Hidden beauty in $\mathcal{N} = 4$ supersymmetric gauge theory

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Collaboration with

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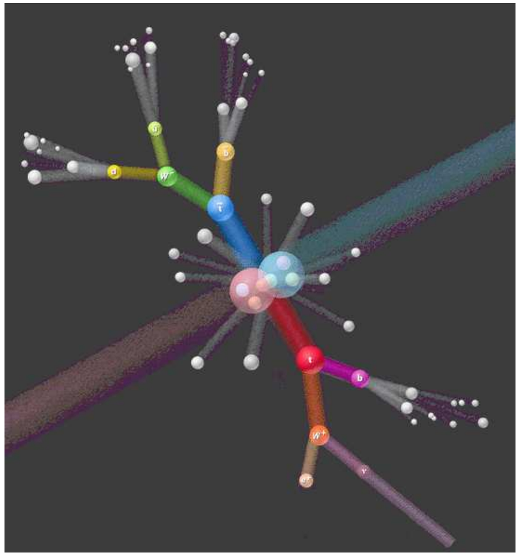
Outline

- Why scattering amplitudes?
- Hidden symmetries of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory
- Correlation function/scattering amplitude duality
- New symmetry of correlation functions
- Open questions

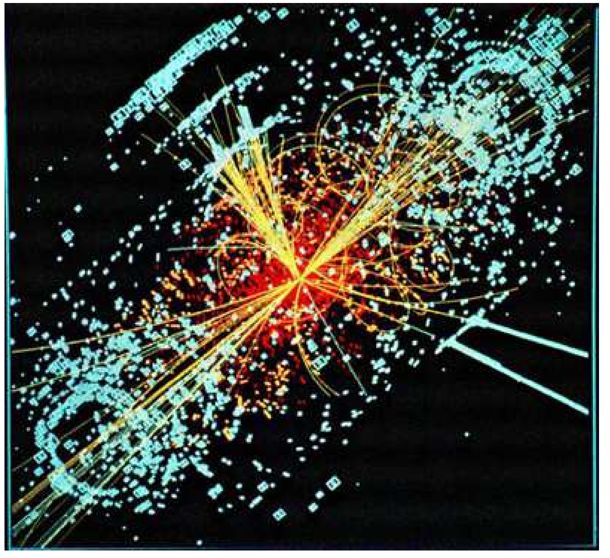
Why scattering amplitudes? Hep-PH motivation:

Search for the Higgs boson at LHC:

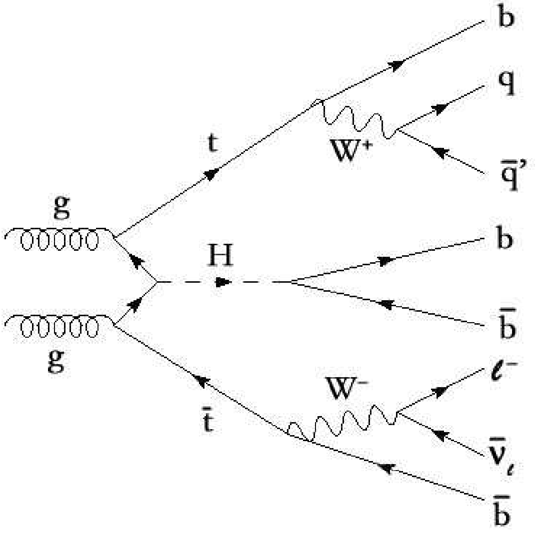
$$\text{gluon} + \text{gluon} \to \text{top quark} + \text{antitop quark} + \text{Higgs}$$



Theory



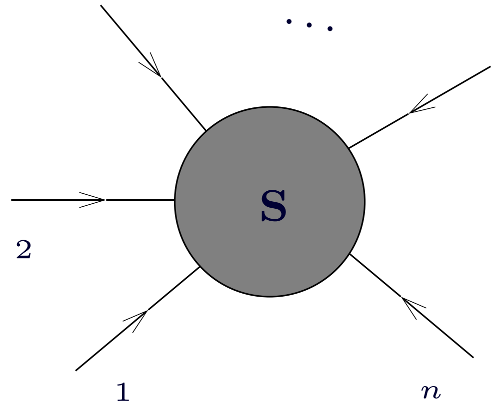
Experiment



Feynman diagram

- ✔ Lots of produced particles in the final state leading to large background
- ✔ Identification of Higgs boson requires detailed understanding of scattering amplitudes
- ✔ Theory should provide solid basis for a successful physics program at the LHC

