Hidden beauty in $\mathcal{N} = 4$ supersymmetric gauge theory

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Outline

✔ Why scattering amplitudes?

✔ Hidden symmetries of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory

✔ Correlation function/scattering amplitude duality

✔ New symmetry of correlation functions

✔ Open questions
Why scattering amplitudes? Hep-PH motivation:

Search for the Higgs boson at LHC:

\[ \text{gluon} + \text{gluon} \rightarrow \text{top quark} + \text{antitop quark} + \text{Higgs} \]

✔ Lots of produced particles in the final state leading to large background

✔ Identification of Higgs boson requires detailed understanding of scattering amplitudes

✔ Theory should provide solid basis for a successful physics program at the LHC
Why scattering amplitudes? Hep-TH motivation:

- On-shell matrix elements of the $S-$matrix:
  - ✓ Well-understood and simple IR divergent part, but extremely complicated finite part

- In planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory they have a remarkable structure:
  - ✓ Simpler than QCD amplitudes but they share many properties
  - ✓ All-order conjectures and a proposal for strong coupling via AdS/CFT
  - ✓ New hidden symmetry – dual superconformal invariance

- Duality with correlation functions of gauge invariant operators (yet another hidden symmetry)
General properties of amplitudes in gauge theories

Tree amplitudes:

✔ Well defined in $D = 4$ dimensions (free from UV and IR divergences)

✔ Respect the classical (Lagrangian) symmetries

✔ *Gluon tree amplitudes* are the same in all gauge theories (QCD, ..., $\mathcal{N} = 4$ SYM)

Loop amplitudes:

✔ Loop corrections are not universal (gauge theory dependent)

✔ Suffer from IR divergences $\rightarrow$ are not well defined in $D = 4$ dimensions

✔ The classical conformal symmetry is broken

Three questions in this lecture:

✔ Do scattering amplitudes in $\mathcal{N} = 4$ SYM have hidden (non-Lagrangian) symmetries?

✔ How powerful are these symmetries to completely determine the scattering amplitudes?

✔ Which dual models describe the all-loop amplitudes?
Maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory

✓ Uniquely specified by the gauge group, e.g. number of colors $N_c$ for $SU(N_c)$

✓ Conformally invariant (UV finite) quantum field theory

✓ Weak/strong coupling duality (AdS/CFT correspondence)

Particle content:

- massless spin-1 gluon ( = the same as in QCD)
- 4 massless spin-1/2 gluinos ( = cousins of the quarks)
- 6 massless spin-0 scalars

Interaction between particles:

All having the same dimensionless coupling $g$ and related to each other by supersymmetry
Gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

✓ On-shell matrix elements of the $S$–matrix:

\[ A_n = \begin{array}{c} \cdot \cdots \cdot \\ 1 \quad 2 \quad n \end{array} \]

\[ A_n = \begin{array}{c} \text{Quantum numbers of scattered gluons:} \\
\text{Color:} \quad a_i = 1, \ldots, N_c^2 - 1 \\
\text{Light-like momenta:} \quad (p_i^\mu)^2 = 0 \\
\text{Polarization state (helicity):} \quad h_i = \pm 1 \end{array} \]

✓ Color-ordered planar ($N_c \to \infty$) gluon amplitudes:

\[ A_n = \text{tr} \left[ T^{a_1} T^{a_2} \cdots T^{a_n} \right] A_{n}^{h_1, h_2, \ldots, h_n} (p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]} \]

✗ Supersymmetry all-loop Ward identity:

\[ A_{n}^{++ \cdots +} = A_{n}^{-+ \cdots +} = 0 \]

✗ Classification of amplitudes according to the total helicity $h = \sum_1^n h_i$

\[ \text{MHV} = \{A_{n}^{-- + \cdots +}, A_{n}^{-+ - \cdots +}, \ldots\}, \quad \text{NMHV} = \{A_{n}^{--- + \cdots +}, A_{n}^{+-+- \cdots +}, \ldots\}, \quad \ldots \]

MHV = Maximally Helicity Violating, NMHV = next-to-MHV, ...
Hints for hidden symmetry

Gluon amplitudes at tree level:

\[ \sum_{\substack{i \leq 1 \cdots n \geq 7 \geq 0 \geq 3 \geq 2 \geq 1}} \]

\[ S = \text{Number of external gluons} \]

\[ \text{Number of } \text{‘tree’ diagrams} \]

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<th>Number of external gluons</th>
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<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
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<td>220</td>
<td>2485</td>
<td>34300</td>
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✔ Number of diagrams grows factorially for large number of external gluons/number of loops

✔ ... but the final expression looks remarkably simple (details to come)

\[ A_{\text{tree}}^n (1^-2^-3^+ \ldots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \delta^{(4)} (\sum_i p_i) \]

