Embedding oscillatory modes of quarks in baryons in QCD

Looking to construct a bridge

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Abstract

The regularities implied at large distances by complete gauge invariance in QCD were the topic of my lectures in the two initial parts of the tri-annual '50th Birthday Celebrations' of this school .
This lecture shall be devoted to oscillatory modes of (valence) u,d,s quarks in baryons (also evident in $q\,\overline{q}\,'$ mesons) , based on the representation derived in ref. [1-1980] and extended to the discussion of paired oscillator modes and associated Bogoliubov transformations in ref. [2-2010] .
The embedding into the base field dynamics of QCD with three light flavors is layed out as a bridge in its present raw construction hopefully to be completed as a task of the future .

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1 - Introduction

In order to put the main topic of this lecture into perspective let me begin citing ref. [1-1980] , my construction of Poincaré invariant oscillatory modes of three valence quarks (u , d) restricted to the two nonstrange flavors in nonstrange baryons . A small collection of references to previous work in this direction is given there ([1-1980]) . The disappearing of perfectly gauge invariant explicit dependence on color of quark- and gauge boson-fields is by the confined nature of the oscillator wave functions – restricted to the center of mass system relative soace coordinates – relegated to an outer factor

(1) $\varepsilon_{\, c_\alpha c_\beta c_\gamma} \;\; ; \;\; c_{\,1,2,3} \; = $ **red , green , blue**

$\alpha, \beta, \gamma \; = \; 1, 2, 3 \quad :$ **numbering individual quark** *positions*

The color factor in eq. 1 : $\varepsilon_{\, c_\alpha c_\beta c_\gamma}$ **, fully antisymmetric in its three color indices , must be fully gauge invariant with respect to the local** $SU3_{\,c}$ **gauge group and thus reduced from the 3 positions** $\vec{x}_{\,1,2,3}$ **to a common space point through parallel transport** $q \; q \; q$ **(triple) QCD string factors .The detailed form of the QCD string factors is discussed in section 2 – premises , for which I cite my previous two Erice lectures in refs. [3-2011] , [4-2012] in order to maintain consistency of notation .**

A sketch of the $q \, \overline{q}$ **(bilocal- ior double-)) and** $q \; q \; q$ **(triple-) QCD string factors , also called 'bond structures' , is given in figure 1 - I below** $\longrightarrow$

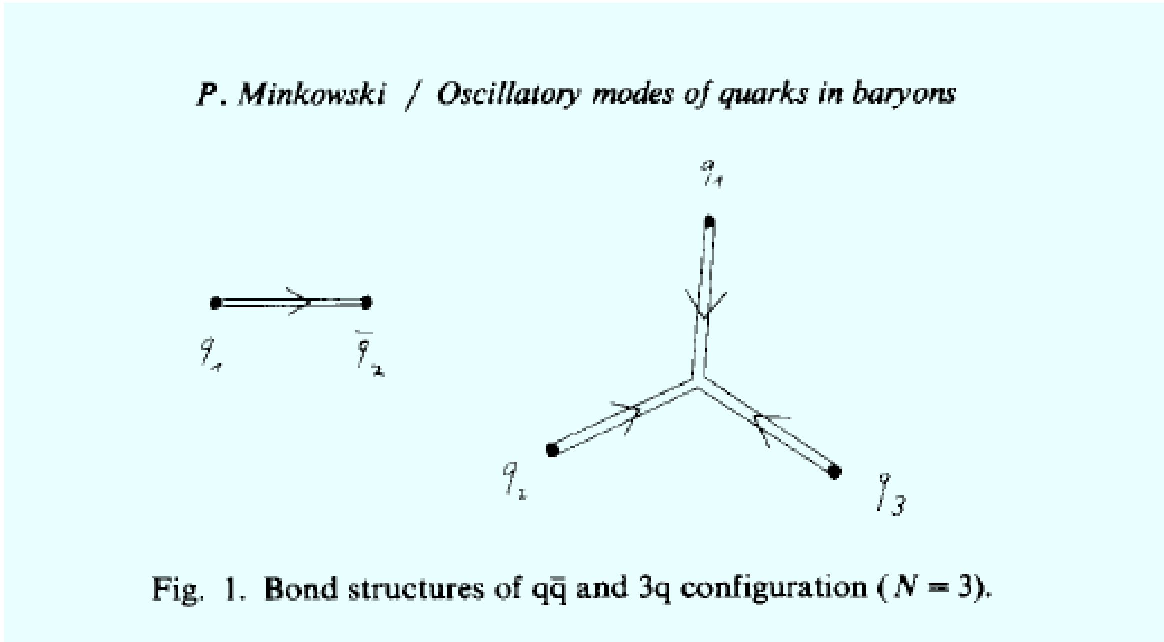
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Fig. 1. Bond structures of $q\bar{q}$ and 3q configuration ($N = 3$).

Fig. 1 - I : Bond structures of $q\overline{q}$ and $3q$ configurations $(\; N = 3 \;)$ from ref. [1-1980] $\longleftrightarrow$

The next step revived questions related to oscillatory modes of a pair of independent oscillators and their eventual connection to Bogoliubov transformations in November 2010 in ref. [2-2010] .
Finally on the road leading to the present outline a discussion with the group of Willibald Plessas from the University of Graz during the Oberwölz Symposium 2012 : Quantum Chromodynamics: History and Prospects 516. WE-Heraeus-Seminar , Oberwölz, Styria, Austria. 3. - 8. September 2012 $\longrightarrow$

(discussion) arose as to whether the baryon modes of light flavors u, d, s of quarks with low total spin were presently more or less completely accounted for – according to the new PDG-review [5-2011] – contrasting with the situation in 1980 with respect to only u and d flavors .

Looking at the exhaustive tables of ref. [5-2011] *included today* the answer is obviously to the negative , yet these tables were not available in the web-version of the PDG tables [6-2012] upon my last search before the Oberwölz Symposium 2012 .

Out of this situation the chiallenge took shape to count the oscillatory modes in baryons in their own right in analytic ways , the content of this lecture . The notefile to it assembles the results obtained in ref. [7-2013] to date .

$\longrightarrow$

2-1 – Assembling elements of the QCD Lagrangean density – premises

We face the theoretical abstraction of QCD with $N_{fl} = 6$, representing strong interactions – adaptable to two or three light flavors (u , d , s) of quarks and antiquarks. $\leftrightarrow$

quarks : color is counted in $\pi^{0} \rightarrow \gamma\gamma$ (assuming global color- and flavor-projections to commute) yet see ref. [8-2001]

spin and flavor are clearly seen in $q\overline{q}$ and $3q$, $3\overline{q}$ spectroscopy (a pre-condition to count color) .

$$\mathcal{L} = \left[\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \, \delta_{c'\dot{c}} \\ + W^{r}_{\mu} \left(\frac{1}{2}\lambda_{r} \right)_{c'\dot{c}} \end{array} \right\} \gamma^{\mu}_{\dot{\mathcal{S}}'\mathcal{S}} \, q^{c}_{\mathcal{S}\,f} \begin{array}{c} \\ \\ - m_{f}\, \overline{q}^{\,\dot{c}}_{\dot{\mathcal{S}}\,\dot{f}}\, q^{c}_{\mathcal{S}\,f} \end{array} \right]$$

(2)

$$- \frac{1}{4\,g^{2}} B^{\mu\nu\,r} B^{r}_{\mu\nu} + \Delta\mathcal{L}$$

$W^{r}_{\mu} \equiv - v^{r}_{\mu}$: for identification of convention for potentials

quarks : c' , $c = 1, 2, 3$ color , $f = 1, \cdots, 6$ flavor

$\mathcal{S}', \mathcal{S} = 1, \cdots, 4$ spin , m_{f} mass

$\longrightarrow$

In eq. 2 , the $\mathcal{D}$ associated gauge connection fields – where $\mathcal{D} = \mathcal{D}\,(\,\mathcal{G}\,)$ denotes a general , irreducible representation of the local gauge group $\mathcal{G} = SU3_{\,c}$ – appear in the form appropriate for quarks : $\mathcal{D} = \{\,3\,\}$, and antiquarks : $\mathcal{D} = \{\,\overline{3}\,\}$ respectively

(3)
$$\boxed{\begin{array}{l} (\,\mathcal{W}_{\mu}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta}\,(\,x\,) = W^{\,r}_{\mu}\,(\,x\,)\,(\,d_{\,r}\,)_{\,\alpha\beta} \;\leftrightarrow\; \mathcal{W}_{\mu}\,(\,\mathcal{D}\,) = -\,\mathcal{W}_{\mu}\,(\,\mathcal{D}\,)^{\,\dagger} \\ d_{\,r} = -\,d^{\,\dagger}_{\,r} = \frac{1}{i}\,J_{\,r} \in Lie\,(\,\mathcal{D}\,)\;\;;\;\;[\,d_{\,p}\,,\,d_{\,q}\,] = f_{\,pqr}\,d_{\,r} \\ r\,,\,p\,,\,q = 1,\cdots,\,dim\,\mathcal{G}\;\;;\;\;\alpha\,,\,\beta = 1,\cdots,\,dim\,\mathcal{D} \end{array}}$$

For $\mathcal{D}\,(\,SU3_{\,c}\,) = \left\{\,3\,(\overline{3})\,\right\}$ the representation matrices become (the Gell-Mann matrices [9-1964])

$$\left(\,d_{\,r}\,(3) = \tfrac{1}{i}\,\tfrac{1}{2}\,\lambda_{\,r}\,\right)_{\,\alpha\beta}\;;\;r = 1,\cdots,8\;;\;(\,\alpha\,,\,\beta\,) \leftrightarrow \left(\,c^{\,\prime}\,,\,\dot{c}\,\right) = 1,\cdots,3$$

(4)
$$d_{\,r}\,(\overline{3}) = \overline{d}_{\,r}\,(3)$$

with the *conventional* normalization conditions : $-\;tr\,d^{\,r}\,d^{\,s} = \frac{1}{2}\,\delta^{\,rs}$

The quantity proportional to the gauge potentials $W^{\,r}_{\mu}$ for the $\overline{q}\,q$ in eq. 2 is thus identified as

(5)
$$\left[\,W^{\,r}_{\mu}\,\left(\,\tfrac{1}{2}\lambda_{\,r}\,\right)_{\,c^{\prime}\dot{c}} = i\,(\,\mathcal{W}_{\mu}\,(\,\mathcal{D} = \{\,3\,\}\,)\,)_{\,c^{\prime}\dot{c}}\,\right](\,x\,)$$

Here we postpone the discussion of complete connections and extend the QCD Lagrangean density to include the term quadratic in the field strengths $B^{\,r}_{\mu\nu}$ and $\Delta\,\mathcal{L}$ in eq. 2, in Fermi gauges. $\longrightarrow$

gauge bosons : $\mathcal{L}_B = -\frac{1}{4g^2} B^{\mu\nu\,r} B^r_{\mu\nu}$

$$B^r_{\mu\nu} = \partial_\mu W^r_\nu - \partial_\nu W^r_\mu + f_{rst} W^s_\mu W^t_\nu \longleftarrow (W^r_\mu \equiv -v^r_\mu)$$

(6) $$r, s, t = 1, \cdots, dim(\mathcal{G} = SU3_c) = 8$$

Lie algebra labels, $\left[\frac{1}{2}\lambda^r, \frac{1}{2}\lambda^s\right] = i f_{rst} \frac{1}{2}\lambda^t$

perturbative rescaling :

$$W^r_\mu = g W^r_{\mu\,pert}, \; B^r_{\mu\nu} = g B^r_{\mu\nu\,pert}$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.

Completing $\Delta \mathcal{L}$ in Fermi gauges

$$\Delta \mathcal{L} = \left\{ \begin{array}{c} -\frac{1}{2\eta g^2} (\partial_\mu W^{\mu\,r})^2 \\ + \partial^\mu \bar{c}^r (D_\mu c)^r \end{array} \right\} \; ; \quad \eta \text{ : gauge parameter}$$

(7)

ghost fermion fields : $c, \bar{c} \;;\; (D_\mu c)^r = \partial_\mu c^r + f_{rst} W^s_\mu c^t$

gauge fixing constraint : $C^r = \partial_\mu W^{\mu\,r}$

$\longrightarrow$

2-2 – Gauge boson binary bilocal and adjoint (here octet-) string operators

One goal is, to identify – not just some candidate resonance – gluonic mesons, binary and higher modes, and to relate them to the base quantities within QCD . Here we follow ref. [3-2011] .

$$B_{\,[\,\mu_1\,\nu_1\,]\,,\,[\,\mu_2\,\nu_2\,]}\,(\,x_{\,1}\,,\,x_{\,2}\,)\;=$$

$$=\;B^{\;r}_{\,[\,\mu_1\,\nu_1\,]}\,(\,x_{\,1}\,)\,U\,(\,x_{\,1}\,,\,r\,;\,x_{\,2}\,,\,s\,)\,B^{\;s}_{\,[\,\mu_2\,\nu_2\,]}\,(\,x_{\,2}\,) \tag{8}$$

$$r\,,\;s\,,\cdots\;=\;1,\cdots,8\quad;\quad\text{no flavor but spin}$$

$B^{\;r}_{\,[\,\mu\,\nu\,]}\,(\,x\,)$ **denote the local color octet of field strengths.**

The quantity $U\,(\,x\,,\,r\,;\,y\,,\,s\,)$ **in eq. (8) denotes the octet string operator, i. e. the path ordered exponential over a straight line path** $\mathcal{C}$ **from y to x**

$$\begin{aligned} U\,(\,x\,,\,r\,;\,y\,,\,s\,) &= P\,\exp\left(\left.\int_{y}^{x}\right|_{\mathcal{C}} d\,z^{\,\mu}\,\tfrac{1}{i}\,v^{\;t}_{\,\mu}\,(\,z\,)\,\mathcal{F}_{\,t}\right)\Bigg|_{\,r\;s} \\ &= P\,\exp\left(-\left.\int_{y}^{x}\right|_{\mathcal{C}} d\,z^{\,\mu}\,W^{\;t}_{\,\mu}\,(\,z\,)\,\left(\,\tfrac{1}{i}\,\mathcal{F}_{\,t}\,\right)\right)\Bigg|_{\,r\;s} \end{aligned} \tag{9}$$

$$(\,\mathcal{F}_{\,t}\,)_{\,rs}\;=\;i\,f_{\,r\;t\;s}\quad;\quad(\,ad_{\,t}\,)_{\,rs}=\tfrac{1}{i}\,(\,\mathcal{F}_{\,t}\,)_{\,rs}\;=\;f_{\,r\;t\;s}$$

The path ordered exponential as a matrix function of the argument is to be performed before the matrix elements, denoted $\big|_{\,..}$ **in eq. 9 , are taken. The local limit becomes** $\longrightarrow$

(10) $$B_{[\mu_1 \nu_1],[\mu_2 \nu_2]}(x_1 = x_2 = x) = (:) B^{r}_{[\mu_1 \nu_1]}(x) B^{r}_{[\mu_2 \nu_2]}(x)(:)$$

no flavor but spin

2-3 – $\overline{q}\, q$ bilinears and triplet-string operators

$$B^{q}_{[\dot{\mathcal{A}}\, \dot{f}_1,\, \mathcal{B}\, f_2]}(x_1, x_2) = \overline{q}^{\,\dot{c}_1}_{\mathcal{B}\, \dot{f}_2}(x_1)\, U(x_1, c_1; x_2, \dot{c}_2)\, q^{\,c}_{\mathcal{A}\, f_1}(x_2)$$

flavor *and* spin

(11) $$U(x, c_1; y, \dot{c}_2) = P \exp\left(\int_y^x \Big|_{\mathcal{C}} dz^{\mu} \tfrac{1}{i} v^{t}_{\mu}(z) \tfrac{1}{2} \lambda_t \right)\Bigg|_{c_1 \dot{c}_2}$$
$$= P \exp\left(- \int_y^x \Big|_{\mathcal{C}} dz^{\mu} W^{t}_{\mu}(z) \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t \right) \right)\Bigg|_{c_1 \dot{c}_2}$$

with the local limit

(12) $$B^{q}_{[\dot{\mathcal{B}}\, \dot{f}_2,\, \mathcal{A}\, f_1]}(x_1 = x_2 = x) = (:)\, \overline{q}^{\,\dot{c}}_{\mathcal{B}\, \dot{f}_2}(x)\, q^{\,c}_{\mathcal{A}\, f_1}(x)\, (:)$$

The symbols $(:)$ in eqs. 10 and 12 should indicate that normal ordering of regulating the local limits is required and further that such normal ordering is *not* unique , and dependent on quark masses in the case of the $\overline{q}\, q$ bilinears .

