Cabling Procedure for the HOMFLY Polynomials

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Current knowledge of the gauge field theories

- 2-dimensional theories - e.g CFT
- 4-dimensional theories - e.g. SW

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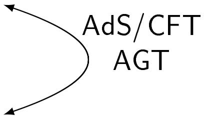

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AdS/CFT
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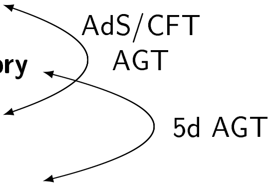

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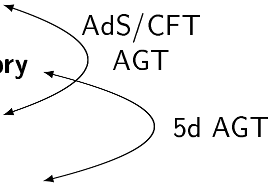

In the last few years many connections between CS and the other theories such as **Integrable Systems**, **Topological Strings Theories** and **Conformal Field Theories** has been found.

Chern-Simons theory

- 3-dimensional topological gauge theory - Chern-Simons theory
- $S_{CS} = \frac{k}{4\pi} \int d^3x \ \left(\mathcal{A} d \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)$
- Wilson-loop averages:

$$\left\langle W_Q^{\mathcal{K}} \right\rangle = \left\langle tr_Q Pexp\left(\oint_{\mathcal{K}} \mathcal{A} dx \right) \right\rangle_{CS(N,q)}$$

Q is a representation of the group $SU(N)$

$\mathcal{K}$ is a contour (i.e. a knot)

$q = exp(\frac{2\pi i}{k+N})$

[1] E.Witten, Comm.Math.Phys. **121** (1989) 351

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  $q = exp(\frac{2\pi i}{k+N})$
- In 1989 E.Witten[1] suggested that the Wilson-loop averages of the CS theory are equal to the knot polynomials:

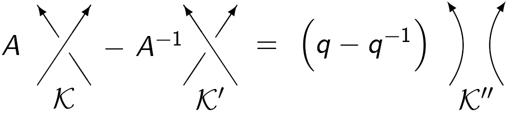
$$H_Q^{\mathcal{K}} = \left\langle W_Q^{\mathcal{K}} \right\rangle_{CS(N,q)}$$

[1] E.Witten, Comm.Math.Phys. **121** (1989) 351

HOMFLY polynomials

HOMFLY polynomials are topological invariants and the Laurent polynomials in variables A and q.

The fundamental HOMFLY can be evaluated using the skein relations



$$A\ H_{\square}^{\mathcal{K}} - A^{-1}\ H_{\square}^{\mathcal{K}'} = \left(q - q^{-1}\right)\ H_{\square}^{\mathcal{K}''}$$

In the Chern-Simons theory $q = e^{\frac{2\pi i}{k+N}}$ and $A = q^N$

Color in the knot theory

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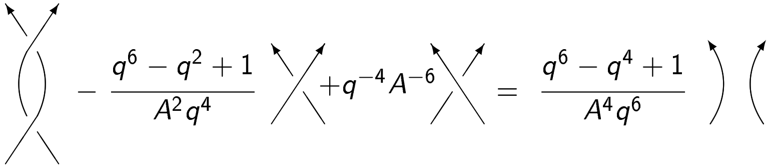
- Colored skein relations

  For the representation [2]:



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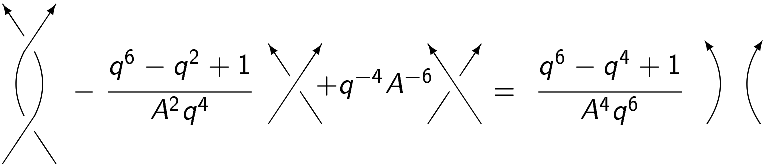
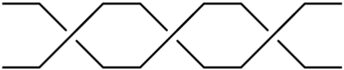


- Cabling procedure

  To define it we will first provide more details about the other methods of evaluating the knot polynomials.