Why scattering amplitudes? Hep-TH motivation:

$\mathcal{A}_n =$ 

- ✔ On-shell matrix elements of the $S-$matrix:
  - ✗ Well-understood and simple IR divergent part, but extremely complicated finite part
- ✔ In *planar* $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory they have a remarkable structure:
  - ✗ Simpler than QCD amplitudes but they share many properties
  - ✗ All-order conjectures and a proposal for strong coupling via AdS/CFT
  - ✗ New hidden symmetry – dual superconformal invariance
- ✔ Duality with correlation functions of gauge invariant operators (yet another hidden symmetry)

General properties of amplitudes in gauge theories

Tree amplitudes:

- ✔ Well defined in $D=4$ dimensions (free from UV and IR divergences)
- ✔ Respect the classical (Lagrangian) symmetries
- ✔ *Gluon tree amplitudes* are the same in all gauge theories (QCD, ..., $\mathcal{N}=4$ SYM)

Loop amplitudes:

- ✔ Loop corrections are not universal (gauge theory dependent)
- ✔ Suffer from IR divergences $\to$ are not well defined in $D=4$ dimensions
- ✔ The classical conformal symmetry is broken

Three questions in this lecture:

- ✔ Do scattering amplitudes in $\mathcal{N}=4$ SYM have hidden (non-Lagrangian) symmetries?
- ✔ How powerful are these symmetries to completely determine the scattering amplitudes?
- ✔ Which dual models describe the all-loop amplitudes?

Maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory

- ✔ Uniquely specified by the gauge group, e.g. number of colors N_c for $SU(N_c)$
- ✔ Conformally invariant (UV finite) quantum field theory
- ✔ Weak/strong coupling duality (AdS/CFT correspondence)

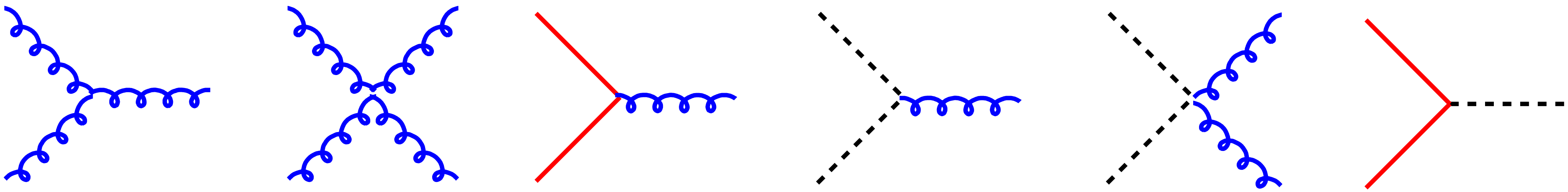
Particle content:

 massless spin-1 gluon (= the same as in QCD)

 4 massless spin-1/2 gluinos (= cousins of the quarks)

 6 massless spin-0 scalars

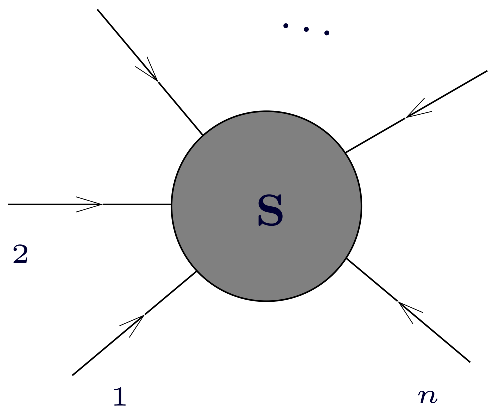
Interaction between particles:



All having the same dimensionless coupling g and related to each other by supersymmetry

Gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

✔ On-shell matrix elements of the $S-$matrix:

$\mathcal{A}_n \;=$ 

- Quantum numbers of scattered gluons:

  Color: $a_i = 1, \ldots, N_c^2 - 1$

  Light-like momenta: $(p_i^\mu)^2 = 0$

  Polarization state (helicity): $h_i = \pm 1$

✔ Color-ordered planar ($N_c \to \infty$) gluon amplitudes:

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1} T^{a_2} \ldots T^{a_n}\right] A_n^{h_1, h_2, \ldots, h_n}(p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}$$

✗ Supersymmetry all-loop Ward identity:

$$A_n^{++\ldots+} = A_n^{-+\ldots+} = 0$$

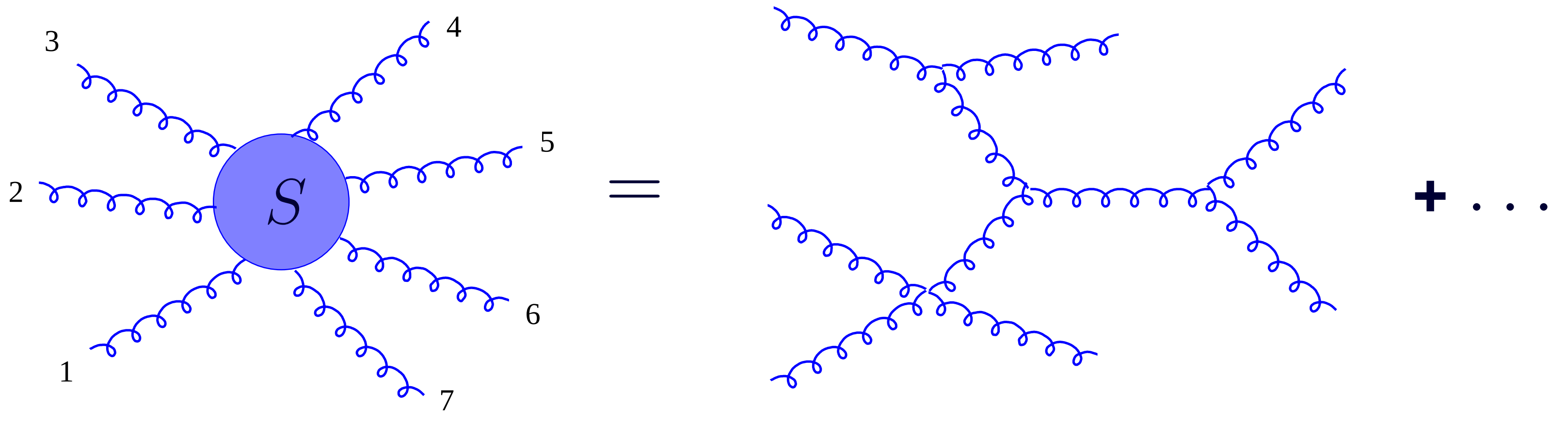
✗ Classification of amplitudes according to the total helicity $h = \sum_1^n h_i$

$$\mathrm{MHV} = \{A_n^{--+\ldots+}, A_n^{-+-\ldots+}, \ldots\}\,, \qquad \mathrm{NMHV} \quad = \{A_n^{---+\ldots+}, A_n^{-+--\ldots+}, \ldots\}\,, \quad \ldots$$

MHV = Maximally Helicity Violating, NMHV = next-to-MHV, ...

Hints for hidden symmetry

Gluon amplitudes at tree level:



Number of external gluons	4	5	6	7	8	9	10
Number of 'tree' diagrams	4	25	220	2485	34300	559405	10525900

- ✔ Number of diagrams grows factorially for large number of external gluons/number of loops
- ✔ ... but the final expression looks remarkably simple (details to come)

$$A_n^{\text{tree}}(1^-2^-3^+\dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\delta^{(4)}\Big(\sum_i p_i\Big)$$

Where does this simplicity come from?