$\longrightarrow$

2-4 – Connection and curvature - forms
preparing the ensuing analysis of regularity conditions

Lets begin this (sub-)section rewriting the bilocal (formally) unitary operators forming the gauge connection dependent octet- (eq. 9) and triplet (eq. 11) QCD strings , substituting an equivalent , matrix oriented notation

octet string : $U (x , r \; ; \; y , s) \; = \; P \exp \left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t} (z) \left(\frac{1}{i} \mathcal{F}_t \right) \right) \Bigg|_{r s}$

triplet string : $U (x , c_1 ; y , \dot{c}_2) \; = \; P \exp \left(- \int_y^x \Big|_{\mathcal{C}} d z^{\mu} W_{\mu}^{t} (z) \left(\frac{1}{i} \frac{1}{2} \lambda_t \right) \right) \Bigg|_{c_1 \dot{c}_2}$

with the substitutions $\longrightarrow$

$$\left. \begin{array}{ll} \text{octet string} : & U (x , r \; ; \; y , s) \to \left(U \left(x \overset{C}{\leftarrow} y \right) \right)_{rs} \\ \text{triplet string} : & U (x , c_1 ; y , \dot{c}_2) \to \left(U \left(x \overset{C}{\leftarrow} y \right) \right)_{c_1 \dot{c}_2} \end{array} \right\} \to \begin{array}{r} (U (x , C , y ; \mathcal{D}))_{\alpha\beta} \\ \in \mathcal{D} (\mathcal{G}) \end{array}$$

with $\mathcal{G} \; = \;$ **simple compact gauge group** ; $\mathcal{D}$: **general irreducible representation of** $\mathcal{G}$

(13)

Here $\mathcal{G} \; = \; SU3_{\,c}$ **and** $\mathcal{D}$ **is the octet- , triplet representation for the respective QCD** $\mathcal{D}$ **- strings.** $\longrightarrow$

Further let us consider matrix valued connection 1-forms , which define the bilocal matrix valued operators $(U(x, C, y; \mathcal{D}))_{\alpha\beta} \in \mathcal{D}(\mathcal{G})$ as given in eq. 13 . To this end the form of octet and triplet strings in eq. 13 is repeated below

$$\text{octet string}: U(x, r \; ; \; y, s) = P \exp\left(-\int_y^x \Big|_{\mathcal{C}} dz^{\mu} W_{\mu}^{t}(z) \left(\tfrac{1}{i} \mathcal{F}_t\right)\right)\Bigg|_{rs}$$

$$\text{triplet string}: U(x, c_1; y, \dot{c}_2) = P \exp\left(-\int_y^x \Big|_{\mathcal{C}} dz^{\mu} W_{\mu}^{t}(z) \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t\right)\right)\Bigg|_{c_1 \dot{c}_2}$$

(14)

The two matrices in brackets to the right of the integrand expressions in eq. 14 form an antihermitian basis of the Lie algebra representation $Lie(\mathcal{D})$ for $\mathcal{D} =$ adjoint and $\mathcal{D} =$ triplet representations of $\mathcal{G} = SU3_c$ respectively

$$d_t \equiv d_t(\mathcal{D}) \leftrightarrow (d_t)_{\alpha\beta} = \begin{cases} \left(\tfrac{1}{i} \mathcal{F}_t\right)_{rs} & \textbf{for} \quad Lie(\mathcal{D}) = \textbf{adjoint} \\ \left(\tfrac{1}{i} \tfrac{1}{2} \lambda_t\right)_{c_1 \dot{c}_2} & \textbf{for} \quad Lie(\mathcal{D}) = \textbf{triplet} \end{cases} \tag{15}$$

$$d_t = -d_t^{\dagger} \;\; ; \;\; t = 1, \cdots, dim\,\mathcal{G} \;\; ; \;\; \alpha, \beta = 1, \cdots, dim\,\mathcal{D} \;\; \textbf{for general } \mathcal{D}$$

$\longrightarrow$

From eqs. 14 and 15 we construct a – hopefully – consistent notation as appropriate for matrix valued $\mathcal{D}$ connections, 1-forms and strings, as well as derived 2- and higher forms. First eq. 15 is subject to the (matrix-) commutation relations

$$\left[\, d_{\,r} \, , \, d_{\,s} \, \right] \, = \, f_{\,rst} \, d_{\,t} \quad ; \quad \forall \, \mathcal{D} \, (\, \mathcal{G} \,) \quad ; \quad r, s, t \, = \, 1, \cdots, dim \, \mathcal{G} \qquad \longrightarrow$$

(16) $\left(\, d_{\,t} \, (\, \mathcal{D} \, = \, \text{adjoint representation} \,) \, \right)_{\,sr} \, = \, \left(\, ad_{\,t} \, \right)_{\,sr} \, = \, f_{\,str} \quad : \quad$ **independent of** $\mathcal{D}$

$f_{\,str}$: **totally antisymmetric, real structure constants of** $Lie \, (\, \mathcal{G} \,)$

In physics the antihermitian matrix code with respect to the representations $Lie \, (\, \mathcal{D} \,)$ is (mostly) replaced by the hermitian one [a]

(17)
$$\left(\, d_{\,t} \, \equiv \, \frac{1}{i} \, h_{\,t} \, \right) \, \left(\, Lie \, (\, \mathcal{D} \,) \, \right) \Big|_{\,\alpha\beta} \quad ; \quad h_{\,t} \, = \, h_{\,t}^{\,\dagger} \quad ; \quad \left[\, h_{\,r} \, , \, h_{\,s} \, \right] \, = \, i \, f_{\,rst} \, h_{\,t}$$
$$\alpha, \beta \, = \, 1, \cdots, dim \, \mathcal{D}$$

Eq. 16 serves to define $\longrightarrow$

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[a] A (partial) collection of historical and textbook references to the topics pertaining to 'Continous transformation groups and differential geometry' is presented aunder R-H R-H references and labelled by the symbols 1H , 2H $\cdots$.

matrix valued connections built from a basis of $Lie\,(\,\mathcal{D}\,)$ representation matrices as defined in eq. 16 for general irredcible representations $\mathcal{D}\,(\,\mathcal{G}\,)$ denoted $\mathcal{W}_{\,\mu}\,(\,z\,,\,\mathcal{D}\,)$

$$\mathcal{W}_{\,\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \;=\; W_{\,\mu}^{\,r}\,(\,z\,)\,(\,d_{\,r}\,)_{\,\alpha\beta\;Lie\,(\,\mathcal{D}\,)} \quad ; \quad \left[\begin{array}{l} r \;=\; 1,\cdots,dim\,(\,\mathcal{G}\,) \\ \alpha\,,\,\beta \;=\; 1,\cdots,dim\,(\,\mathcal{D}\,) \end{array}\right] \tag{18}$$

$$\mathcal{W}_{\,\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \quad\longrightarrow\quad \mathcal{W}_{\,\mu} \quad \textbf{for compact matrix notation}$$

In the following it is to be understood, that $\mathcal{W}_{\,\mu}$ is extended to a general collection of representations $\bigcup\,\mathcal{D}$ – thought to be carried by real and spurious spin $\frac{1}{2}$ fields – care beeing taken that asymptotic freedom in the ultraviolet is not upset.

From eq. 18 we define the associated matrix valued connection 1-form displayed alongside the base definition repeated from eq. 18 in eq. 19 below

$$\begin{array}{lcll} \mathcal{W}^{\,(1)}\,(\,z\,,\,\mathcal{D}\,)\Big|_{\,\alpha\beta} & = & d\,z^{\,\mu}\;\mathcal{W}_{\,\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} & \longrightarrow\quad \mathcal{W}^{\,(1)} \\ \mathcal{W}_{\,\mu}\,(\,z\,,\,\mathcal{D}\,)\,|_{\,\alpha\beta} & = & W_{\,\mu}^{\,r}\,(\,z\,)\,(\,d_{\,r}\,)_{\,\alpha\beta\;Lie\,(\,\mathcal{D}\,)} & \longrightarrow\quad \mathcal{W}_{\,\mu} \end{array} \tag{19}$$

and the matrix valued field-strength tensor

$$\mathcal{W}_{\,\mu\nu}\,(\,z,\,\mathcal{D}\,)\,|_{\,\alpha\beta} \;=\; \left\{\begin{array}{c} \partial_{\,\mu}\,\mathcal{W}_{\,\nu}\,(\,z,\,\mathcal{D}\,) \;-\; \partial_{\,\nu}\,\mathcal{W}_{\,\mu}\,(\,z,\,\mathcal{D}\,) + \\ +\,[\,\mathcal{W}_{\,\mu}\,(\,z,\,\mathcal{D}\,)\,,\,\mathcal{W}_{\,\nu}\,(\,z,\,\mathcal{D}\,)\,] \end{array}\right\}_{\alpha\beta} \;\longrightarrow\; \mathcal{W}_{\,\mu\nu} \tag{20}$$

⟶

together with their associated curvature 2-form

$$\mathcal{W}^{(2)}(z,\mathcal{D})\big|_{\alpha\beta} = \tfrac{1}{2}\, dz^{\mu} \wedge dz^{\mu}\, \mathcal{W}_{\mu\nu}(z,\mathcal{D})\big|_{\alpha\beta} \longrightarrow \mathcal{W}^{(2)}$$

$$\mathcal{W}_{\mu\nu}(z,\mathcal{D})\big|_{\alpha\beta} = \left\{ \begin{array}{c} \partial_{\mu}\mathcal{W}_{\nu}(z,\mathcal{D}) - \partial_{\nu}\mathcal{W}_{\mu}(z,\mathcal{D}) + \\ +\,[\mathcal{W}_{\mu}(z,\mathcal{D}),\, \mathcal{W}_{\nu}(z,\mathcal{D})] \end{array} \right\}_{\alpha\beta} \longrightarrow \mathcal{W}_{\mu\nu}$$

$$= W^{\,r}_{\mu\nu}(z)\,(d_r)_{\alpha\beta\; Lie(\mathcal{D})}$$

$$W^{\,r}_{\mu\nu} = \partial_{\mu} W^{\,r}_{\nu} - \partial_{\nu} W^{\,r}_{\mu} + f_{rst}\, W^{\,s}_{\mu} W^{\,t}_{\nu} \;;$$ **independent of** $\mathcal{D}$

(21)

Two remarks are in place here

1) In order to distinguish field strength from potentials (connections) the following equivalent but different notations for the field strength shall be used

(22) $$\mathcal{W}_{\mu\nu} \equiv \mathcal{B}_{\mu\nu} \;;\; \mathcal{W}^{(2)} \equiv \mathcal{B}^{(2)} \;;\; W^{\,r}_{\mu\nu} \equiv B^{\,r}_{\mu\nu}$$

2) From the last relation in eq. 21 it may appear redundant to extend connections and curvatures to matrix valued form with respect to a wide collection of irreducible representations $\mathcal{D}(\mathcal{G})$ **. This however is tantamount to neglecting nontrivial global regularity conditions in the infrared .**

⟶

We end this subsection (2-4) displaying the bilocal (parallel transport-) operators defined in eq. 13 using the shorthand notation in eq. 21

$$(U (x , C , y ; \mathcal{D}))_{\alpha\beta} \quad = \quad P \exp \left(- \int_{y}^{x} \Big|_{\mathcal{C}} \mathcal{W}^{(1)} (z , \mathcal{D}) \right) \Big|_{\alpha\beta}$$

(23) $\downarrow \qquad \downarrow$

$$U (x , C , y) \quad = \quad P \exp \left(- \int_{y}^{x} \Big|_{\mathcal{C}} \mathcal{W}^{(1)} \right) \quad [\textbf{for} \quad (\bigcup \mathcal{D})]$$

$\longrightarrow$

2-5 – The U1- or singlet axial current anomaly

The U1-axial central anomaly involves the local chiral current projections from $B^{\,q}_{[\,\dot{\mathcal{B}}\,\dot{f}_2\,,\,\mathcal{A}\,f_1\,]}(x)$ **in eq. 12**

$$\left(j^{\pm}_{\mu}\right)_{\dot{f}_2\,f_1}(x) \;=\; B^{\,q}_{[\,\dot{\mathcal{B}}\,\dot{f}_2\,,\,\mathcal{A}\,f_1\,]}(x)\,\left(\gamma_\mu\,\tfrac{1}{2}\,(\,\P\pm\gamma_5\,)\right)_{\mathcal{B}\,\dot{\mathcal{A}}}$$

$$(24) \qquad = \;(:)\,\overline{q}^{\,\dot{c}}_{\dot{f}_2}\,\gamma^{\pm}_{\mu}\,q^{\,c}_{f_1}(x)\,(:)$$

$$\gamma_5 = \gamma_{5\,R} = \tfrac{1}{i}\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 \;\;;\;\; \gamma^{\pm}_{\mu} = \gamma_\mu\,\tfrac{1}{2}\,(\,\P\pm\gamma_5\,)$$

The equations of motion for the fermion fields are *and superficially imply* **(upon** $f_1 \leftrightarrow f_2$ **)**

$$\begin{aligned} \not{\partial}\,q^{\,c}_{f_2} &= \tfrac{1}{i}\left(\not{v}^{\,c\,\dot{c}'} + \delta^{\,c\,\dot{c}'}\,m_{f_2}\right)q^{\,c'}_{f_2} \\ \overline{q}^{\,\dot{c}}_{\dot{f}_1}\,\overleftarrow{\not{\partial}} &= \overline{q}^{\,\dot{c}'}_{\dot{f}_1}\,\tfrac{1}{i}\left(-\not{v}^{\,c'\,\dot{c}} - \delta^{\,c'\,\dot{c}}\,m_{f_1}\right) \end{aligned} \quad ; \;\; \text{no sums over } \dot{f}_1\,,\,f_2 \;\rightarrow$$

$$\partial^{\,\mu}\left(j^{\pm}_{\mu}\right)_{\dot{f}_1\,f_2} = \tfrac{1}{2\,i}\left(\left(m_{f_2}-m_{f_1}\right)S_{\dot{f}_1\,f_2} \mp \left(m_{f_2}+m_{f_1}\right)P_{\dot{f}_1\,f_2}\right)$$

$$S_{\dot{f}_1\,f_2} = (:)\,\overline{q}^{\,\dot{c}}_{\dot{f}_1}\,q^{\,c}_{f_2}\,(:) \;\;,\;\; P_{\dot{f}_1\,f_2} = (:)\,\overline{q}^{\,\dot{c}}_{\dot{f}_1}\,\gamma_5\;q^{\,c}_{f_2}\,(:)$$