Where does this simplicity come from?
Tree MHV amplitude

✔ Spinor-helicity formalism:

✗ Parameterization of four-momentum \( p^\mu = (p_0, p_1, p_2, p_3) \)

\[
\hat{p}^\alpha\dot{\alpha} = p_0 \sigma_0 + \sum_{i=1}^{3} p_i \sigma_i = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}
\]

✗ On-shell gluon momentum

\[
p_\mu^2 = 0 \implies \det |\hat{p}| = 0 \implies \hat{p}^\alpha\dot{\alpha} = \lambda^\alpha(p) \tilde{\lambda}^{\dot{\alpha}}(p)
\]

✗ Commuting spinors: \((\hat{p})^{\dot{\alpha}\alpha} \lambda_\alpha = \tilde{\lambda}_\dot{\alpha} (\hat{p})^{\dot{\alpha}\alpha} = 0\) defined up to a phase:

\[
\lambda^\alpha(p) \quad [\text{helicity } -\frac{1}{2}], \quad \tilde{\lambda}^{\dot{\alpha}}(p) \quad [\text{helicity } +\frac{1}{2}]
\]

✔ Amplitudes are homogeneous functions of \( \lambda_i = \lambda(p_i), \tilde{\lambda}_i = \tilde{\lambda}(p_i) \) \( (i = 1, \ldots, n) \)

\[
A_{n}^{\text{MHV}}(1^- 2^- 3^+ \ldots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \delta^{(4)} \left( \sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \right), \quad \text{with } \langle ij \rangle \equiv \lambda_i^\alpha \varepsilon_{\alpha\beta} \lambda_j^{\beta}
\]

What are the symmetries of this amplitude?
Conformal symmetry of the amplitude

$\mathcal{N} = 4$ SYM is a quantum field theory with (super)conformal $SU(2, 2|4)$ symmetry:

✔ Conformal symmetry acts locally in $x$–space (e.g. inversion $x_\mu \rightarrow x_\mu / x^2$)

✔ Conformal symmetry acts non-locally in $p$–space (via Fourier transform)

✔ Realization of conformal symmetry on amplitudes by 2nd-order operators [Witten]

$$k_{\alpha \dot{\alpha}} = \sum_i \frac{\partial^2}{\partial \lambda_i^\alpha \partial \tilde{\lambda}_i^{\dot{\alpha}}} \implies k_{\alpha \dot{\alpha}} A_{n}^{\text{MHV}} = 0$$

✔ Can be extended to the full $SU(2, 2|4)$ superconformal invariance

$$g \cdot A_{n}^{\text{MHV}} = 0, \quad g = \{p, m, d, k, q, \bar{q}, s, \bar{s}\} \in SU(2, 2|4)$$

Less trivial to verify for NMHV, $N^2$MHV, ... amplitudes [Korchemsky,ES]

✔ Conformal symmetry alone is not powerful enough to fix the tree amplitudes. What else?
\( \mathcal{N} = 4 \) amplitudes have a much bigger, dual (super)conformal symmetry \[ \text{[Drummond, Henn, Korchemsky, ES]} \]

✔ Simplest example:

\[
\left| \hat{A}_{n}^{\text{MHV}} \right|^2 = \frac{(s_{12})^4}{s_{12}s_{23} \cdots s_{n1}}, \quad (\text{with} \quad s_{ij} = (p_i + p_j)^2)
\]

✔ Introduce dual variables (not a Fourier transform!)

\[
x \quad p_i = x_i - x_{i+1}, \quad x_{n+1} \equiv x_1 \quad \Rightarrow \quad \text{solves} \quad \sum_1^n p_i = 0
\]

\[
x \quad p_i^2 = 0 \iff (x_i - x_{i+1})^2 = 0
\]

\[
x \quad s_{i,i+1} = (x_i - x_{i+2})^2
\]

MHV amplitude in dual space

\[
\left| \hat{A}_{n}^{\text{MHV}} \right|^2 \quad \frac{[(x_1 - x_3)^2]^3}{(x_2 - x_4)^2(x_3 - x_5)^2 \cdots (x_n - x_2)^2}
\]

Looks like an \( n \)-point correlation function in \( x \)-space, BUT the \( x \)'s are momenta!
Dual (super)conformal symmetry II

✔ Conformal inversion in dual $x-$space

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2} \quad \Rightarrow \quad s_{i,i+1} \rightarrow \left(x_i^2 x_{i+2}^2\right)^{-1} s_{i,i+1}$$

Acts locally on the momenta \(\Rightarrow\) not related to the conformal symmetry of $\mathcal{N} = 4$ SYM

✔ The tree-level MHV (super)amplitude is covariant under dual conformal inversion

$$I \left[ A_{n}^{\text{MHV}} \right] = \frac{x_2^2 x_4^2 x_5^2 \cdots x_n^2}{x_1^2 x_3^2} \times A_{n}^{\text{MHV}}$$