Tree MHV amplitude

- Spinor-helicity formalism: [Xu,Zhang,Chang]
  - Parameterization of four-momentum $p^\mu = (p_0, p_1, p_2, p_3)$

$$\hat{p}^{\alpha\dot{\alpha}} = p_0\sigma_0 + \sum_{i=1}^{3} p_i\sigma_i = \begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

  - On-shell gluon momentum

$$p_\mu^2 = 0 \qquad \Longrightarrow \qquad \det \|\hat{p}\| = 0 \qquad \Longrightarrow \qquad (\hat{p})^{\alpha\dot{\alpha}} = \lambda^\alpha(p)\tilde{\lambda}^{\dot{\alpha}}(p)$$

  - Commuting spinors: $(\hat{p})^{\dot{\alpha}\alpha}\lambda_\alpha = \tilde{\lambda}_{\dot{\alpha}}(\hat{p})^{\dot{\alpha}\alpha} = 0$ defined up to a phase:

$$\lambda^\alpha(p) \quad [\text{helicity } -\tfrac{1}{2}]\,, \qquad \tilde{\lambda}^{\dot{\alpha}}(p) \quad [\text{helicity } +\tfrac{1}{2}]$$

- Amplitudes are homogeneous functions of $\lambda_i = \lambda(p_i)$, $\tilde{\lambda}_i = \tilde{\lambda}(p_i)$ $(i = 1, \ldots, n)$ [Parke,Taylor]

$$A_n^{\text{MHV}}(1^-2^-3^+\ldots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\delta^{(4)}\Big(\sum_1^n \lambda_i^\alpha\tilde{\lambda}_i^{\dot{\alpha}}\Big)\,, \qquad (\text{with } \langle ij\rangle \equiv \lambda_i^\alpha\epsilon_{\alpha\beta}\lambda_j^\beta)$$

What are the symmetries of this amplitude?

Conformal symmetry of the amplitude

$\mathcal{N}=4$ SYM is a quantum field theory with (super)conformal $SU(2,2|4)$ symmetry:

- ✔ Conformal symmetry acts locally in $x-$space (e.g. inversion $x_\mu \to x_\mu/x^2$)
- ✔ Conformal symmetry acts non-locally in $p-$space (via Fourier transform)
- ✔ Realization of conformal symmetry on amplitudes by 2nd-order operators [Witten]

$$k_{\alpha\dot\alpha} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha \partial\tilde\lambda_i^{\dot\alpha}} \qquad \Longrightarrow \qquad k_{\alpha\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

- ✔ Can be extended to the full $SU(2,2|4)$ superconformal invariance

$$g\cdot\mathcal{A}_n^{\rm MHV} = 0\,, \qquad g = \{p,\, m,\, d,\; k,\, q,\, \bar q,\, s,\, \bar s\} \in SU(2,2|4)$$

Less trivial to verify for NMHV, N^{2}MHV,... amplitudes [Korchemsky,ES]

- ✔ *Conformal symmetry alone is not powerful enough to fix the tree amplitudes. What else?*

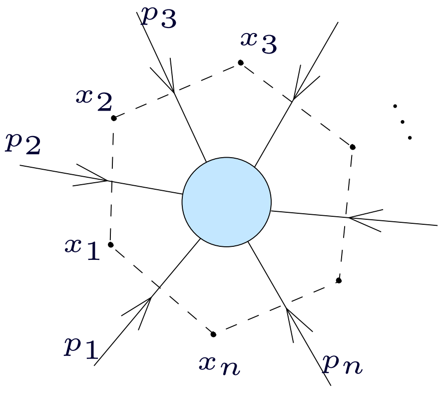
Dual (super)conformal symmetry I

$\mathcal{N}=4$ amplitudes have a much bigger, dual (super)conformal symmetry [Drummond, Henn, Korchemsky, ES]

- ✔ Simplest example:

$$\left|\hat{A}_n^{\rm MHV}\right|^2 = \frac{(s_{12})^4}{s_{12}s_{23}\ldots s_{n1}}\,, \qquad \text{(with} \quad s_{ij}=(p_i+p_j)^2)$$

- ✔ Introduce dual variables (not a Fourrier transform!)