(25)

In eq. 25 m_f **denotes the** *real* , *nonnegative* **quark mass for flavor f.** $\longrightarrow$

From eq. 25 the relations for vector and axial vector currents *superficially* follow

(26)
$$
\begin{aligned}
(j_{\mu})_{\dot{f}_1 f_2} &= \left(j_{\mu}^{+}\right)_{\dot{f}_1 f_2} + \left(j_{\mu}^{-}\right)_{\dot{f}_1 f_2} \\
(j_{\mu}^{5})_{\dot{f}_1 f_2} &= \left(j_{\mu}^{+}\right)_{\dot{f}_1 f_2} - \left(j_{\mu}^{-}\right)_{\dot{f}_1 f_2} \\
\partial^{\mu}(j_{\mu})_{\dot{f}_1 f_2} &= \tfrac{1}{i}\left(m_{f_2} - m_{f_1}\right) S_{\dot{f}_1 f_2} \\
\partial^{\mu}(j_{\mu}^{5})_{\dot{f}_1 f_2} &= \left(m_{f_2} + m_{f_1}\right) i\, P_{\dot{f}_1 f_2}
\end{aligned}
$$

As it follows from the original derivation by Adler and Bell and Jackiw [10-1969] in QED, the vecor current Ward identities in eq. 26 can be implemented also in QCD , leaving the axial current ones reduced to the flavor non-singlet case , leaving the U1 axial current divergent anomalous

$$
\partial^{\mu}(j_{\mu})_{\dot{f}_1 f_2} = \tfrac{1}{i}\left(m_{f_2} - m_{f_1}\right) S_{\dot{f}_1 f_2} \quad \surd
$$

(27)
$$
\left\{ \begin{array}{c} j_{\mu}^{5} \\ P \end{array} \right\}^{NS}_{\dot{f}_1 f_2} = \left\{ \begin{array}{c} j_{\mu}^{5} \\ P \end{array} \right\}_{\dot{f}_1 f_2} - \frac{1}{N_{fl}} \delta_{\dot{f}_1 f_2} \sum_f \left\{ \begin{array}{c} j_{\mu}^{5} \\ P \end{array} \right\}_{\dot{f} f}
$$

$\longrightarrow$

and similarly

(28)
$$\left\{ \begin{array}{c} j^5_\mu \\ P \end{array} \right\}^S_{\dot{f}_1 f_2} = \sum_f \left\{ \begin{array}{c} j^5_\mu \\ P \end{array} \right\}_{\dot{f} f}$$

2-6 – Quark masses and splittings : m_f and $\Delta m_f = m_f - \langle m \rangle$

In the subtitle above $\langle m \rangle$ stands for the mean quark mass

(29)
$$\langle m \rangle = \frac{1}{N_{fl}} \sum_f m_f$$

The identities for vector currents in eqs. 26 and 27 can be extended separating the conributions proportional to Δm_f and $\langle m \rangle$

(30)
$$\begin{aligned} \partial^\mu \left(j_\mu \right)_{\dot{f}_1 f_2} &= \frac{1}{i} \left(\Delta m_{f_2} - \Delta m_{f_1} \right) S_{\dot{f}_1 f_2} \quad \surd \\ \partial^\mu \left(j^5_\mu \right)^{NS}_{\dot{f}_1 f_2} &= \left(\Delta m_{f_2} + \Delta m_{f_1} \right) i P^{NS}_{\dot{f}_1 f_2} \quad \surd \\ \partial^\mu \left(j^5_\mu \right)^S_{\dot{f}_1 f_2} &= 2 \langle m \rangle \, i P^S \quad \not\surd \; [\longrightarrow + \delta_5] \end{aligned}$$

[a] $$\delta_5 = \left(2 N_{fl} \right) \frac{1}{32\pi^2} B^r_{\mu\nu} \widetilde{B}^{\mu\nu r} \Big|_{\rightarrow ren.gr.inv} \; ; \; \widetilde{B}^r_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} B^{\sigma\tau r}$$ $\longrightarrow$

---

[a] δ_5 was – as far as I know – introduced by Murray Gell-Mann in lectures $\sim$ 1970 in Hawaii .

We shall return to the question of how the local operator $ch_2 \, (\, B \,) \, \equiv \, \frac{1}{32 \pi^{\, 2}} \, (:) \, B^{\, r}_{\, \mu \, \nu} \, \widetilde{B}^{\, \mu \, \nu \, r} \, (:)$ **is to be normalized and rendered renormalization group invariant [11-1991] . Here we just assume this to have been achieved and denote the U1-axial anomaly, the first of the central two, in its general form (eq. 30)**

(31)
$$\begin{aligned} & \left\{ \, \partial^{\, \mu} \, \left(j^{\, 5}_{\, \mu} \right)^{\, S} \, = \, 2 \, \langle \, m \, \rangle \, i \, P^{\, S} \, + \, \delta_{\, 5} \, \right\} \, (\, x \,) \\ & \delta_{\, 5} \, = \, \left(\, 2 \, N_{\, fl} \, \right) \, \frac{1}{32 \pi^{\, 2}} \, (:) \, B^{\, r}_{\, \mu \, \nu} \, \widetilde{B}^{\, \mu \, \nu \, r} \, (:) \, \Big|_{\, \rightarrow ren.gr.inv} \end{aligned}$$

From here it is conceptually clear how the scale- (or trace-) anomaly arises but strictly within QCD . The renormalizability of a field theory in the limit of uncurved space-time gives rise to a local , symmetric and *conserved* **energy momentum tensor , implying exact Poincaré invariance**

(32)
$$\begin{aligned} & \{ \, \vartheta_{\, \mu \, \nu} \, = \, \vartheta_{\, \nu \, \mu} \, \} \, (\, x \,) \\ & \partial^{\, \nu} \, \vartheta_{\, \mu \, \nu} \, = \, 0 \end{aligned}$$

In connection with the normal ordering questions it is important to admit in the precise form of the energy momentum tensor a nontrivial vacuum expected value , which $\longrightarrow$

in view of exact Poincaré invariance must be of the form

$$\langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle = \frac{1}{4} \eta_{\mu\nu} \tau$$

$$\left\{ \begin{array}{c} \eta_{\mu\nu} = diag(1, -1, -1, -1) \\ \tau \end{array} \right\} \text{ independent of } x \longrightarrow$$

(33)

$$\Delta \vartheta_{\mu\nu}(x) = \vartheta_{\mu\nu}(x) - \langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle \times \left\{ \begin{array}{l} \widehat{\P} \\ \text{or } | \Omega \rangle \langle \Omega | \end{array} \right.$$

with $\partial^{\nu} \Delta \vartheta_{\mu\nu}(x) = 0 \; ; \; \langle \Omega | \Delta \vartheta_{\mu\nu}(x) | \Omega \rangle = 0$

In eq. 33 $\widehat{\P}$ denotes the unit operator in the entire Hilbert space of states , while $P_{\Omega} = | \Omega \rangle \langle \Omega |$ stands for the projector on the ground state .

Furthermore from the two local , *conserved* tensors in eq. 33 only $\Delta \vartheta_{\mu\nu}(x)$ with vanishing vacuum expected value is acceptable as representing the conserved 4 momentum *operators* in the integral form

(34)
$$\widehat{P}_{\mu} = \int_t d^3 x \, \Delta \vartheta_{\mu 0}(t, \vec{x})$$

All these arguments ***notwithstanding*** **to subtract any eventual vacuum expected values of local operators , often put forward as mathematical prerequisites , it is wise** ***not to do so*** **in the presence of spontaneous parameters , the dynamical origin of spontaneous symmetry breaking, e.g. chiral symmetries in the limit or neighbourhood of some** $m_f \rightarrow 0$ **.**

Using the (classical) equations of motion pertaining to the Lagrangean in eqs. 2 - 7

$$(D_\nu B^{\mu\nu})^r = j^{\mu\, r}(\overline{q}, q) \; ; \quad B \rightarrow B_{pert}$$

$$(D_\varrho B^{\mu\,\nu})^r = \partial_\varrho B^{\mu\,\nu\,r} + f_{rst} W^s_\varrho B^{\mu\,\nu\,t}$$

$$j^r_\mu(\overline{q}, q) = g\, \overline{q}^{\,\dot{c}}_{\dot{\mathcal{A}}\,\dot{f}} (\gamma_\mu)_{\dot{\mathcal{A}}\,\mathcal{B}} \tfrac{1}{2} (\lambda^r)_{c\dot{c}'}\, q^{c'}_{\mathcal{A}\,f}$$

(35)

$$i\,(\gamma^\mu D_\mu q)^c_{\mathcal{A}\,f} = m_f\, q^c_{\mathcal{A}\,f} \quad \textbf{and} \quad q \rightarrow \overline{q}$$

$$(D_\mu q)^c_{\mathcal{A}\,f} = \left[\partial_\mu \delta_{c\dot{c}'} + \tfrac{1}{i} W^t_\mu \tfrac{1}{2} \left(\lambda^t \right)_{c\dot{c}'} \right] q^{c'}_{\mathcal{A}\,f}$$

$$W^r_\mu \equiv -v^r_\mu = g \left(W^r_\mu \right)_{pert} \equiv -g \left(v^r_\mu \right)_{pert}$$

the associated form of the energy momentum becomes $\rightarrow$

(36)
$$\vartheta^{(cl)}_{\mu\,\nu} = \left[\begin{array}{c} \frac{1}{4\,g^{2}} \left[\; B^{\,t}_{\mu\,\varrho} B^{\varrho\,t}_{\;\;\nu} - \frac{1}{4} \eta_{\mu\,\nu} B^{\,t}_{\sigma\,\varrho} B^{\varrho\,\sigma\,t} \;\right] + \\ + \frac{1}{2} \left[\overline{q}_{\dot{f}} \gamma_{\mu} \frac{i}{2} \overleftrightarrow{D}_{\nu}\; q_{f} + \mu \leftrightarrow \nu \right] \end{array} \right]$$

and using once more the fermion part of the equations of motion the trace of the classical energy momentum tensor becomes

(37)
$$\begin{aligned} \vartheta^{\mu\,(cl)}_{\;\;\mu} &= \sum_{f} m_{f}\, S_{\dot{f}\,f} \\ S_{\dot{f}_1\,f_2} &= (:)\, \overline{q}^{\,\dot{c}}_{\dot{f}_1}\, q^{\,c}_{f_2}\, (:) \end{aligned}$$

2-7 – The scale- or trace- anomaly

From the classical soft fermionic contribution to the trace of the energy momentum tensor there is a clear conjecture , also by Murray Gell-Mann , of the anomalous contribution , which subsequently became the scale- or trace- anomaly within QCD

(38)
$$\begin{aligned} \vartheta^{\mu}_{\;\;\mu} &= \sum_{f} m_{f}\, S_{\dot{f}\,f} + \delta_{0} \\ \delta_{0} &= -\left(-2\,\beta(g)\,/\,g^{3} \right) \left[\frac{1}{4} (:)\, B^{\,t}_{\mu\,\nu} B^{\mu\,\nu\,t} (:) \right]_{\rightarrow\, ren.gr.inv} \end{aligned}$$

$\rightarrow$

2-8 – The two central anomalies alongside : scale- or trace- and U1-axial anomaly

We collect the two anomalous identities in eqs. 38 and 20

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_f m_f \, S_{\dot{f}\, f} + \delta_0 \right\} (x)$$

$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \, \langle m \rangle \, i \, P^{S} + \delta_5 \right\} (x)$$

(39)

$$\delta_0 = - \left(-2 \, \beta (g) / g^3 \right) \left[\tfrac{1}{4} \, (:) \, B^{t}_{\mu \nu} \, B^{\mu \nu \, t} \, (:) \right]_{\rightarrow ren.gr.inv}$$

$$\delta_5 = \left(2 \, N_{fl} \right) \frac{1}{8 \pi^2} \left[\tfrac{1}{4} \, (:) \, B^{t}_{\mu \nu} \, \widetilde{B}^{\mu \nu \, t} \, (:) \right]_{\rightarrow ren.gr.inv}$$

$$- \beta / g^3 = \frac{1}{16 \pi^2} \, b_0 + O(X) \; ; \; X = g^2 / (16 \, \pi^2$$

$\beta(g)$: **Callan-Symanzik rescaling function in QCD**

The qualification 'central' for the anomalies in eq. 39 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson *renormalized* **Lagrangean density x dependent**

(40)

$$\mathcal{L}_{g.b.} = - \frac{1}{g^2} \, \frac{1}{4} \, (:) \, B^{t}_{\mu \nu} \, B^{\mu \nu \, t} \, (:) + \vartheta \, \frac{1}{8 \pi^2} \, \frac{1}{4} \, (:) \, B^{t}_{\mu \nu} \, \widetilde{B}^{\mu \nu \, t} \longrightarrow$$

$$g^2 \rightarrow g^2 (x) \; ; \; \vartheta \rightarrow \vartheta (x)$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions $\rightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(41) $$\frac{1}{4} \, (:) \, B^{\,t}_{\mu\,\nu} \, B^{\,\mu\,\nu\,t} \, (:) \quad , \quad \frac{1}{4} \, (:) \, B^{\,t}_{\mu\,\nu} \, \widetilde{B}^{\,\mu\,\nu\,t}$$

We will use the following definitions relative to the rescaling function β

$$- \beta \, / \, g \; = \; X \, B \, (\, X \,) \quad ; \quad B \, (\, X \,) \; = \; b_{\,0} \, A \, (\, X \,)$$

$$B \, (\, X \,) \; \sim \; \sum_{\,n=0}^{\,\infty} \, b_{\,n} \, X^{\,n} \quad , \quad A \, (\, X \,) \; \sim \; \sum_{\,n=0}^{\,\infty} \, a_{\,n} \, X^{\,n}$$

$$\kappa \; = \; g^{\,2} \, / \, (\, 16 \, \pi^{\,2} \,) \quad \textbf{generic} \quad \longrightarrow \quad X \, , \, Y$$