Braid representation

To describe the knot it is convenient to use its braid representation:



Then in the Reshetikhin-Turaev formalism[2] for each crossing the $\mathcal{R}$-matrix should be inserted and the answer for the HOMFLY polynomials then is given by the trace over the product of all the $\mathcal{R}$-matrices:

$$H_Q^{\mathcal{K}} = Tr_{Q^{\otimes 2}} \prod_i \mathcal{R}_i$$

[2] N.Yu.Reshetikhin and V.G.Turaev, Comm. Math. Phys. **127** (1990) 1-26

Current knowledge about the $\mathcal{R}$-matrices

$\mathcal{R}$-matrices depend on many parameters:

- Representations of the group $SU(N)$ placed on the different strands
- Number of the strands in the braid
- The pair of the crossing strands

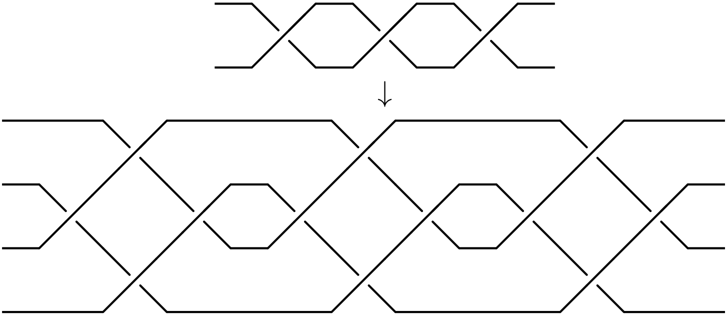
The exact matrix form has been constructed for any $\mathcal{R}$-matrix in the fundamental representation.

Cabling procedure

The main idea behind the cabling procedure is that

$$H^{\mathcal{K}}_{Q^{\otimes m}} = H^{\mathcal{K}^m}_{Q}$$

$\mathcal{K}^m$ is a knot $\mathcal{K}$ were the strand is replaced with the m parallel strands:



Projectors

The cabling of the knot gives the answer in the reducible representation $Q^{\otimes m}$ therefore to find the answer in some other irreducible or reducible representation T the operator called projector $P_T^{Q^{\otimes m}}$ should be constructed.

These projectors can be described as a combination of the added crossings between the m strands in the cable.

In other words the knot polynomials (Wilson-loop average) in the representation T can be represented as a linear combination of the knot polynomials in the fundamental representations but for the knots with $|T|$ times more strands.

Answers for the projectors

The general answer for the projectors is known in the matrix form.

The answer which describes the connections between the colored and the fundamental knot polynomials is known only in the several simplest examples.

$$P_2 = -\frac{R_1 + q^{-1}}{q + q^{-1}} \qquad P_{11} = \frac{R_1 - q}{q + q^{-1}}$$

There is also a suggestion of the recursive procedure which in principle should be able to construct any projector.

Conclusion

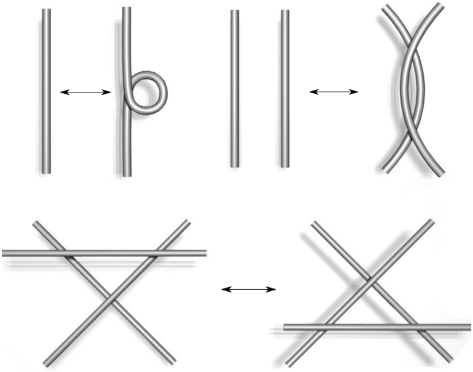
- The exact matrix form of the $\mathcal{R}$-matrices in the fundamental representation has been constructed.
  This allows in principle to find the HOMFLY polynomial of any knot or link in the fundamental representation.
- The matrix form of the projectors has been constructed.
  This allows in principle to find the HOMFLY polynomial of any knot or link in any representation.
- The method to find the projectors which describes the connections between the fundamental and the colored HOMFLY polynomials has been suggested.

The collected data should be analyzed to find the properties of the Wilson averages and the connections between the different knot polynomials.

THANK YOU FOR YOUR ATTENTION!

Yang-Baxter Equation

HOMFLY polynomials satisfy three Reidemeister moves



The third Reidemeister move is the Yang-Baxter equation which solutions are $\mathcal{R}$-matrices.

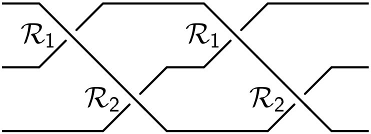
Character Expansion

The eigenvectors of these $\mathcal{R}$-matrices with the different eigenvalues correspond to the different irreducible representations from the decomposition of the $Q^{\otimes m}$. Thus instead of using the $\mathcal{R}$-matrices acting on the vectors one could use one for the irreducible representations. Then the answer for the Wilson-loop average (Knot polynomial) will be an expansion into the Schur-functions (characters of different irreducible representations of the studied group). The coefficients of this expansion are not topologically invariant, but the whole sums are:

$$H_Q^{\mathcal{K}}(A|q) = \langle W_Q^{\mathcal{K}} \rangle = \sum_T h_Q^T S_T^*(A,q)$$

$\mathcal{R}$-matrices

For the braid with m strands there are $m-1$ different $\mathcal{R}$ matrices corresponding to the crossings between different pairs of strands.