- ✗ $p_i = x_i - x_{i+1}\,, \quad x_{n+1}\equiv x_1 \qquad \Rightarrow$ solves $\sum_1^n p_i = 0$
- ✗ $p_i^2 = 0 \Longrightarrow (x_i - x_{i+1})^2 = 0$
- ✗ $s_{i,i+1} = (x_i - x_{i+2})^2$

- ✔ MHV amplitude in dual space

$$\left|\hat{A}_n^{\rm MHV}\right|^2 = \frac{[(x_1-x_3)^2]^3}{(x_2-x_4)^2(x_3-x_5)^2\ldots(x_n-x_2)^2}$$

Looks like an $n-$point correlation function in $x-$space, *BUT the x's are momenta!*

Dual (super)conformal symmetry II

- ✔ Conformal inversion in dual $x-$space

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad \Longrightarrow \qquad s_{i,i+1} \to \left(x_i^2 x_{i+2}^2\right)^{-1} s_{i,i+1}$$

Acts locally on the momenta $\Longrightarrow$ *not related* to the conformal symmetry of $\mathcal{N}=4$ SYM

- ✔ The tree-level MHV (super)amplitude is covariant under dual conformal inversion

$$I\left[\mathcal{A}_n^{\rm MHV}\right] = \frac{x_2^2 x_4^2 x_5^2 \ldots x_n^2}{x_1^2 x_3^2} \times \mathcal{A}_n^{\rm MHV}$$

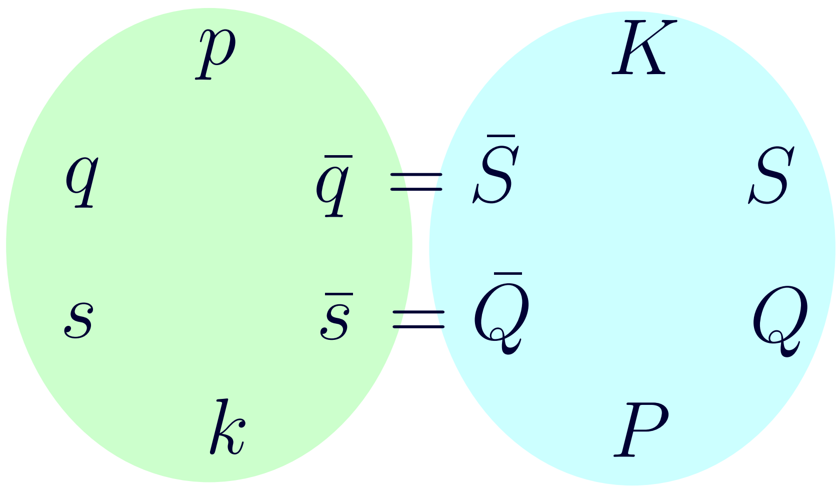
- ✔ Dual conformal symmetry can be extended to dual superconformal $\widetilde{SU}(2,2|4)$ symmetry

$$G \cdot \mathcal{A}_n^{\rm MHV} = 0\,, \qquad G = \{P,\, M,\, D,\, K,\, Q,\, \bar{Q},\, S,\, \bar{S}\} \in \widetilde{SU}(2,2|4)$$

- ✔ ***Dual superconformal symmetry is a property of all tree-level (super)amplitudes (MHV, NMHV, N^{2}MHV,...) in $\mathcal{N}=4$ SYM theory*** [Drummond,Henn,Korchemsky,ES],[Brandhuber,Heslop,Travaglini]

Symmetries of tree amplitudes

- ✔ The relationship between conventional and dual superconformal symmetries:



- ✔ Same symmetries appear at strong coupling from invariance of $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

- ✔ (Infinite-dimensional) closure of the two symmetries has Yangian structure [Drummond,Henn,Plefka]

- ✔ All tree $\mathcal{N} = 4$ amplitudes are uniquely fixed by:
  - ✗ supersymmetric BCFW recursion relations [Brandhuber,Heslop,Travaglini],[Drummond,Henn],[Bianchi,Elvang,Freedman],[Arkani-Hamed,Cachazo,Kaplan]
  - ✗ or equivalently, by symmetries + analytic properties [Korchemsky,ES],[Beisert et al],[Arkani-Hamed et al],[Mason,Skinner]

What happens to these symmetries at loop level?