(42) $$b_{\,0} \; = \; \frac{1}{3} \, \left(\, 33 \, - \, 2 \, N_{\,fl} \, \right) \quad , \quad a_{\,0} \; = \; 1 \quad , \quad a_{\,n} \; = \; b_{\,n} \, / \, b_{\,0}$$

$$b_{\,1} \; = \; \frac{2}{3} \, \left(\, 153 \, - \, 19 \, N_{\,fl} \, \right)$$

$$b_{\,2} \; = \; \frac{1}{54} \, \left(\, 77139 \, - \, 15099 \, N_{\,fl} \, + \, 325 \, N_{\,fl}^{\,2} \, \right)$$

$$b_{\,3} \; \sim \; 29243 \, - \, 6946.3 \, N_{\,fl} \, + \, 405.089 \, N_{\,fl}^{\,2} \, + \, 1.49931 \, N_{\,fl}^{\,3}$$

References in conjunction with this section (2 - premises) are presented in five (partial) collections :

1 : (R) directly related to the two central anomalies

2 : (rBsquare) establishing the one renormalization group invariant quantity of dimension $\left[\, M^{\,4} \, \right]$

3 : (r-sp-1) a recent paper by Guido Altarelli and references cited therein

4 : (r-A2x) a selection of papers and textbooks for the entire realm of QCD

5 : (r-condx) : Condensation phenomena and field theory realizations

$\longrightarrow$

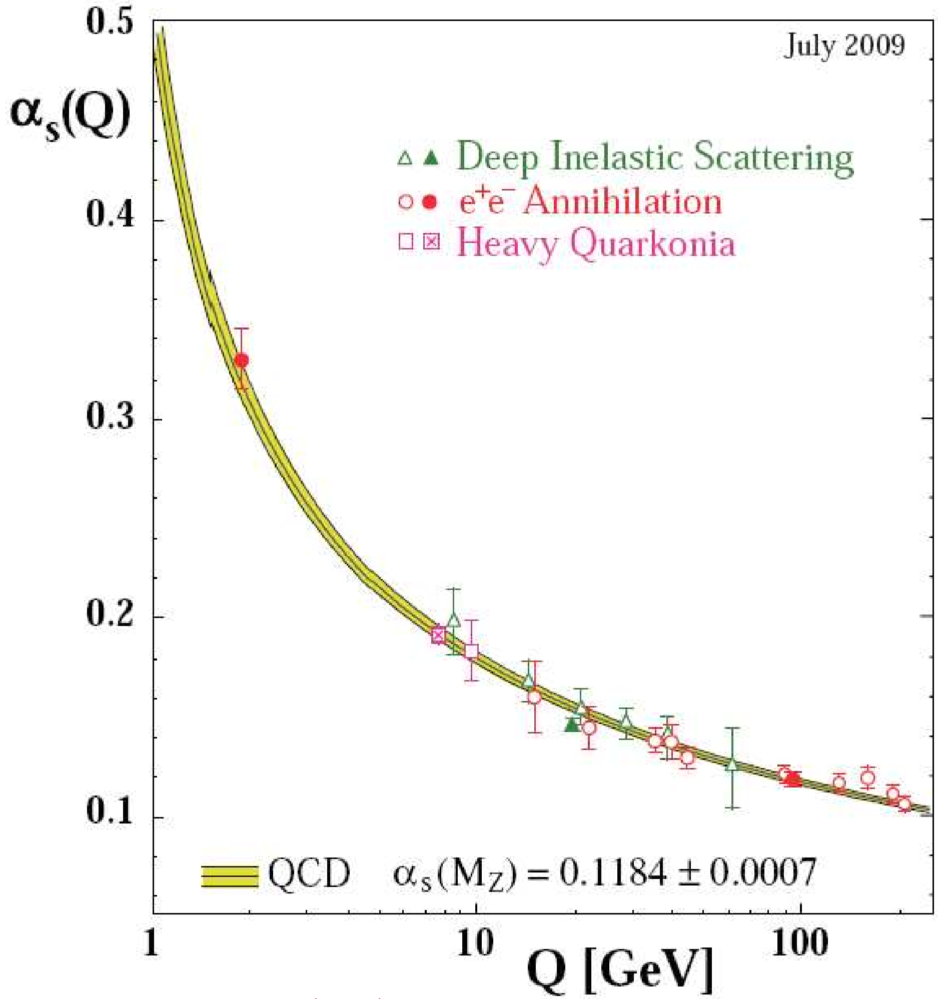

Fig. A21 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from ref. [12-2009] .**

This ends section 2 – premises

$\longrightarrow$

3-1 – Perturbative renormalization rescaling

We begin with the renormalization of external operaters and gauge boson fields, denoting unrenormalized fields and operators by the suffix $^{(0)}$

(43)
$$J_\alpha = (Z_J)^{-1} J_\alpha^{(0)} \quad , \quad \mathcal{O} = (Z_\mathcal{O})^{-1} \mathcal{O}^{(0)}$$
$$g = (Z_3)^{3/2} (Z_1)^{-1} g^{(0)} \quad , \quad \eta = (Z_3)^{-1} \eta^{(0)}$$
$$W^r_{\mu\,pert} = (Z_3)^{-\frac{1}{2}} \left(W^r_{\mu\,pert} \right)^{(0)}$$

and continue with the ghost fields c^r , $\overline{c}^s$ **as defined in** $\Delta\mathcal{L}$ **in eq. 7**

(44)
$$c^r = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (c^r)^{(0)} \quad , \quad \overline{c}^r = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (\overline{c}^r)^{(0)}$$
$$g = (Z_3)^{1/2} \widetilde{Z}_2 \left(\widetilde{Z}_1 \right)^{-1} g^{(0)}$$

The renormalization constant $Z_1 = Z_{1\,(3W)}$ **refers to the 3 gauge boson vertex of the derivative type** $W\,W\,\partial\,W$ **as shown in figure 1, while** $\widetilde{Z}_1 = \widetilde{Z}_1\,(\overline{c}\,c\,W)$ **refers to the ghost-W vertex, shown in figure 2. The 4 gauge boson vertex involves the analogous renormalization constant** $Z_{1(4W)}$ **and the** $\overline{q}\,q\,W$ **vertex** $Z_{1(\overline{q}qW)}$ **, shown in figures 3 and 4 respectively below .** $\longrightarrow$

4 irreducible vertex diagrams

W-3 : (i)

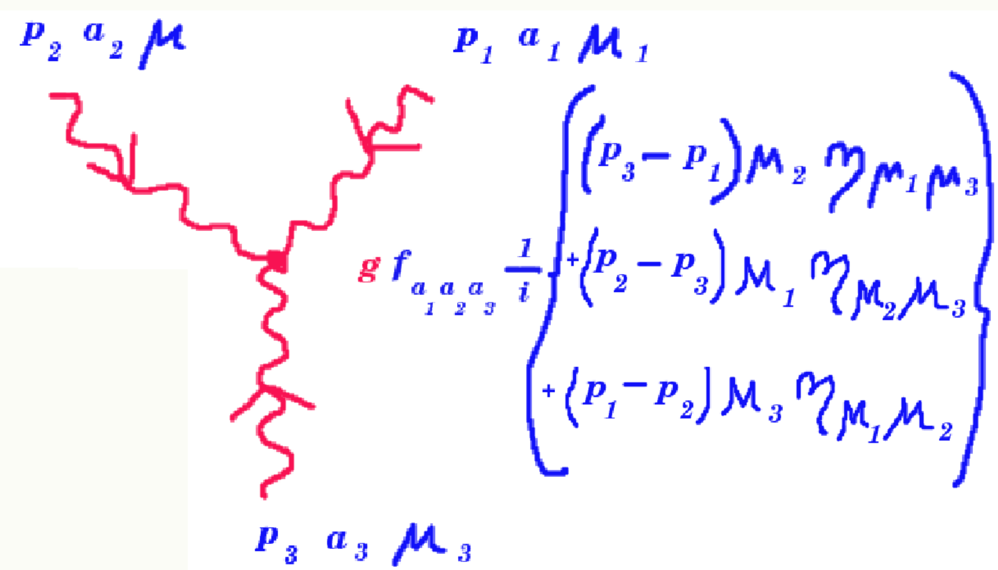

Fig. 1 : $Z_1 \leftrightarrow$ W-3 vertex of the type $W\,W\,\partial\,W$

$\longrightarrow$

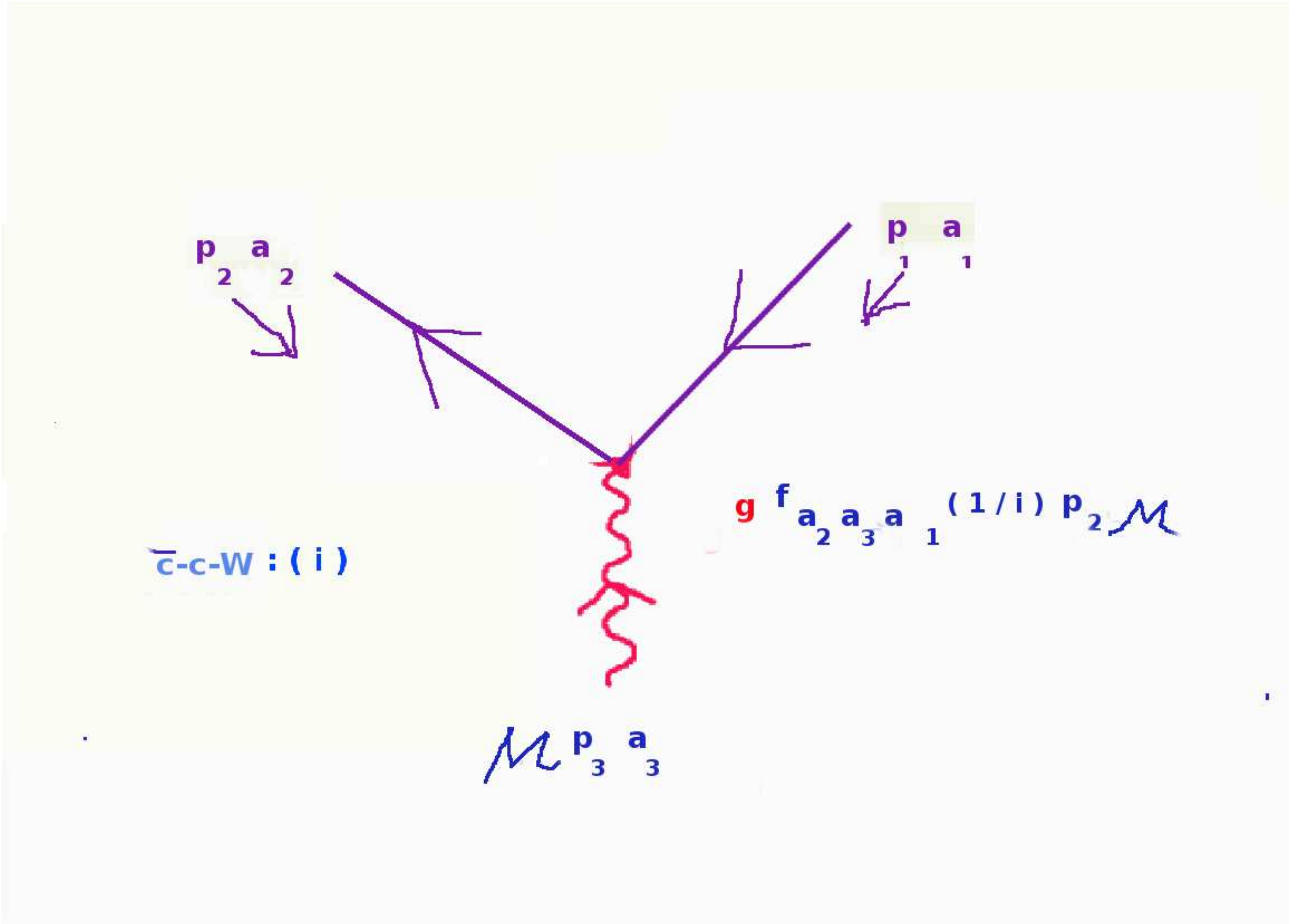

Fig. 2 : $\widetilde{Z}_1 \leftrightarrow$ cbar-c W-3 vertex of the type $\partial\, \overline{c}\, c\, W$

$\longrightarrow$

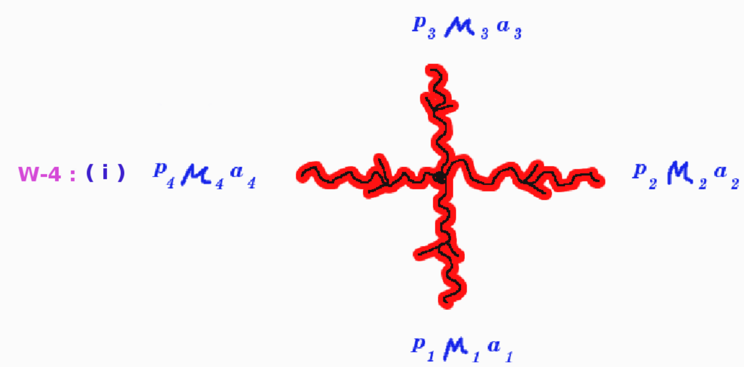

Fig. 3 : $Z_{1(4W)} \leftrightarrow$ **W-4 vertex for gauge fields**

$$g^2 \left(\begin{array}{r} r_{[a_1 a_2][a_3 a_4]} \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \end{array} \right] \\ + r_{[a_1 a_3][a_2 a_4]} \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \\ + r_{[a_1 a_4][a_2 a_3]} \left[\begin{array}{r} \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \end{array} \right)^{\to r}$$

$$r_{[a_1 b_1][a_2 b_2]} = f_{d a_1 b_1} f_{d a_2 2_2}$$

$\longrightarrow$

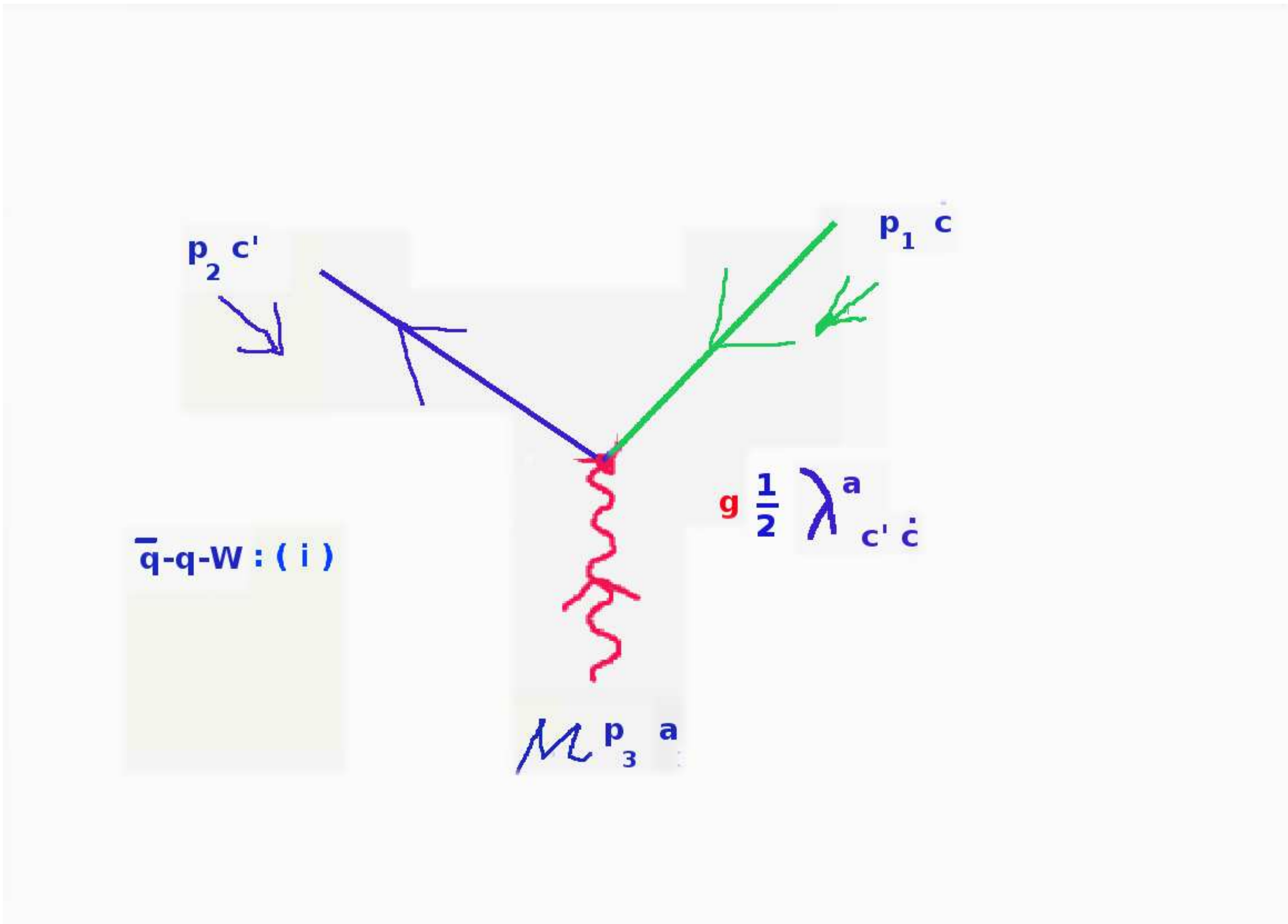

Fig. 4 : $Z_{1(\overline{q}qW)}$ $\leftrightarrow$ W-3 vertex for $\overline{q}\,q$ to gauge fields