$\mathcal{R}_1 = \mathcal{R}_0 \otimes I \qquad \mathcal{R}_2 = I \otimes \mathcal{R}_0$

All of them have the same eigenvalues but they are not the same ones.

Diagonal $\mathcal{R}$-matrices

For the crossing between representations T_1 and T_2 the eigenvalues are described by the irreducible representations in the decomposition $Q_i \vdash T_1 \otimes T_2$.

$$|\lambda_i| = q^{\varkappa_{Q_i}}, \quad \varkappa_{Q_i} = \frac{1}{2} \sum_{\{i,j\} \in Q_i} (i - j)$$

If there are more than two strands then the following decomposition should be studied:

$$T_1 \otimes T_2 \otimes .. \otimes T_m = (\sum_i Q_i) \otimes .. \otimes T_m = \sum_j \bar{Q}_j$$

The eigenvalues for the $\bar{Q}_j$ are the same ones as for the corresponding Q_i

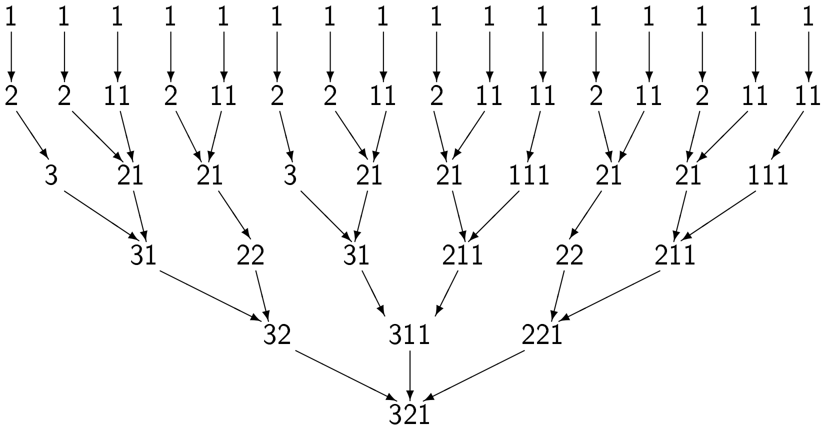
Non-diagonal $\mathcal{R}$-matrices

The general formula is known only for the fundamental representations.
The matrix is block-diagonal with blocks 1×1 equal to q or $-q^{-1}$ and 2×2 equal to the

$$b_j = \begin{pmatrix} -\frac{1}{q^j [j]_q} & \frac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} \\ \frac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} & \frac{q^j}{[j]_q} \end{pmatrix}$$

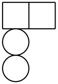
Where $[j]_q$ is the quantum j: $[j]_q \equiv \frac{q^j - q^{-j}}{q - q^{-1}}$.
Then the paths on the tree of the diagrams should be considered

Tree of the diagrams

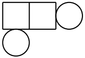


Non-diagonal $\mathcal{R}$-matrices

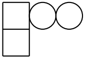
$\mathcal{R}_{k-1}$ is described by the level k in the tree. The paths can go as singlets or doublets. In doublets two paths differ only on the level k



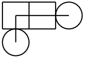
$2 \to 211$
$-q^{-1}$ in R



$2 \to 31$
b_3 block in R



$11 \to 31$
q in R



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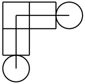


$21 \to 311$
b_4 block in R

If the length of a hook, connecting the two added cells is equal to j then block b_{j-1} should be used