Planar gluon amplitudes at weak coupling

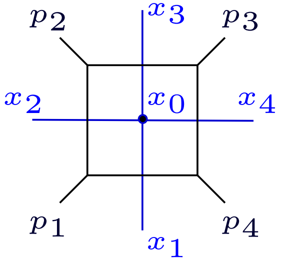
- ✔ *Loop corrections to all MHV amplitudes are described by a single scalar function* [Parke,Taylor]

$$A_n^{\rm MHV}(p_i) = A_n^{\rm (tree)}(p_i) M_n^{\rm MHV}(\{s_{ij}\};g)$$

- ✔ Examle: four-gluon amplitude at one loop

$$A_4/A_4^{\rm (tree)} = 1 + \frac{g^2 N_c}{8\pi^2} I^{(1)}(s,t) + O(g^4)$$ [Green,Schwarz,Brink]

  - ✗ Scalar box in dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

$I^{(1)}(s,t) =$ 

$$\sim \int \frac{d^D x_0 \, x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

  - ✗ Invariant under conformal transformations, e.g. $x_i \to x_i/x_i^2$, in ***dual*** space if $D = 4$
  - ✗ This symmetry *is not related* to the conformal symmetry of $\mathcal{N} = 4$ SYM
  - ✗ All scalar integrals contributing to A_4 are dual conformal! [Drummond,Henn,Smirnov,ES],[Bern et al]

- ✔ Dual conformal symmetry is broken by IR divergences (anomalous Ward identity)

Correlation functions of BPS operators

- ✔ Dual conformal symmetry is natural for correlation functions of gauge invariant operators
- ✔ Protected superconformal operators made from 6 scalars $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$

$$\mathcal{O}(x) = \mathrm{Tr}(\phi_{12}\phi_{12})\,, \qquad \tilde{\mathcal{O}}(x) = \mathrm{Tr}(\bar{\phi}^{12}\bar{\phi}^{12})$$

Scaling dimensions do not receive quantum corrections

- ✔ Simplest non-trivial correlation function

$$G_4 = \langle 0|T(\mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\mathcal{O}(x_3)\tilde{\mathcal{O}}(x_4))|0\rangle = \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}\mathcal{F}(u,v;g)$$

Conformal cross-ratios

$$u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

- ✔ The limit $x_i \to x_j$ corresponds to the standard OPE:
  - ✗ Extract anomalous dimensions of all twist-two operators [Dolan,Osborn]
  - ✗ Confirm maximal transcedentality conjecture to 3 loops [Lipatov et al],[Eden et al]

Correlation functions on the light cone

✔ New limit: let *all neighboring points* become light-like separated

$$x^2_{i,i+1} \to 0\,, \qquad x_i \neq x_{i+1}\,, \qquad (i=1,\dots,n) \qquad \Rightarrow \quad u,\ v \to 0$$

✔ The light-cone limit of G_4 is singular:

(i) For $x^2_{i,i+1} \to 0$ the correlator develops pole singularities already at tree level

$$G_4^{(\text{tree})} \sim \frac{N_c^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \text{subleading terms}$$

but we consider the ratio

$$\mathcal{F}_4 \equiv \lim_{x^2_{i,i+1}\to 0} G_4(x_i)/G_4^{(\text{tree})}(x_i)$$

(ii) In addition, loop integrals develop logarithmic light-cone singularities ($u, v \to 0$)

$$\mathcal{F}_4 = 1 + \frac{g^2 N_c}{8\pi^2}\,\frac{i}{\pi^2}\int \frac{d^4x_0\, x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} + \ldots = 1 - \frac{g^2 N_c}{8\pi^2}\ln u \ln v + \ldots\,.$$