$\longrightarrow$

3-2 – The complete coupling constant renormalization equations

The minimal renormalization constants defined in eqs. 43 and 44, referring beyond the vertices shown in figures 1-4 to the two point functions for gauge connections , Z_3 , and ghost fields , $\widetilde{Z}_2$, thus need extension (also) to q , $\overline{q}$ fields and their two point function, Z_2 , as shown below

$$g = (Z_3)^{1/2} (Z_3 / Z_1) \, g^{(0)} \quad ; \; W^{\,r}_{\mu \, pert} = (Z_3)^{-\frac{1}{2}} \left(W^{\,r}_{\mu \, pert} \right)^{(0)}$$

$$g = (Z_3)^{1/2} \left(\widetilde{Z}_2 / \widetilde{Z}_1 \right) g^{(0)} \quad ; \; (c, \overline{c})^{r} = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} ((c, \overline{c})^{r})^{(0)}$$

$$g^2 = Z_3 \left(Z_3 / Z_{1(4W)} \right) \left(g^{(0)} \right)^2$$

$$g = (Z_3)^{1/2} \left(Z_2 / Z_{1(\overline{q}qW)} \right) g^{(0)} \; ; \quad q^{\,c}_{\mathcal{S}\, f} = (Z_2)^{-\frac{1}{2}} \left(q^{\,c}_{\mathcal{S}\, f} \right)^{(0)}$$

(45) $$\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\, \dot{f}} = (Z_2)^{-\frac{1}{2}} \left(\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\, \dot{f}} \right)^{(0)}$$

Maintaining gauge invariance through renormalization thus implies the relations

(46) $$Z_3 / Z_1 = \widetilde{Z}_2 / \widetilde{Z}_1 = Z_2 / Z_{1(\overline{q}qW)} \; ; \; Z_3 / Z_{1(4W)} = (Z_3 / Z_1)^2$$

It remains to include quark mass renormalization and associated rescaling. $\longrightarrow$

3-3 – Renormalization equations for quark masses m_f

We perform quark mass renormalization 'at zero mass' [A218-1973] , with the following relations , using the q , $\overline{q}$ fields and their field renormalization constant, Z_2 , as defined in eqs. 45 and 46, through the extension

(47) $$(\mathcal{L}_m)^{(0)} = -\sum_f Z_2 Z_m m_f \;\; ; \;\; m_f = (Z_m)^{-1} m_f^{(0)}$$

The coefficients of the quark mass rescaling functions are known to four loops as calculated by Chetyrkin and independently by others [13-1997] . They are given in eq. 106 in Appendix 1.

Here we subsum the complete set of renormalized rescaling functions, which follows from the renormalization constants defined in eqs. 44 - 46

$$\left(\begin{array}{l} \mu^2 \partial_{\mu^2} + (\beta(g)/g)\, g^2 \partial_{g^2} + \\ + \gamma_m m_f \partial_{m_f} - 2\gamma_3 (\eta \partial_\eta) - \gamma_{J\mathcal{O}} \end{array} \right) C_{J\mathcal{O}}^{T\,(\Pi)} (z; \mu, g, m_\beta, \eta) = 0$$

(48) $$\left\{ \begin{array}{l} \beta(g) = -g\, b(g^2) \\ \gamma_m(g^2) \equiv -\chi_m \\ \gamma_{J\mathcal{O}}(g^2, (\eta)) \\ \gamma_3(g^2, \eta) \end{array} \right\} = \mu^2 d/d\mu^2 \left\{ \begin{array}{c} 2 \log \left((Z_3)^{3/2} (Z_1)^{-1} \right) \\ \log \left((Z_m)^{-1} \right) \\ \log \left(Z_{\mathcal{O}} / Z_J^2 \right) \\ \log \left((Z_3)^{1/2} \right) \end{array} \right\}$$

$\longrightarrow$

We can always return to the standard form of the renormalization group equation

(49) $$\left(\begin{array}{l} \mu \, \partial_{\mu} + \beta(g) \, \partial_g + (2\gamma_m) \, m_f \, \partial_{m_f} \\ -2(2\gamma_3)(\eta \, \partial_{\eta}) - (2\gamma_{J\mathcal{O}}) \end{array} \right) C^{T\,(\Pi)}_{J\,\mathcal{O}} \left(z \, ; \, \mu \, , \, g \, , \, m_{\beta} \, , \, \eta \right) = 0$$

without changing the definition of β, transforming the lower relation in eq. 48 , which amounts to double the coefficients of the quantities $\gamma_m \, , \, \gamma_3 \, , \, \gamma_{J\mathcal{O}}$, as displayed in eq. 49

(50) $$\left\{ \begin{array}{l} \beta(g) = -g \, b(g^2) \\ 2\gamma_m(g^2) \equiv -2\chi_m \\ 2\gamma_{J\mathcal{O}}(g^2, (\eta)) \\ 2\gamma_3(g^2, \eta) \end{array} \right\} = \mu \, d / d\mu \left\{ \begin{array}{c} \log \left((Z_3)^{3/2} (Z_1)^{-1} \right) \\ \log \left((Z_m)^{-1} \right) \\ \log \left(Z_{\mathcal{O}} / Z_J^2 \right) \\ \log \left((Z_3)^{1/2} \right) \end{array} \right\}$$

$\longrightarrow$

rescaling equations in the $\overline{\text{MS}}$ scheme for coupling constant g and quark masses m_f

$\beta(g)$ and $\gamma_m(\kappa)$, $\kappa = g^2/(16\pi^2)$

We turn to the scale (-square) evolution of the rationalized coupling constant and quark masses, introducing the variables

(51)
$$\tau = \log\left(\mu^2/\mu_0^2\right) \;\; ; \;\; \kappa = g^2/\left(16\pi^2\right) \;\; \rightarrow$$
$$\left\{ \widehat{\beta} = (\beta/g)\,\kappa = \widehat{\beta}(\kappa)\,,\; \gamma_m(\kappa) \right\}$$

The renormalization group equation in the form given in eq. 48 then becomes

(52)
$$\begin{pmatrix} \partial_\tau + \widehat{\beta}\,\partial_\kappa + \gamma_m\, m_f\,\partial_{m_f} \\ -2\,\gamma_3\,(\eta\,\partial_\eta) - \gamma_{J\,\mathcal{O}} \end{pmatrix} C_{J\,\mathcal{O}}^{\,T\,(\Pi)}\left(z\,;\mu\,,g\,,m_\beta\,,\eta\right) = 0$$

The partial derivative terms in brackets in eq. 52 determine the sliding scale equations with respect to the quantities $\tau\,,\overline{\kappa}\,,\,\overline{m}_f\,,\overline{\eta}$ in the sense of its initial value structure. We restrict the discussion to $\tau\,,\overline{\kappa}(\tau)\,,\,\overline{m}_f(\tau)$ here for simplicity .

(53)
$$\dot{\overline{\kappa}} = \widehat{\beta}(\overline{\kappa})\,,\; \dot{\overline{m}}_f = \gamma_m(\overline{\kappa})\,\overline{m}_f \;\; ; \;\; \bullet = d/d\tau$$
$$\overline{\kappa}(\tau = 0) = \kappa\,,\; \overline{m}_f(\tau = 0) = m_f$$

$\longrightarrow$

sliding coupling constant : $\overset{\bullet}{\overline{\kappa}} = \widehat{\beta}\,(\,\overline{\kappa}\,)$

We associate integration variables , initial values and endpoint variables in the following way

(54) $$\theta \leftrightarrow [\,\tau\,,\,0\,]\ ,\ \lambda \leftrightarrow [\,\overline{\kappa}\,,\,\kappa\,]$$

The integration of the rescaling differential equation becomes

(55) $$\int_0^{\tau} d\,\theta\ =\ \tau\ =\ \int_{\kappa}^{\overline{\kappa}} d\,\lambda\,\left(\widehat{\beta}\,(\,\lambda\,)\right)^{-1}\ \longrightarrow\ \tau\ =\ \int_{\overline{\kappa}}^{\kappa} d\,\lambda\,\left(-\,\widehat{\beta}\,(\,\lambda\,)\right)^{-1}$$

The first two coefficients in the expansion of $\widehat{\beta}$ in powers of λ necessitate a twofold subtraction for $\lambda \downarrow 0$ for the integral on the right hand side of eq. 55 to converge at the lower limit of integration

(56) $$\widehat{\beta}\,(\,\lambda\,)\ =\ \sum_{n=0}^{\infty}\widehat{\beta}_{\,n}\,\lambda^{\,n+2}\ \ ;\ \ \widehat{\beta}_{\,n}\ \equiv\ -\,b_{\,n}$$

The first four coefficients $b_{\,n}\ ;\ n\ =\ 0, \cdots, 3$ are known , (ref. [14-1997]) and given in eq. 105 in Appendix 1 .

We perform this subtraction splitting $\widehat{\beta}$

(57) $$\widehat{\beta}\ =\ \widehat{\beta}_{\,(2)}\ +\ \Delta_{\,(2)}\,\widehat{\beta}\ \ ;\ \ \left[\begin{array}{l}\widehat{\beta}_{\,(2)}\ =\ -\ \left(\,b_{\,0}\lambda^{\,2}\ +\ b_{\,1}\,\lambda^{\,3}\,\right)\\ \Delta_{\,(2)}\,\widehat{\beta}\ =\ -\ \sum_{n=2}^{\infty} b_{\,n}\,\lambda^{\,n+2}\end{array}\right.$$

The subtraction, displayed in eq. 57 $\longrightarrow$

gives rise to the substitutions

$$\zeta_{(2)} = \frac{\Delta_{(2)} \widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 b_0^{-1} \lambda^2 (1 + O(\lambda))$$

(58)
$$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \Big/ \left(1 + \zeta_{(2)}\right) = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

$$\psi_{(2)} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 (b_0)^{-2} (1 + O(\lambda))$$

ψ_2 as defined in eq. 58 has a regular power series expansion in λ, fully determined by $\widehat{\beta}$ and hence the sought subtractions involve only $\widehat{\beta}_{(2)}$, i.e. the beta function truncated to the first two terms.

We proceed in the reduction of $\left(-\widehat{\beta}\right)^{-1}$ using the substitutions in eq. 58

$$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

(59)
$$\begin{aligned}\left(-\widehat{\beta}_{(2)}\right)^{-1} &= b_0^{-1} \lambda^{-2} \left(1 + b_1^{(0)} \lambda\right)^{-1}; \quad b_1^{(0)} = b_1 / b_0 \\ &= b_0^{-1} \lambda^{-2} \left(1 - b_1^{(0)} \lambda\right) \left(1 - \left(b_1^{(0)} \lambda\right)^2\right)^{-1}\end{aligned}$$

$\longrightarrow$

The last expression in eq. 59 decomposes into

(60)
$$\left(- \widehat{\beta}_{(2)} \right)^{-1} = b_0^{-1} \lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} + \phi_{(2)}$$
$$\phi_{(2)} = b_0^{-1} \left(b_1^{(0)} \right)^2 \left(1 - \left(b_1^{(0)} \lambda \right)^2 \right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$

$\phi_{(2)}$ defined in eq. 60 , as ψ_2 defined in eq. 58 , has a regular power series expansion in λ .

