-reggeon vertex

$$\Gamma^{\mu'\nu'}_{\mu\nu} = \frac{\kappa}{4} \left(\Gamma_{\mu\mu'} \Gamma_{\nu\nu'} + \Gamma_{\mu\nu'} \Gamma_{\nu\mu'} \right)$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p^A_{\mu'} p^B_{\mu} + p^{A'}_{\mu} p^B_{\mu'}}{p^A p^B} + \frac{q^2}{2} \frac{p^B_{\mu} p^B_{\mu'}}{(p^A p^B)^2}$$

Reggeon–reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} \left(C_\rho C_\sigma - N_\rho N_\sigma \right) , \; N = \sqrt{q_1^2 q_2^2} \left(\frac{p^A}{k p^A} - \frac{p^B}{k p^B} \right)$$

14 Graviton trajectory in supergravity

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\alpha}{\pi}\int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,,\;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k,q-k)^2\left(\frac{1}{k^2}+\frac{1}{(q-k)^2}\right) - q^2 + \frac{N}{2}(k,q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5\gamma_\nu\partial_\rho\psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha|q|^2\left(\ln\frac{|q|^2}{\lambda^2} + \frac{N-4}{2}\ln\frac{|\Lambda|^2}{|q|^2}\right)$$

15 Double-logarithms in (super) gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2 \,\ln \frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\;\; \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for supergravity (BLS (2012))

$$f_\omega = 1 + b\frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - b\,\frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\,,\;\; b = \alpha|q|^2\,,\;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,\;\; \xi = \alpha\,|q|^2\,\ln^2\frac{s}{|q|^2}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2 - 26N + 36)\,\frac{\xi^3}{6!} + \ldots$$

Solution in terms of the parabolic cylinder function

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln d^{(N)}(x)\,,\;\; d^{(N)}(x) = e^{\frac{x^2}{4}}\,D_{\frac{6-N}{2}}(x)\,,\;\; x = \frac{\omega}{\sqrt{b}}$$

16 Double-logarithmic eikonal phase

Scattering amplitude in the eikonal picture

$$A_{DL}(s,t) = -2is\, s^{-\alpha|q|^2 \ln \frac{|q|^2}{\mu^2}} \int d^2\rho\, e^{i\vec{q}\vec{\rho}} \left(e^{i\delta_{DL}(\vec{\rho},\, \ln s)} - 1\right) ,$$

Double logarithmic approximation for the phase

$$\delta_{DL}(\vec{\rho},\, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \int \frac{d^2q}{|q|^2}\, e^{-i\vec{q}\vec{\rho}}\, \Phi(\xi)$$

Eikonal phase in $N = 8$ SUSY at small impact parameters

$$\delta_{DL}^{N=8}(\vec{\rho},\, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \pi \ln \frac{1}{\alpha\lambda^2 \ln^2(\rho^2 s)} ,\;\; \rho^2 \ll \alpha \ln^2 s$$

Two loop result in an agreement with exact calculations

$$A_4^{N=8} = \frac{\kappa^2 s^2}{|q|^2} (-i\pi s)\alpha^2 |q|^2 \frac{\ln^3 \frac{s}{|q|^2}}{3}$$

17 Discussion

1. Locality in rapidity and Gribov effective action.
2. BFKL Pomeron in QCD.
3. Integrability of the BKP equation.
4. Effective action for high energy QCD.
5. Pomeron and graviton in Maldacena model.
6. Effective action for high energy gravity.
7. Graviton trajectory and effective vertices.
8. Double logarithmic amplitudes in (super) gravity.
9. Eikonal amplitude with double logarithmic phase.