Duality correlation functions/scattering amplitudes

- ✔ In the light-cone limit we observe a surprising duality: [Alday,Eden,Korchemsky,Maldacena,ES]

$$\lim_{x_{i,i+1}^2 \to 0} \ln\left(G_n/G_n^{(\mathrm{tree})}\right) = 2\ln\left(A_n^{\mathrm{MHV}}/A_n^{(\mathrm{tree})}\right) + O(1/N_c^2)$$

- ✔ Generalizes Wilson loops/scattering amplitudes duality [Alday,Maldacena], [Drummond,Henn,Korchemsky,ES]
- ✔ Both objects are divergent and require regularization (UV for G_n, IR for A_n)
- ✔ The duality can be formulated in terms of integrands:
  - ✗ Loop corrections via Lagrangian insertions

$$g^{2\ell}\frac{d^\ell}{dg^{2\ell}}G_n = \prod_{i=5}^{4+\ell}\int d^4x_i\,\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\mathcal{L}_{\mathcal{N}=4}(x_{n+1})\ldots\mathcal{L}_{\mathcal{N}=4}(x_{n+\ell})\rangle^{(\mathrm{tree})}$$

  - ✗ The $\ell-$loop integrand is an $n-$point correlator with ℓ Lagrangian insertions at tree level:

$$G_{n+\ell}^{(\mathrm{tree})} = \langle \mathcal{O}(1)\ldots\mathcal{O}(n)\mathcal{L}_{\mathcal{N}=4}(n+1)\cdots\mathcal{L}_{\mathcal{N}=4}(n+\ell)\rangle^{(\mathrm{tree})}$$

  - ✗ Duality with the integrand of the $\ell-$loop amplitude (no regularization needed!):

$$\lim_{x_{i,i+1}^2 \to 0} \ln\left(G_{n+\ell}^{(\mathrm{tree})}/G_n^{(\mathrm{tree})}\right) = 2\ln\left(Int_\ell[A_n^{\mathrm{MHV}}]/A_n^{(\mathrm{tree})}\right) + O(1/N_c^2)$$

  - ✗ Matches exactly the integrand of the momentum-twistor construction [Arkani-Hamed et al]

Hidden permutation symmetry of the integrand

- ✔ General form of the integrand for $n = 4$ predicted by $\mathcal{N} = 4$ superconformal symmetry:

$$\langle \mathcal{O}(1) \dots \mathcal{O}(4) \mathcal{L}(5) \dots \mathcal{L}(4+\ell) \rangle^{(\text{tree})} \sim \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \times \frac{P^{(\ell)}(x_1, \dots, x_{4+\ell})}{\prod_{1 \le i < j \le 4+\ell} x_{ij}^2}$$

- ✔ The numerator $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of conformal weight $(1-\ell)$ at each point, invariant under $S_{4+\ell}$ permutations of x_i. [Eden,Heslop,Korchemsky,ES]