Thus we rearrange the expressions in eq. 59

$$\left(- \widehat{\beta} \right)^{-1} = \left(- \widehat{\beta}_{sing.} \right)^{-1} + \left(- \widehat{\beta}_{reg.} \right)^{-1} \; ; \; \widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)} \widehat{\beta}$$
$$\left(- \widehat{\beta}_{sing.} \right)^{-1} = b_0^{-1} \lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} \; , \; \left(- \widehat{\beta}_{reg.} \right)^{-1} = \phi_{(2)} - \psi_{(2)}$$

---

(61)
$$\phi_{(2)} = b_0^{-1} \left(b_1^{(0)} \right)^2 \left(1 - \left(b_1^{(0)} \lambda \right)^2 \right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$
$$\psi_{(2)} = \left(- \widehat{\beta}_{(2)} \right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 \, (b_0)^{-2} \, (1 + O (\lambda))$$
$$\zeta_{(2)} = \frac{\Delta_{(2)} \widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 b_0^{-1} \lambda^2 \, (1 + O (\lambda))$$

→

We recall that in the decomposition $\widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)} \widehat{\beta}$, retained in eq. 61 , $\widehat{\beta}_{(2)}$ and $\Delta_{(2)} \widehat{\beta}$ denote the beta function truncated to the first wo terms and the remainder term respectively . It further follows from eq. 61

(62) $$\left(- \widehat{\beta}_{reg.} \right)^{-1} = b_0^{-1} \left((b_1 / b_0)^2 - b_2 / b_0 \right) (1 + O(\lambda))$$

We return to eq. 55 and integrate the singular part $\left(- \widehat{\beta}_{sing.} \right)^{-1}$

$$\begin{aligned} \tau &= \int_{\overline{\kappa}}^{\kappa} d\lambda \left(- \widehat{\beta}(\lambda) \right)^{-1} \\ &= \int_{\overline{\kappa}}^{\kappa} d\lambda \left\{ \left(- \widehat{\beta}_{reg.}(\lambda) \right)^{-1} + \left(- \widehat{\beta}_{sing.}(\lambda) \right)^{-1} \right\} \end{aligned}$$

$$\int_{\overline{\kappa}}^{\kappa} d\lambda \left(- \widehat{\beta}_{sing.}(\lambda) \right)^{-1} = b_0^{-1} \left[\begin{array}{c} \overline{\kappa}^{-1} - \kappa^{-1} \\ - b_0^{-1} b_1 \left(\log \left(\overline{\kappa}^{-1} \right) - \log \left(\kappa^{-1} \right) \right) \end{array} \right]$$

(63)

Eq. 55 thus allows the separation of variables $\overline{\kappa}$ and κ $\longrightarrow$

(64)
$$\tau = F(\overline{\kappa}) - G_{reg.}(\overline{\kappa}) - (F(\kappa) - G_{reg.}(\kappa))$$
$$F(\overline{\kappa}) = b_0^{-1}\overline{\kappa}^{-1} - (b_1/b_0^2)\log(\overline{\kappa}^{-1})$$
$$G_{reg}(\overline{\kappa}) = \int_0^{\overline{\kappa}} d\lambda \left(-\widehat{\beta}_{reg.}(\lambda)\right)^{-1}$$

the substitution $\tau \to t = \tau - \log(\mu_0^2/\Lambda^2)$ and
inverting the functional relation $t = t(\overline{\kappa}) \longleftrightarrow \overline{\kappa} = \overline{\kappa}(t)$

We rewrite eq. 64 separating sliding scale parts and associated scale μ and initial value parts associated with scale μ_0, as indicated in eqs. 51 - 53

(65)
$$\left[\begin{array}{l} \tau + b_0^{-1}\kappa^{-1} - (b_1/b_0^2)\log\left(b_0^{-1}\kappa^{-1}\right) - G_{reg.}(\kappa) + \\ \qquad + (b_1/b_0^2)\log\left(b_0^{-1}\overline{\kappa}^{-1}\right) + G_{reg.}(\overline{\kappa}) \end{array}\right] = b_0^{-1}\overline{\kappa}^{-1}$$

The substitutions $\kappa \to b_0\kappa$ and $\overline{\kappa} \to b_0\overline{\kappa}$ in the inverse of the arguments of the logarithm terms in eq. 65 cancels out in the difference. Next we adjust the scale parameter $\mu_0 \to \Lambda$ $\longrightarrow$

setting

$$-b_0^{-1}\kappa^{-1} + (b_1/b_0^2)\log\left(b_0^{-1}\kappa^{-1}\right) + G_{reg.}(\kappa) = \log\left(\mu_0^2/\Lambda^2\right)$$

(66) $$t = \tau - \log\left(\mu_0^2/\Lambda^2\right) = \log\left(\Lambda^2/\mu^2\right)$$

$$x = b_0\overline{\kappa} \;;\; G_{reg.}(x/b_0) = G(x) \;;\; b_1/b_0^2 = a$$

Further we substitute the sliding scale variable $\overline{\kappa} \longrightarrow b_0\overline{\kappa} = x$. With these substitutions eq. 65 becomes

(67) $$t + a\log\left(x^{-1}\right) + G(x) = x^{-1}$$

$$G(x) = x\sum_{n=0}^{\infty} G_n x^n$$

In order to establish the asymptotic 'ultraviolet' expansion for $t \rightarrow \infty$ and $Y = x^{-1} \rightarrow \infty$ we introduce the notation , for clarity, including a change of variables $t = \exp(L)$; $L = \log(t)$

(68) $$\log(t) = L \;;\; x = x(L) \longrightarrow Y = Y(L) \equiv (x(L))^{-1}$$

$$\log(Y) = Z(L) \equiv \log\left((x(L))^{-1}\right)$$

$\longrightarrow$

Eq. 67 takes the form

(69)
$$L + \log \left[1 + \frac{a\,Z(L) + G(x = Y^{-1})}{\exp(L)} \right] = Z(L)$$

$$t = \exp(L) \;\; ; \;\; Y = \exp(Z) \longleftarrow Z = Z(L)$$

Eq. 69 can now be solved by iteration in the asymptotic limit $L \rightarrow \infty$

$$L = \log\left[\log\left(\Lambda^2/\mu^2\right)\right] \rightarrow \infty$$

$$Z(L) = lim_{n\rightarrow\infty} Z_n(L)$$

(70)
$$L + \log \left[1 + \frac{a\,Z_n(L) + G\left(x_n = Y_n^{-1}\right)}{\exp(L)} \right] = Z_{n+1}(L)$$

anchor : $Z_0(L) = L$

The first anchor takes care of the largest contribution to to Z , i.e. L . $\longrightarrow$

Thus we obtain $Z_1(L)$ **and** $Y_1 = \exp(Z_1)$ **from rq. 70**

(71)
$$Z_1(L) = L + \log\left[1 + aL\exp(-L) + \frac{G(\exp(-L))}{\exp(L)}\right]$$

$$Y_1(L) = e^L + aL + G(\exp(-L)) \quad ; \quad t = e^L$$

Comparing with the recursive equations (eq. 70 , repeated below

$$L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \to \infty$$

$$Z(L) = lim_{n\to\infty} Z_n(L)$$

(72)
$$L + \log\left[1 + \frac{aZ_n(L) + G\left(x_n = Y_n^{-1}\right)}{\exp(L)}\right] = Z_{n+1}(L)$$

anchor : $Z_0(L) = L$

it follows that all contributions from the function G , i.e. resulting from the subleading terms proportinal in the beta function truncated by the first to terms $b_0\kappa^2 + b_1\kappa^3$ **give vanishing contributions to** $Y(L) \equiv (b_0\overline{\kappa})^{-1}(L)$ **for** $L \to \infty$ **. This can be verified recursively.** $\longrightarrow$

Thus modulo terms vanishing for $L \to \infty$ we can solve the simpler functional equation, omitting G from eqs. 67, 71 and 72

(73) $$t + a \log \left(\widetilde{Y} \right) = \widetilde{Y} \ ; \ \widetilde{Y} = x^{-1} \ ; \ Y - \widetilde{Y} \to 0 \ \textbf{for} \ t \to \infty$$

or equivalently

(74) $$L + \log \left(1 + a \widetilde{Z} e^{-L} \right) = \widetilde{Z} \ ; \ \widetilde{Y} = e^{\widetilde{Z}} \ ; \ t = e^{L}$$

The same argument which led to the elimination of the term proportional to G in eq. 72 as far as nonvanishing contributions for Y are concerned in the limit $L \to \infty$ imply that expanding the logarithm on the left hand side of eq. 74 only the first term need be retained

(75) $$\log \left(1 + a \widetilde{Z} e^{-L} \right) \sim a \widetilde{Z} e^{-L}$$

leading to the equivalent approximate functional equation simplifying eq. 74

(76) $$\begin{aligned} & Z \sim \widetilde{Z} \sim L / \left(1 - a e^{-L} \right) \sim L + a e^{-L} L \ \longrightarrow \\ & Y \sim \widetilde{Y} \sim e^{L} \left(1 + a e^{-L} L \right) = e^{L} + a L + [\, 0 \,] \ \textbf{for} \ L \to \infty \end{aligned}$$

In eq. 76 the $\sim$ symbol is meant to imply modulo additive terms to Y , vanishing for $L \to \infty$, proving the remarkable fact that the potential constant in the asymptotic expansion for Y vanishes . $\longrightarrow$

We conclude this subsection comparing side by side the functional dependence of $t \equiv \log\left(\mu^2/\Lambda^2\right) = t\left(\overline{\kappa}\left(\mu\right)\right)$ **and** $Y \equiv \left(b_0\,\overline{\kappa}\right)^{-1} = Y\left(t\right)$**.**

The first relation is given in eq. 67

(77)
$$
\begin{array}{l}
t = x^{-1} - a \log\left(x^{-1}\right) - G\left(x\right) \\
\hline
G\left(x\right) = x \sum_{n=0}^{\infty} G_n\, x^n \;;\; x = b_0\,\overline{\kappa}\left(\mu^2\right) \\
t = \log\left(\mu^2/\Lambda^2\right) \longrightarrow \\
\hline
\Lambda^2 = \mu^2 \left[\exp\left(-\dfrac{1}{b_0\,\overline{\kappa}}\right)\right]\left[\dfrac{1}{b_0\,\overline{\kappa}}\right]^a \exp\left[\; G\left(b_0\,\overline{\kappa}\right)\;\right] \\
\hline
a = b_1/b_0^2
\end{array}
$$

The second relation is displayed in eq. 76 , as an asymptotic expansion for $t = e^L \rightarrow \infty$

(78)
$$
\begin{array}{rcl}
Y & = & Y\left(L\right) \equiv \left(b_0\,\overline{\kappa}\right)^{-1}\left(L\right) \;\textbf{for}\; L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \rightarrow \infty \\
 & \sim & e^L + a\,L + \left[\,0\,\right]
\end{array}
$$

$\longrightarrow$

Eqs. 77 and 78 constitute the essence of this subsection, giving rise to some remarks :

1) The central scale Λ

with all subtle properties as defined, in the $\overline{\text{MS}}$ scheme, constitutes a renormalization group invariant finite mass scale, given a region of sliding scale coupling constant $\kappa = g^2 / (16\pi^2)$ generically or $\overline{\kappa}$ in the perturbative region region here, nonvanishing but appropriately small.

This scale does *not* depend on quark masses , only on the number of quark flavors , while derivations from measurable quantities (in the asymptiotic region) suffer from systematic errors , which do depend on quark massses .

The combined analysis of the sliding scale coupling constant , as of 2009 by Bethke , is shown in Fig. A21 .

I quote here a discussion of Λ for $N_{fl} = 3$ by Bodenstein et al., ref. [15-2011] , from data in the mass scale region of the τ lepton

$$\Lambda_{N_{fl}=3} = 382 \pm 24 \text{ MeV} \longleftrightarrow \alpha_s = 4\pi\kappa(\mu = m_\tau) = 0.344 \pm 0.014 \quad (79)$$

The reconstruction of Λ from the ’perturbatively accessible region’ implies a hard breaking of scale invariance , and hence equivalently a nonvanishing trace of the co-renormalizable and renormalized energy momentum density operator. This shall be discussed in the next subsection.

$\longrightarrow$

2) sliding scale quark masses

The rescaling equations (eqs. 51 - 53) extend to the quark mass parameters , again universally and quark mass independently rescaling m_f , keeping ratios invariant

(80)
$$r_{f_1 f_2} = m_{f_1} / m_{f_2}$$

Here no detailed discussion of extracting sliding scale quark masses including heavy flavors c , b is given. For heavy flavor comparisons with initially the reaction $e^+ e^- \to f \overline{f} -$ flavored hadrons , the QCD sum rule approach is the main tool , as pioneered by Shifman, Vainshtein and Zakharov [16-1979] , together with lattice QCD .

The sliding quark masses as for 3 flavors are shown in figure 6 below , restricted to two loop approximation . References [15-2011] - [17-2011] are representative for present refinement to four loops in the rescaling functions and derived results , e.g.

(81)
$$\overline{m}_b (\overline{m}_b) = \begin{cases} 4163 \pm 16 \text{ MeV} & \text{ref. [18-2010]} \\ 4171 \pm 7 \text{ MeV} & \text{ref. [19-2011]} \\ 4177 \pm 11 \text{ MeV} & \text{ref. [17-2011]} \end{cases}$$

$$\overline{m}_c (3 \text{ GeV}) = 986 \pm 13 \text{ MeV} , \text{ ref. [18-2010]}$$

$$\overline{m}_c (\overline{m}_c) = 1262 \pm 17 \text{ MeV} , \text{ ref. [17-2011]}$$

$\longrightarrow$

3-4 – Reinterpreting the central anomalies

$$\tfrac{1}{4}\left[B^{\,r}_{\mu\nu} B^{\,\mu\nu\, r} \right]_{\infty} = \mathcal{B}^{\,+}_{\infty} \text{ \& } \tfrac{1}{4}\left[B^{\,r}_{\mu\nu} \widetilde{B}^{\,\mu\nu\, r} \right]_{\infty} = \mathcal{B}^{\,-}_{\infty}$$

The central anomalies (eq. 39) become, using the field strength bilinears (re-)normalized at $\mu = \infty$, and separating gauge- and quark antiquatrk parts in the lowest order constant $b_0 = b_{0\,g.b.} - \frac{2}{3} N_{fl}$ of the coupling constant rescaling function $\beta (g)$

$$\left\{ \begin{array}{ll} \vartheta^{\,\mu}{}_{\mu} & = \sum_q \left[m_q (\mu) \, \overline{q} \, q (\mu) + \frac{1}{12\pi^2} \mathcal{B}^{\,+}_{\infty} \right] - b_{0\,g.b.} \frac{1}{8\pi^2} \mathcal{B}^{\,+}_{\infty} \\ \partial^{\,\nu} \left(\overline{q} \gamma_{\nu} \gamma_5 q \right) & = 2 \, m_q (\mu) \, \overline{q} \, i \gamma_5 q (\mu) \qquad + \quad 2 \quad \frac{1}{8\,\pi^2} \mathcal{B}^{\,-}_{\infty} \end{array} \right\} \quad (x)$$

$$- \beta / g^3 = b_0 / (16 \pi^2) + O (\kappa) \; ; \; \kappa = g^2 / (16 \pi^2)$$

$$b_0 = b_{0\,g.b.} - \frac{2}{3} N_{fl} \; ; \; b_{0\,g.b.} = \frac{11}{3} C_2 (adj (Lie - SU3_c)) = \frac{11}{3} 3$$

(82)

In eq. 82 normal ordering symbols are omitted for brevity of notation. Furthermore the renormalization scale for the field strengths bilinears

$$\frac{1}{4}\left[B^{\,r}_{\mu\nu} B^{\,\mu\nu\, r} \right]_{\infty} = \mathcal{B}^{\,+}_{\infty} \; ; \; \frac{1}{4}\left[B^{\,r}_{\mu\nu} \widetilde{B}^{\,\mu\nu\, r} \right]_{\infty} = \mathcal{B}^{\,-}_{\infty}$$

is set to $\mu = \infty$ which does lead to a conceptual clarification of these quantities most importantly for a selfdual or anti-selfdual classical background field configuration. $\longrightarrow$

In figure 5 , below , the behaviour of m_u , $\frac{1}{2}(m_u + m_d)$, m_d as functions of the sliding scale Q is shown in a range of euclidean values of Q , believed to be within the perturbatively accessible region. The curves are all proportional to *one* universal function .

$>$ Date: Fri, 21 June 2013 16:50:53 +0800

$>$ Greetings from World Scientific!