✔ Dual conformal symmetry can be extended to dual superconformal $\widetilde{SU}(2, 2|4)$ symmetry

$$G \cdot A_{n}^{\text{MHV}} = 0, \quad G = \{P, M, D, K, Q, \bar{Q}, S, \bar{S}\} \in \widetilde{SU}(2, 2|4)$$

✔ **Dual superconformal symmetry is a property of all tree-level (super)amplitudes** (MHV, NMHV, $N^2$ MHV,...) in $\mathcal{N} = 4$ SYM theory

[Drummond,Henn,Korchemsky,ES],[Brandhuber,Heslop,Travaglini]
Symmetries of tree amplitudes

✔ The relationship between conventional and dual superconformal symmetries:

\[
\begin{align*}
  p & = \bar{q} \\
  q & = \bar{s} \\
  s & = \bar{k}
\end{align*}
\]

\[
\begin{align*}
  K & = \bar{S} \\
  S & = \bar{Q} \\
  Q & = \bar{P}
\end{align*}
\]

✔ Same symmetries appear at strong coupling from invariance of AdS$_5 \times$S$^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

✔ (Infinite-dimensional) closure of the two symmetries has Yangian structure [Drummond,Henn,Plefka]

✔ All tree $\mathcal{N} = 4$ amplitudes are uniquely fixed by:
  × supersymmetric BCFW recursion relations [Brandhuber,Heslop,Travaglini],[Drummond,Henn],[Bianchi,Elvang,Freedman],[Arkani-Hamed,Cachazo,Kaplan]
  × or equivalently, by symmetries + analytic properties [Korchemsky,ES],[Beisert et al],[Arkani-Hamed et al],[Mason,Skinner]

What happens to these symmetries at loop level?
Planar gluon amplitudes at weak coupling

- Loop corrections to all MHV amplitudes are described by a single scalar function

$$A_{n \text{MHV}}(p_i) = A_{n \text{(tree)}}(p_i) M_{n \text{MHV}}(\{s_{ij}\}; g)$$

- Example: four-gluon amplitude at one loop

$$A_4/A_4^{\text{(tree)}} = 1 + \frac{g^2 N_c}{8\pi^2} I^{(1)}(s, t) + O(g^4)$$

- Scalar box in dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

$$I^{(1)}(s, t) = \int \frac{d^D x_0 x_1^2 x_2^2 x_3^2 x_4^2}{x_0^2 x_1^2 x_2^2 x_3^2 x_4^2}$$

- Invariant under conformal transformations, e.g. $x_i \rightarrow x_i/x_i^2$, in dual space if $D = 4$

- This symmetry is not related to the conformal symmetry of $\mathcal{N} = 4$ SYM

- All scalar integrals contributing to $A_4$ are dual conformal!

- Dual conformal symmetry is broken by IR divergences (anomalous Ward identity)
Correlation functions of BPS operators

- Dual conformal symmetry is natural for correlation functions of gauge invariant operators.

- Protected superconformal operators made from 6 scalars $\phi_{AB} = \frac{1}{2} \epsilon_{ABCD} \bar{\phi}^{CD}$

$$\mathcal{O}(x) = \text{Tr}(\phi_{12} \phi_{12}), \quad \tilde{\mathcal{O}}(x) = \text{Tr}(\bar{\phi}^{12} \bar{\phi}^{12})$$

Scaling dimensions do not receive quantum corrections.

- Simplest non-trivial correlation function

$$G_4 = \langle 0|T(\mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4))|0\rangle = \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \mathcal{F}(u, v; g)$$

Conformal cross-ratios

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

- The limit $x_i \to x_j$ corresponds to the standard OPE:

  - Extract anomalous dimensions of all twist-two operators [Dolan,Osborn]
  - Confirm maximal transcendentality conjecture to 3 loops [Lipatov et al],[Eden et al]
Correlation functions on the light cone

✔ New limit: let all neighboring points become light-like separated

\[ x_{i,i+1}^2 \rightarrow 0, \quad x_i \neq x_{i+1}, \quad (i = 1, \ldots, n) \quad \Rightarrow \quad u, v \rightarrow 0 \]

✔ The light-cone limit of \( G_4 \) is singular:

(i) For \( x_{i,i+1}^2 \rightarrow 0 \) the correlator develops pole singularities already at tree level

\[
G_4^{(\text{tree})} \sim \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \text{subleading terms}
\]

but we consider the ratio

\[
\mathcal{F}_4 \equiv \lim_{x_{i,i+1}^2 \rightarrow 0} \frac{G_4(x_i)}{G_4^{(\text{tree})}(x_i)}
\]