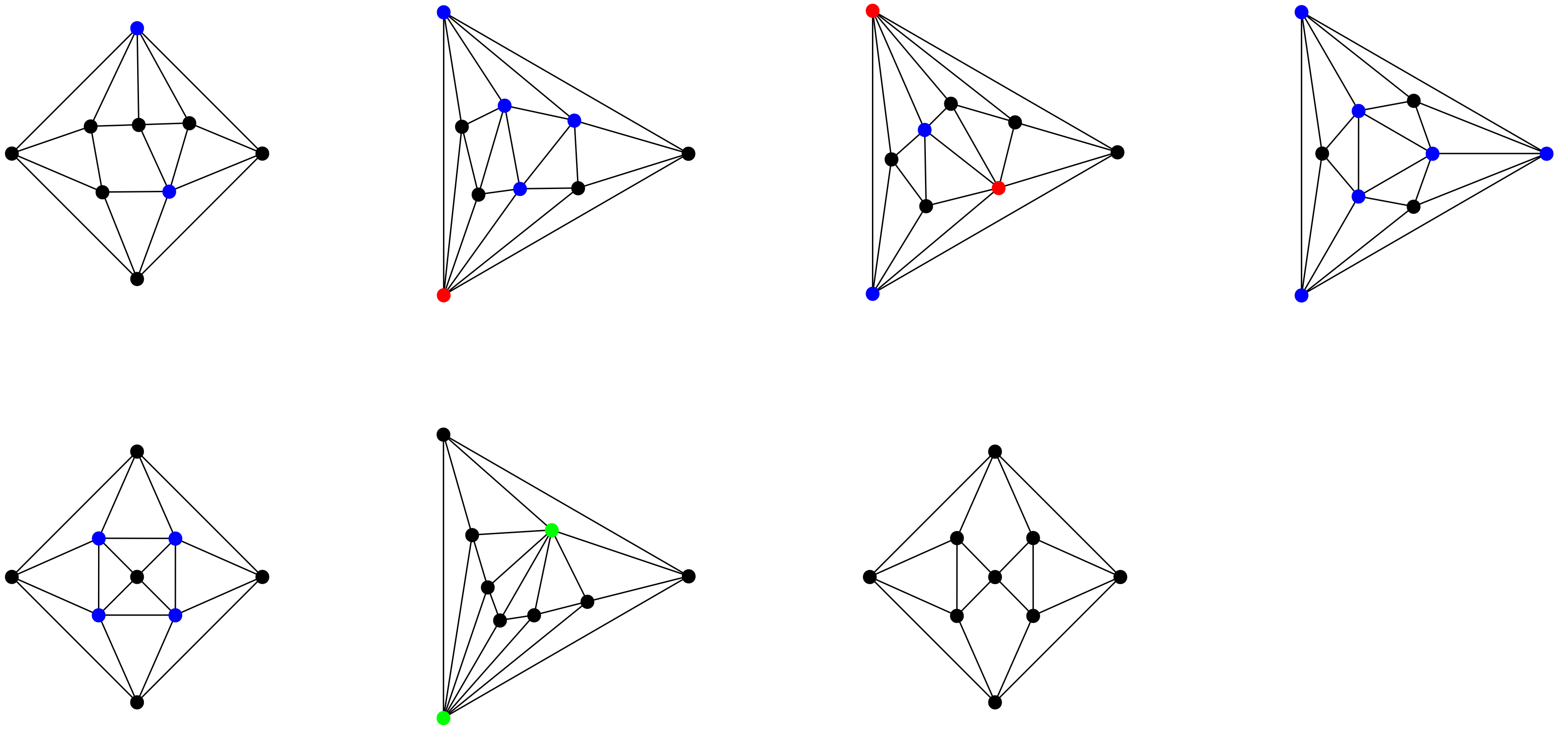
- ✔ Examples at 1 and 2 loops:

$$P^{(1)}(x_1, \dots, x_5) = 1\,, \qquad P^{(2)}(x_1, \dots, x_6) = \frac{1}{48} \sum_{\sigma \in S_6} x_{\sigma(1)\sigma(2)}^2 x_{\sigma(3)\sigma(4)}^2 x_{\sigma(5)\sigma(6)}^2$$

- ✔ The $\ell-$loop integrand is obtained by drawing pictures (graph theory). Example:

Five loops (planar)

- ✔ We find only 7 planar integrand graph topolgies:



- ✔ All coefficients in the planar sector fixed by a simple log singularity criterion; a few arbitrary constants remain in the non-planar sector
- ✔ Direct calculation of the five-loop Konishi anomalous dimension [Eden,Heslop,Korchemsky,Smirnov,ES]

Open questions

- ✔ What is the origin of dual (super)conformal symmetry?
- ✔ Is it related (equivalent?) to the integrability of the $\mathcal{N} = 4$ SYM Hamiltonian?
- ✔ Understand *why* the duality correlators/amplitudes works. Are there some hidden symmetries of the correlator, which fix it to a unique form?
- ✔ What happens in the non-planar sector?
- ✔ Which of these features survive in gauge theories with less supersymmetry?
- ✔ Amplitudes are IR divergent, so not directly related to physical observables. Look for IR-safe observables, e.g. energy-energy correlations.
- ✔ In the year 2113: conformal collider physics?