$>$ Murray Gell-Mann recently gave a public lecture, titled The Story of Quarks

$>$ www.worldscientific.com], at the National University of Singapore. It was

$>$ organized by the Institute of Advanced Studies as part of the

$>$ Distinguished Public Lecture series. His lecture has been published in

$>$International Journal of Modern Physics A (IJMPA) Volume 28, Issue 13 . (ref. [20-2013]

We return to the main theme of this discussion in the next section $\longrightarrow$

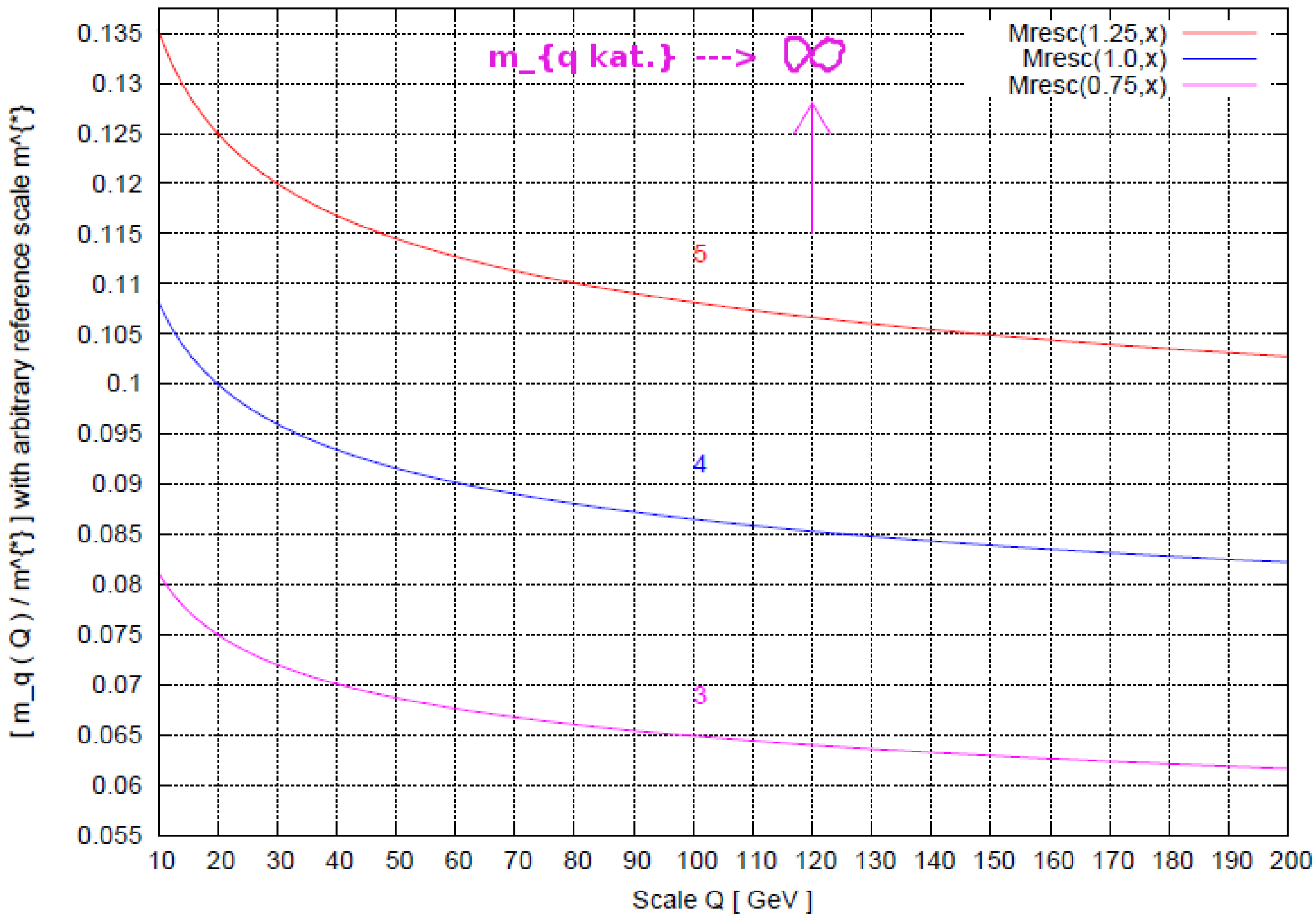

Fig. 5 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$$

→

4 – Ideas forging and foregoing - the dynamics of genuinely oscillatory modes [1-1980]

It is worth noting , that Erwin Schrödinger turned to the discussion of oscillatory modes of single- and by reduction of c.m. coordinates – of a pair mode of oscillatory motion , in the last (4th) paper in ref. [21-1926] .

However in the above paper he (E.S.) makes the assumption , that associated forces arise from the exchange of a photon , i.e. involve a local electromagnetic exchange interaction as giving rise to an equally local second order wave equation , responsible for oscillatory (pair-) modes . This is incorrect , contrary to the structure embedded in QCD , up to the present incomplete level of completion , which remains a future task .

Thus we concentrate on the present topic and lay out the ideas in ref. [1-1980] .

We relate the bond structure of quark-antiquark (meson) and $N-$ quark (baryon) systems, subject to an SU(N) unbroken color gauge group , to the long-range dynamics involving the oscillatory modes in the phase space of the center of mass position and momentum variables $[\, N \equiv N_c \rightarrow 3 \,]$ to be clear.

These canonical baricentric 3-vector variables are shown in eq. 83 $\longrightarrow$

$$
\begin{array}{ccc}
\vec{\pi}_{1} = \frac{1}{\sqrt{2}}\,(\,\vec{p}_{1} - \vec{p}_{2}\,) & , & \vec{z}_{1} = \frac{1}{\sqrt{2}}\,(\,\vec{x}_{1} - \underline{x}_{2}\,) \\
\vec{\pi}_{2} = \frac{1}{\sqrt{6}}\,(\,\vec{p}_{1} + \vec{p}_{2} - 2\,\vec{p}_{3}\,) & , & \underline{z}_{2} = \frac{1}{\sqrt{6}}\,(\,\underline{x}_{1} + \vec{x}_{2} - 2\,\vec{x}_{3}\,) \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\vec{\pi}_{\nu} = (\nu(\nu+1))^{-1/2} \left(\begin{array}{l} \sum_{\alpha=1}^{\nu} \vec{p}_{\alpha}) \\ -\nu\,\vec{p}_{\nu+1} \end{array} \right) & , & \vec{z}_{\nu} = (\nu(\nu+1))^{-1/2} \left(\begin{array}{l} \sum_{\alpha=1}^{\nu} \vec{x}_{\alpha} \\ -\nu\,\vec{x}_{\nu+1} \end{array} \right) \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\vec{\pi}_{N-1} = \cdots & , & \vec{z}_{N-1} = \cdots \\
\hline
\vec{\pi}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{p}_{\alpha} \to 0 & , & \vec{z}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{x}_{\alpha} \to 0
\end{array}
\tag{83}
$$

The last line in eq. 83 refers to c.m. momentum and position . $\longrightarrow$

The bond structures of $q\overline{q}$ and $3q$ configurations are shown in figure 1 - I , repeated below

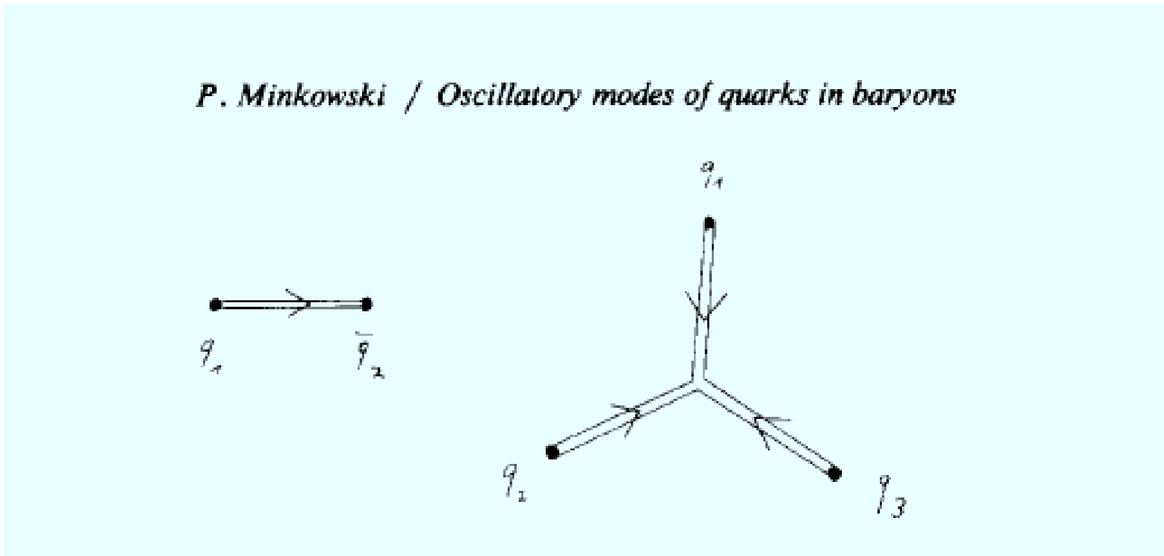

Fig. 1. Bond structures of qq̄ and 3q configuration ($N = 3$).

Fig. 1 - I : Bond structures of $q\overline{q}$ and $3q$ configurations $(N = 3)$ $\longleftrightarrow$

Of course we are interested in $N = 3$ but one goal of this investigation is to shed light on the N dependence of the ratio of baryonic to mesonic inverse Regge slopes

(84) $\Lambda_N / \Lambda \ (= 1)$ for vanishing quark masses $M_\alpha \rightarrow 0 \ ; \ \alpha = 1, \cdots, N$ $\longrightarrow$

$$\Delta\,\mathcal{M}^{\,2}_{\,baryon} \quad = \quad 2\,\Lambda_{\,N}\,\sum_{\alpha=1}^{3N-3}\Delta\,\nu_{\,\alpha}$$

(85) $$\Delta\,\nu_{\,\alpha} \;=0\,,\;\pm 1\,,\;\pm 2\,,\cdots$$

$$\Delta\,\mathcal{M}^{\,2}_{\,meson} \quad = \quad 2\,\Lambda \qquad \sum_{\alpha=1}^{3}\Delta\,\nu_{\,\alpha}$$

In eq. 84 quark masses are denoted $M_{\,\alpha}\;;\;\alpha$: quark flavor , as throughout section 4 , in order to distinguish them from – the oscillator state cofiguration space variable dependent masses – denoted $m\;,\;\overline{m}\;\cdots$.

$\longrightarrow$

and the connection of this ratio to the intermediary range quark-antiquark potential – mainly studied for the $c\overline{c}$ charmonium (binding) spectrum – determined by the conditions

(86) $$Min_{\alpha=1,\cdots,N} M_{\alpha} \ll \left| V_{NR}(z_{\beta}) \right| \ll Min_{\gamma=1,\cdots,N-1} \left| z_{\gamma} \right|^{-1}$$

We reformulate as starting point the snowball effect for a $q\overline{q}$ equal mass pair , described (in the c.m. system) by the Lagrangean

(87) $$\begin{aligned} \mathcal{L}_{(2)} &= -m_1 \left(1 - v_1^2\right)^{-1/2} - m_2 \left(1 - v_2^2\right)^{-1/2} \\ &= -2m \left(1 - v^2\right)^{-1/2} \end{aligned}$$

$$\vec{v}_j = \dot{\vec{x}}_j ; j = 1,2 \;\; ; \;\; m = m(z) \;\; \textbf{for (just here)} \;\; M_{q_1} = M_{q_1} = 0$$

Eq. 87 is only valid in the c.m. frame , where the analog of energy conservation takes the form

(88) $$\begin{aligned} \mathcal{H}_{(2)} &= \vec{v}\,\mathcal{L}_{(2),\vec{v}} - \mathcal{L}_{(2)} = \frac{2m}{\sqrt{1-v^2}} \\ (\vec{p})_1 - (\vec{p})_2 &= 2\,\vec{p}_{c.m.} = \mathcal{L}_{(2),\vec{v}} = \mathcal{H}_{(2)}\,\vec{v} \\ \left(\mathcal{H}_{(2)}\right)^2 v^2 &= \left(\mathcal{H}_{(2)}\right)^2 - 4m^2 = 4\,p_{c.m.}^2 \end{aligned}$$

⟶

Eqs. 87 and 88 are understood as approximations for large distances . They can be interpreted classically or quantum mechanically .

We note that in the discussion ongoing of $q\overline{q}$ oscillator modes, we do not use the orthogonal normalization as displayed in eq. 83 . This is so because other conventions had been used before , as for myself to the year 1976 . while working at Caltech . The definitions used are shown in the next equation

$$\overline{m} = 2\,m \quad , \quad -\Delta_{z} + \overline{m}^{\,2}(z) = H^{2}_{(2)}$$

(89) $$\left(\mathcal{H}_{(2)}\,\vec{v}\right)^{\cdot} = \mathcal{H}_{(2)}\,\tfrac{1}{2}\,\ddot{\vec{y}} = -\frac{1}{\mathcal{H}_{(2)}}\,grad_{\,y}\,\overline{m}^{\,2} \quad ; \quad \vec{y} = \vec{x}_{1} - \vec{x}_{2}$$

$$\longrightarrow \mathcal{H}^{2}_{(2)}\,\tfrac{1}{4}\,(\dot{\vec{y}})^{2} = 4\,p^{2}_{c.m.} + \overline{m}^{\,2} = \mathcal{M}^{2} = \text{constant}$$

adopting the long range approximate nature of the harmonic oscillator relations – for the $q\overline{q} -$ bond

(90) $$\overline{m}^{\,2} \sim_{|y|\rightarrow\infty} \left(\tfrac{1}{2}\,\Lambda\right)^{2} y^{2}\left[1 + O\left(M_{q}/|y|\right) + \cdots\right]$$

In eq. 89 we have substituted the variable $\vec{y}$ for $z \rightarrow \vec{z}$.

A few remarks shall follow $\longrightarrow$

1) The lessons from the $q\overline{q}-$ bond are limited

The exclusive relative *distance*-dependence contained in the asymptotic term in eq. 90

(91)
$$\overline{m}^{\,2} \sim \tfrac{1}{2}\,\Lambda^{\,2}\,y^{\,2}$$

generates genuine oscillatory modes for the $q\overline{q}-$ bond , yet no multi-position dependent generalization can accomplish the same for the $3\,q-\;(\,N\,q-\,)$ bonds .

2) but not empty

For the $q\overline{q}-$ bond it follows

(92)
$$\ddot{\vec{y}} = -\left(\frac{\Lambda}{\mathcal{H}_{(2)}}\right)\;y \quad , \quad \Lambda\,\left[\,-\,\Delta_{\,\xi} + \xi^{\,2}\,\right] = H^{\,2}_{(2)}$$

$$\xi = \tfrac{1}{2}\,(\,\Lambda\,)^{\,1/2}\,y$$

$$\longrightarrow \omega_{\,cl} = \frac{\Lambda}{\mathcal{H}_{(2)}} \quad , \quad H^{\,2}_{(2)}\,(\,\{\,\nu\,\}\,) = 2\,\Lambda\,\textstyle\sum_{\alpha=1}^{3}\,\nu_{\,\alpha} + 3\,\Lambda$$

$\nu_{\,\beta} = 0\,,\,1\,,\,\cdots\;\;;\;\;\beta = 1,2,3\;\;;$ **oscillator occupation numbers**

$\longrightarrow$

3) The color quantum number has vanished from the description

Vacuum - vacuum amplitedes of two colored local operators are not gauge invariant , provided local gauge inavrince is not conserve 'completely' , to be defined including appropriate generalized boundary conditions , in QCD .