(ii) In addition, loop integrals develop logarithmic light-cone singularities \((u, v \rightarrow 0)\)

\[
\mathcal{F}_4 = 1 + \frac{g^2 N_c}{8\pi^2} \frac{i}{\pi^2} \int \frac{d^4 x_0 x_1^2 x_2^2 x_3^2 x_4^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} + \ldots = 1 - \frac{g^2 N_c}{8\pi^2} \ln u \ln v + \ldots.
\]
Duality correlation functions/scattering amplitudes

In the light-cone limit we observe a surprising duality:

\[
\lim_{x^{2}_{i,i+1}\to 0} \ln \left( \frac{G_{n}}{G^{(\text{tree})}_{n}} \right) = 2 \ln \left( \frac{A^{\text{MHV}}_{n}}{A^{(\text{tree})}_{n}} \right) + O\left(\frac{1}{N_{c}^{2}}\right)
\]

Generalizes Wilson loops/scattering amplitudes duality

Both objects are divergent and require regularization (UV for \(G_{n}\), IR for \(A_{n}\))

The duality can be formulated in terms of integrands:

- Loop corrections via Lagrangian insertions

\[
g^{2\ell} \frac{d^{\ell}}{dg^{2\ell}} G_{n} = \prod_{i=5}^{4+\ell} \int d^{4}x_{i} \langle O(x_{1}) \ldots O(x_{n}) L_{\mathcal{N}=4}(x_{n+1}) \ldots L_{\mathcal{N}=4}(x_{n+\ell}) \rangle^{(\text{tree})}
\]

- The \(\ell\)-loop integrand is an \(n\)-point correlator with \(\ell\) Lagrangian insertions at tree level:

\[
G^{(\text{tree})}_{n+\ell} = \langle O(1) \ldots O(n) L_{\mathcal{N}=4}(n+1) \ldots L_{\mathcal{N}=4}(n+\ell) \rangle^{(\text{tree})}
\]

- Duality with the integrand of the \(\ell\)-loop amplitude (no regularization needed!):

\[
\lim_{x^{2}_{i,i+1}\to 0} \ln \left( \frac{G^{(\text{tree})}_{n+\ell}}{G^{(\text{tree})}_{n}} \right) = 2 \ln \left( \frac{\text{Int}_{\ell}[A^{\text{MHV}}_{n}]}{A^{(\text{tree})}_{n}} \right) + O\left(\frac{1}{N_{c}^{2}}\right)
\]

Matches exactly the integrand of the momentum-twistor construction

[Alday, Eden, Korchemsky, Maldacena, ES]

[Arkani-Hamed et al]
Hidden permutation symmetry of the integrand

✓ General form of the integrand for $n = 4$ predicted by $\mathcal{N} = 4$ superconformal symmetry:

$$\langle \mathcal{O}(1) \ldots \mathcal{O}(4) \mathcal{L}(5) \ldots \mathcal{L}(4 + \ell) \rangle^{(\text{tree})} \sim \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \times \frac{P^{(\ell)}(x_1, \ldots, x_{4+\ell})}{\prod_{1 \leq i < j \leq 4+\ell} x_{i,j}^2}$$

✓ The numerator $P^{(\ell)}$ is a homogeneous polynomial in $x_{i,j}^2$ of conformal weight $(1 - \ell)$ at each point, invariant under $S_{4+\ell}$ permutations of $x_i$. [Eden, Heslop, Korchemsky, ES]

✓ Examples at 1 and 2 loops:

$$P^{(1)}(x_1, \ldots, x_5) = 1, \quad P^{(2)}(x_1, \ldots, x_6) = \frac{1}{48} \sum_{\sigma \in S_6} x_{\sigma(1)}^2 x_{\sigma(2)}^2 x_{\sigma(3)}^2 x_{\sigma(4)}^2 x_{\sigma(5)}^2 x_{\sigma(6)}^2$$

✓ The $\ell$–loop integrand is obtained by drawing pictures (graph theory). Example:
Five loops (planar)

✔ We find only 7 planar integrand graph topologies:

✔ All coefficients in the planar sector fixed by a simple log singularity criterion; a few arbitrary constants remain in the non-planar sector

✔ Direct calculation of the five-loop Konishi anomalous dimension

[Eden, Heslop, Korchemsky, Smirnov, ES]
Open questions

✔ What is the origin of dual (super)conformal symmetry?

✔ Is it related (equivalent?) to the integrability of the $\mathcal{N} = 4$ SYM Hamiltonian?

✔ Understand *why* the duality correlators/amplitudes works. Are there some hidden symmetries of the correlator, which fix it to a unique form?

✔ What happens in the non-planar sector?

✔ Which of these features survive in gauge theories with less supersymmetry?

✔ Amplitudes are IR divergent, so not directly related to physical observables. Look for IR-safe observables, e.g. energy-energy correlations.

✔ In the year 2113: conformal collider physics?