Non-diagonal $\mathcal{R}$-matrices

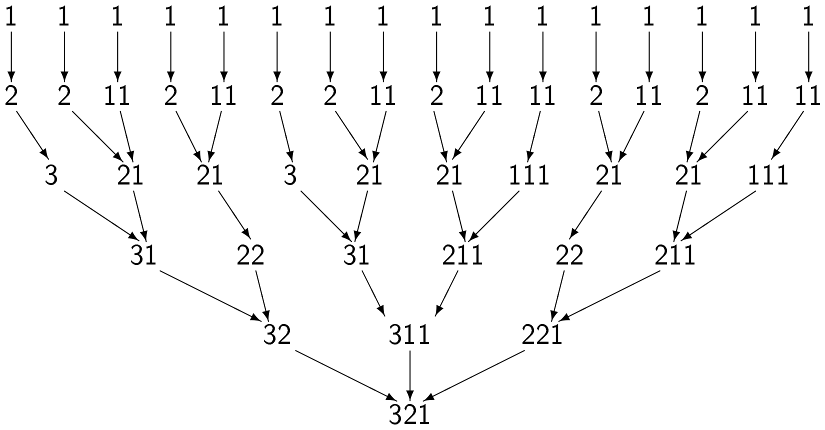
$$R_{3|311} = \begin{pmatrix} -\frac{1}{[3]_q q^3} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & & & & \\ \frac{\sqrt{[2]_q[4]_q}}{[2]_q} & \frac{q^3}{[3]_q} & & & & \\ & & q & & & \\ & & & -q^{-1} & & \\ & & & & -\frac{1}{[3]_q q^3} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} \\ & & & & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & \frac{q^3}{[3]_q} \end{pmatrix}$$

Path Projectors

Projectors can be constructed n the same manner as the $\mathcal{R}$-matrices. For the P_Q in the tree only the paths which pass through the Q should remain.

$$P_{2\otimes1^3|311} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Representation tree



Characteristic equations

The definition using the paths is very constructive but does not provide the connections between the colored HOMFLY and the fundamental ones. To find these connections the definition which uses the R-matrices should be constructed.

This can be done using the operator properties of the R-matrices:

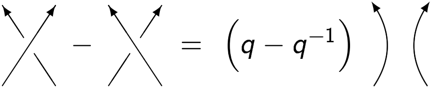
The projector on eigenvalue λ_i of the operator with the characteristic equation $\prod_{i=1}^{n}(\hat{R}-\lambda_i)=0$ is

$$\hat{P}_{\lambda_j}=\prod_{i\neq j}\frac{\hat{R}-\lambda_i}{\lambda_j-\lambda_i}$$

Characteristic equations for the R-matrices

For the fundamental R-matrices the characteristic equation is known from the mathematics and is described by the skein relations:

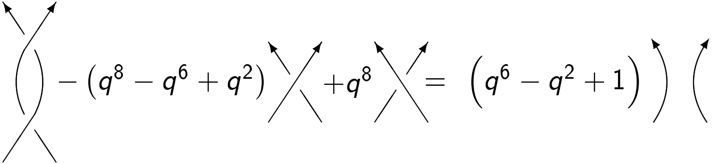
$$R - R^{-1} - \left(q - q^{-1}\right) = (R - q)\left(R + \frac{1}{q}\right) = 0$$



Colored skein relations

From the characteristic equation the analogue of the skein relations for the colored $\mathcal{R}$-matrices can be constructed though it is not very useful:

$$\left(\mathcal{R}_{2\otimes 2}-q^6\right)\left(\mathcal{R}_{2\otimes 2}+q^2\right)\left(\mathcal{R}_{2\otimes 2}-1\right)=0$$



The projectors on [2] and [11]

From the characteristic equation the projectors can be constructed

$$(R-q)\left(R+\frac{1}{q}\right)=0$$

$$\Downarrow$$

$$P_2=-\frac{R_1+q^{-1}}{q+q^{-1}} \qquad P_{11}=\frac{R_1-q}{q+q^{-1}}$$

The projectors on [21]

There are also additional characteristic equations for the combinations of the R-matrices:

$$(R_1 - R_2)\Big((R_1 - R_2)^2 - (q^2 + 1 + q^{-2})\Big) = 0$$

Together with the characteristic equations on each of the R-matrices this gives

$$P_{21} = \frac{(R_1 - R_2)^2}{q^2 + 1 + q^{-2}}$$

Recursive formula for the projector

$$B_{|Q|} = \left(\prod_{i=1}^{|Q|} R_i\right)\left(\prod_{j=1}^{|Q|} R_{|Q|+1-j}\right)$$

The projector on the representation Q made from T with the addition of the cell $\{k,l\}$ is equal to

$$P_{Q=T\cup(k,l)} = \sum_i \prod_{(i,j)\neq(k,l)} \frac{B_{|Q|} - q^{2j-2i}}{q^{2l-2k} - q^{2j-2i}} P_T$$

The product is over all possible additions of cells $\{i,j\}$.