The wave functions on the other hand are not local .

4) Which are the dependences on quantum numbers like (light-) flavors and spin ?

The quantity Λ in eq. 91 , of dimension mass ^{2} is considered universal , i.e. does not depend on any other quantum numbers neither on quark masses, except on the occupation numbers of the oscillatory modesat hand .

Extension to include the $N\,q -$ bond

The key idea arose upon a discussion initiated by H. R. Dicke , concerning the feasibility and appropriateness to envisage a revision of the errors , as established by Loránd (Roland v.) Eötvös in 1918 , in his famous experiments in ref. [22-1918-1961] measuring the equality of inertial and gravitational mass with the help of rotating springs , to which test bodies are attached , while the springs are fixed to one point , say atop a rotating rod . It should be added that the springs must be elongating under the centrifugal force only in one longitudinal direction . $\longrightarrow$

Dicke reports in ref. [22-1918-1961] that Eötvös , in his description of the experiments mentions an important obstacle to overcome , consisting in a precise separation of the mass of the test bodies from a combination of a part of the spring mass with them .
Hence the conclusion from the above situation to the question envisaged was , that in presence of position dependent mass this mass and inertial mass were *not the same* **.**
This led to the Ansatz , as I followd in ref. [1-1980]

$$\mathcal{L}_N = - \sum_{\alpha=1}^{N} \left[m_\alpha^2 - Q^\alpha_{\beta\gamma} \vec{v}_\beta . \vec{v}_\gamma \right]^{1/2}$$

$$\vec{v}_\alpha = \dot{\vec{x}}_\alpha \tag{93}$$

$$m_\alpha = m_\alpha \left[\vec{z}_1, \cdots, \vec{z}_{N-1} \right] \quad , \quad \textbf{gravitational effective masses}$$

$$Q^\alpha_{\beta\gamma} = Q^\alpha_{\beta\gamma} \left[\vec{z}_1, \cdots, \vec{z}_{N-1} \right] \quad , \quad \textbf{inertial effective masses}$$

vald in the c.m. system of the N quarks .
The external quark masses M_q , appropriate multipliers of the scalar densities $\overline{q}q$ composing the mass term in the local (QCD) Lagrangian

$$- \mathcal{L}_m = \sum_{flavors} \frac{z_q}{z_M} M_q \overline{q}^c q^c \tag{94}$$

$\longrightarrow$

appear as constants in the gravitational mass

(95)
$$m_{\alpha}\,[\,M_q\,,\,\underline{z}\,] = M_{\alpha} + m_{\alpha}\,[\,M_q = 0\,,\,\underline{z}\,]$$
$$\underline{z} = (\,\vec{z}_1\,,\,\cdots\,,\,\vec{z}_{N=1}\,)$$

whereas a consistent non-relativistic limit demands

(96)
$$[\,Q^{\alpha}_{\alpha\alpha}\,(\,M_q\,,\,\underline{z}\,)\,]^{1/2} \underset{M_q\to\infty}{\longrightarrow} M_{\alpha} + O\left[\,m_{\beta}\,[\,M_q = 0\,,\,\underline{z}\,]\,\right]$$
$$Q^{\alpha}_{\beta\gamma\,,\,\beta,\gamma\neq\alpha} \underset{M_q\to\infty}{\longrightarrow} O\left[\,m_{\beta}\,[\,M_q = 0\,,\,\underline{z}\,]\,\right]$$

From eqs. 95 and 96 we recognize the problem of separation of mass and binding energy, as relevant, e.g., to the gravitational interaction of the whole N-quark system, appearing.

In the following m- , Q- are approximated by the corresponding quantities for $M_q = 0$, i.e., in the chiral limit with respect to all the quark flavors composing the $N\,q -$ bond .

Then in the harmonic long-range limit Q- is determined from m- through the relation

(97)
$$Q^{\alpha}_{\beta\gamma} = \frac{1}{K_N}\,(\,m_{\alpha}\,)^2\,\delta_{\beta\gamma} \quad ; \quad K_N \; : \; \text{constant}$$

$\longrightarrow$

The kinetic term for the quark – a – depends on all the velocities $\vec{v}_\beta$ and the Lagrangean $\mathcal{L}_N$ in eq. 93 takes the simplified form

(98)

$$\mathcal{L}_N = - \overline{m} \left[1 - \sum_\beta (\vec{v}_\beta)^2 \right]^{1/2} \; ; \; \overline{m} = \sum_{\alpha=1}^N m_\alpha = \overline{m} \left(\vec{x}_\beta - \vec{X} \right)$$

$$\vec{X} = \frac{1}{N} \sum_{\alpha=1}^N \vec{x}_\alpha \to 0$$

The meaning of te constant K_N is the following : under the constraint $\sum_\alpha \vec{v}_\alpha = 0$ the maximum any individual $(\vec{v}_\beta)^2$ can assume for given $\sum_\gamma (\vec{v}_\gamma)^2$ is for the so-called $\lambda -$ mode , shown in figure 6 below .

An inequality for any individual square velocity follows

(99)

$$v_\alpha = \left((\vec{v}_\alpha)^2 \right)^{1/2} \longrightarrow$$

$$v_\alpha^2 \le \frac{N - 1}{N} \sum_\gamma v_\gamma^2 \le \frac{N - 1}{N} K_N c^2 \quad (c = 1)$$

$\longrightarrow$

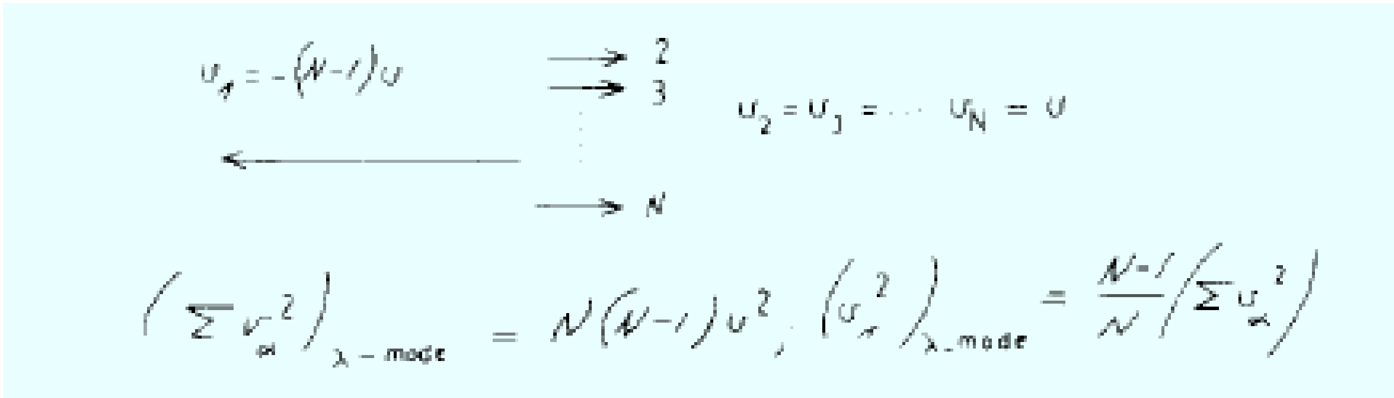

Fig. 6 : $\lambda -$ **mode for N quark bond** $\longleftrightarrow$

$\mathcal{L}_N$ **in eq. 98 delimits its validity to physical values of** v_α^2 **for** $K_N = N/(N-1)$

$$v_\alpha^2 \leq c^2 \longleftrightarrow K_N = \frac{N}{N-1} \tag{100}$$

$\longrightarrow$

The equation for the conserved energy (eq. 88 for $\mathcal{L}_{(2)}$) for $\mathcal{L}_N$ becomes

$$\mathcal{H}_N = \vec{v}_\alpha \vec{p}_\alpha - \mathcal{L}_N$$

(101)
$$\vec{p}_\beta = \mathcal{L}_{N,\vec{v}_\beta} = \frac{\overline{m}}{K_N} \left(\left[1 - \omega^2 \right]^{-1/2} \right)_{,\vec{v}_\beta} = \frac{\mathcal{H}_N}{K_N} \vec{v}_\beta$$

$$\omega^2 = \frac{1}{K_N} \sum_{\gamma=1}^N v_\gamma^2 \; ; \; K_N = \frac{N}{N-1}$$

From eq. 101 we obtain in canonically conjugate oscillator variables

$$(\mathcal{H}_N)^2 = \left[K_N \sum_{\alpha=1}^N (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{p}_\beta = 0} + \overline{m}^2 (x_\gamma - X) \right]$$

(102)
$$= \left[K_N \sum_{\alpha=1}^N (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{p}_\beta = 0} + \; + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^N (\vec{x}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{x}_\beta = 0} \right]$$

$\longrightarrow$

$\mathcal{H}_N$ **is a constant of the motion by the relations displayed in eq. 101 , but it is** $(\mathcal{H}_N)^2 \equiv \mathcal{M}_N^2$ **becomes the genuinely canonical dynamic operator or in the classical framework 'Hamiltonian function' . in the genuinely relativistic situation .**

The structure of $\mathcal{M}_N^2$ **is derived straightforwardly from eqs. 101 and 102**

$$\mathcal{M}_N^2 = \left[K_N \sum_{\alpha=1}^{N} (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^{N} \vec{p}_\beta = 0} + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^{N} (\vec{x}_\alpha)^2 \Big|_{\sum_{\beta=1}^{N} \vec{x}_\beta = 0} \right]$$

(103)

$$= \left[K_N \sum_{\alpha=1}^{N-1} (\vec{\pi}_\alpha)^2 + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^{N-1} (\vec{z}_\alpha)^2 \right]$$

$(\vec{\pi}_\beta , \vec{z}_\beta)$: **baricentric coordinates defined in eq. 83** ; $K_N = \dfrac{N}{N-1}$

$\longrightarrow$

The mass-square spectrum according to $\mathcal{M}_N^{\,2}$ in eq. 103 is given by

(104)
$$\mathcal{M}_N^{\,2}\Big|_{spectrum} = 2\,\Lambda \sum_{\alpha=1}^{3N-3} \nu_{\,\alpha} + 3\,\Lambda\,(\,3\,N - 3\,)$$
$$\nu_{\,\beta} = 0\,,\,1\,,\,\cdots\;\;;\;\;\beta = 1\,,\,2\,,\,\cdots\,,\,3\,N - 3$$

Except for the zero point contribution , i. e. for the oscillation level splittings , it is universal , independent of N , proving the validity of the universal relation in eq. 84 $(\,\Lambda_{\,N}\,/\,\Lambda = 1\,)$. $\longrightarrow$

Taking the $\nu = \sum_{\eta=1}^{6} \nu_\eta = 2\,,\ P = +$ nonstrange baryon states , i. e. for $N_{fl} = 2\,,\ N = 3$ and counting iin a reduced way, all corresponding oscillatory modes with positive parity , compatible with overall Bose symmetry , neglecting color , we obtain the following table of 21 states , not counting isospin and spin degrees of freedom separately

TABLE
πN-partial waves and associated number of baryon resonances with $\nu = 2$, $P = +$

πN partial wave	Number of states with $\nu = 2$, $P = +$	Candidates
P11 $N(\frac{1}{2}^+)$	4	2
P13 $N(\frac{3}{2}^+)$	5	1-2
F15 $N(\frac{5}{2}^+)$	3	2
F17 $N(\frac{7}{2}^+)$	1	1
P31 $\Delta(\frac{1}{2}^+)$	2	1-2
P33 $\Delta(\frac{3}{2}^+)$	3	1
F35 $\Delta(\frac{5}{2}^+)$	2	1
F37 $\Delta(\frac{7}{2}^+)$	1	1
	21	**10-12**

Fig. 7 : Nonstrange baryons with $\nu = 2\,, P = +$ in 1980 $\longleftrightarrow$

$\longrightarrow$

4-17

The candidates were collected in ref. [1-1980] from the PDG tables valid in 1980 . At least then almost 50 % of the resonances so characterized were missing , using this way of counting .

$\longrightarrow$

5 – Epilogue and outlook

The present outline has ts root in my discussion of the oscillatory modes of quarks in baryons in ref. [1-1980] , as well as the presentation of two hadron resonance collections used in ref. [23-2010] , worked out subsequently in a notefile similar to this one in ref. [24-2011] .

In comparing the two collections used in ref. [23-2010] , denoted Ntype=65 $\supset$ Ntype=26 , turn out both to be too small to account for the hadron abundances , as measured at RHIC and LHC , comparing with the Hadron Resonance Gas (HRG) approach (see e.g. ref. [25-2010]) , with noninteracting hadrons.

The detailed description of the collection of hadron resonances used in thermal fits to all , including RHIC and LHC hadron abundances is explicitely presented in ref. [26-2009] . The selection consists of including all hadron resonances in the PDG tables of 2012 [6-2012] and the review within the PDG tables [27-2012] .

Here we propose to consider the three light flavors of quark u , d , s , extending the modes discussed in ref. [1-1980] as a first step .

We define as final goal of the present study , based also on the 'Notes 22.12.2012 $\longrightarrow$ 17.06.2013 in ref. [28-2012] to illustrate eventual new data relative to 1980 and the situation of missing states as it persists today according to the PDG particle tables . The recent inclusion of 1- and 2-star hadron resonances also used in evaluating the Hadron Resonance Gas states in heavy ion collisions [26-2009] , shall be scrutinized as well and compared with spectroscopic theoretical valence quark counting of oscillatory modes in hadrons. For now I wish

— to thank you for your attention — $\longrightarrow$

Appendix 1 - Expansion coefficients of the rescaling functions $\widehat{\beta}$, γ to four loops

$$-\beta / g = X B(X) \; ; \; B(X) = b_0 A(X)$$

$$B(X) \sim \sum_{n=0}^{\infty} b_n X^n \; , \; A(X) \sim \sum_{n=0}^{\infty} a_n X^n$$

$\kappa = g^2 / (16 \pi^2)$ **generic** X

(105)

$$b_0 = \frac{1}{3} \left(33 - 2 N_{fl} \right) \; , \; a_0 = 1 \; , \; a_n = b_n / b_0$$

$$b_1 = \frac{2}{3} \left(153 - 19 N_{fl} \right)$$

$$b_2 = \frac{1}{54} \left(77139 - 15099 N_{fl} + 325 N_{fl}^2 \right)$$

$$b_3 \sim 29243 - 6946.3 N_{fl} + 405.089 N_{fl}^2 + 1.49931 N_{fl}^3$$

$\longrightarrow$

$$-\gamma_m^0 = 4\,,\quad -\gamma_m^1 = \frac{202}{3} - n_{fl}\frac{20}{9} \qquad\qquad \Big|\; -\gamma_m^l \equiv \chi_m^l$$

$$-\gamma_m^2 = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] N_{fl} - \frac{140}{81} N_{fl}^2$$

$$-\gamma_m^3 = \left\{ \begin{array}{l} \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\,\zeta(5) + \\ + \left[-\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\zeta(5) \right] N_{fl} + \\ + \left[\frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right] N_{fl}^2 + \\ + \left[-\frac{332}{243} + \frac{64}{27}\zeta(3) \right] N_{fl}^3 \end{array} \right\} \tag{106}$$

$\longrightarrow$

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