Superqubits

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Introduction

Superqubits

Super entanglement classification

Introduction

Entanglement and information

- In providing nonlocal resources, quantum mechanics distinguishes itself from the classical world.
- Perhaps the best known example of a nonlocal resource is the EPR-pair, introduced by Einstein, Podolsky and Rosen [1935]

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- This fundamental shift is our understanding of reality has more recently precipitated a generalisation of *information theory*: *qubits*

$$0 \quad \text{or} \quad 1 \quad \rightarrow \quad a_A|A\rangle = a_0|0\rangle + a_1|1\rangle \in \mathbb{C}^2$$

$$0010\ldots0101 \quad \rightarrow \quad a_{A_1A_2\ldots A_n}|A_1A_2\ldots A_n\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \ldots \otimes \mathbb{C}^2$$

- Entanglement classification is an essential open problem

Introduction

Supersymmetric quantum information

- Here we propose a supersymmetric generalization of the qubit, the *superqubit*

$$|\Psi\rangle = |A\rangle a_A + |\bullet\rangle a_\bullet$$

- A single superqubit forms a 3-dimensional representation of OSp(1|2) consisting of two commuting "bosonic" components and one anticommuting "fermionic" component.

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Superqubit entanglement

- Super-Bell and super-GHZ states are characterized, respectively, by nonvanishing superdeterminant (distinct from the Berezinian) and superhyperdeterminant

Entanglement: Non-local games

The 3-player game

[Greenberger, Horne, Zeilinger: 1989; Mermin: 1990; Watrous et al: 2004]

- *Players* (Alice, Bob, Charlie. . .): act cooperatively in order to win.
- *Referee*: coordinates the game, question set $\mathcal{Q}$ and answer set $\mathcal{A}$

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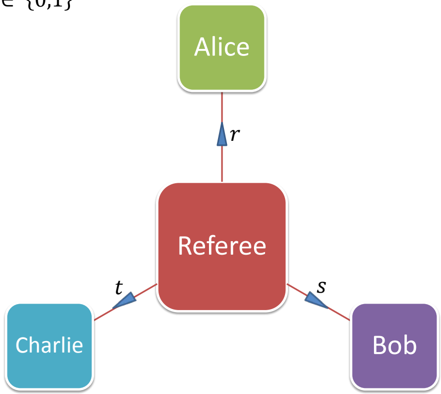
$r, s, t \in \{0,1\}$



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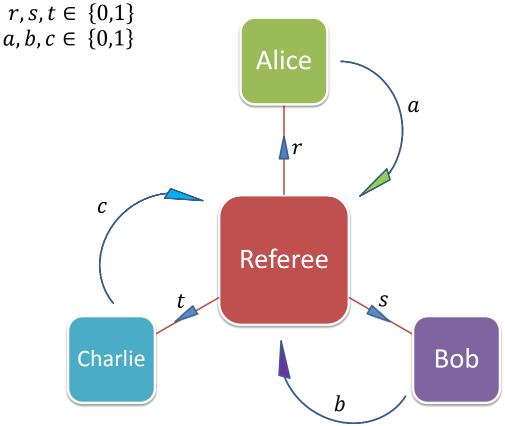

Entanglement: Non-local games

The 3-player game: rules for winning

- The referee ensures that $rst \in \{000, 110, 101, 011\}$ and the players are aware of this.
- The players win if $r \vee s \vee t = a \oplus b \oplus c$, where $\vee$ and $\oplus$ respectively denote disjunction and addition mod 2:

rs	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

The 3-player game: Deterministic strategy

- What is the best possible classical *deterministic* strategy?
- A deterministic strategy amounts to specifying three functions, one for each player, from the question set $\mathcal{Q}$ to the answer set $\mathcal{A}$

$$
\begin{aligned}
a &: \mathcal{Q} \to \mathcal{A}; \quad r \mapsto a(r),\\
b &: \mathcal{Q} \to \mathcal{A}; \quad s \mapsto b(s),\\
c &: \mathcal{Q} \to \mathcal{A}; \quad t \mapsto c(t),
\end{aligned}
$$

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$$\begin{aligned} a &: \mathcal{Q} \to \mathcal{A}; \quad r \mapsto a(r), \\ b &: \mathcal{Q} \to \mathcal{A}; \quad s \mapsto b(s), \\ c &: \mathcal{Q} \to \mathcal{A}; \quad t \mapsto c(t), \end{aligned}$$

The condition that the players win may then be written as,

$$\begin{aligned} a(0) \oplus b(0) \oplus c(0) &= 0, \\ a(1) \oplus b(1) \oplus c(0) &= 1, \\ a(1) \oplus b(0) \oplus c(1) &= 1, \\ a(0) \oplus b(1) \oplus c(1) &= 1. \end{aligned}$$

- Add mod 2 $\to$ contradiction $\Rightarrow$ best one can do is win 75% of the time

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In this language the contradiction with local realism exposed by Mermin translates into the existence of a local strategy utilising the GHZ state that wins the game with $p_{win} > 0.75$.

The 3-player game: Quantum strategy

Alice, Bob and Charlie share an entangled GHZ state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right)$$

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Alice, Bob and Charlie share an entangled GHZ state:

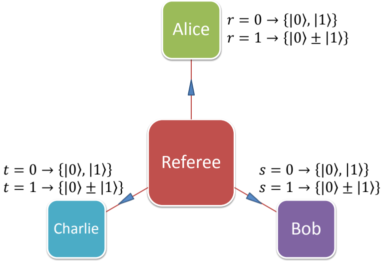
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right)$$

- If $r, s, t = 0$ then Alice , Bob, Charlie measure in the computational basis

$$\{|0\rangle,\ |1\rangle\}$$

- If $r, s, t = 1$ then Alice , Bob, Charlie measure in the Hadamard basis

$$\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$$



The 3-player game: Quantum strategy

Winning probability

1. $rst = 000$: All measure in the computational basis

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$\Rightarrow a \oplus b \oplus c = 0$. Always win.

The 3-player game: Quantum strategy

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2. $rst = \{011, 101, 110\}$: Two measure in the Hadamard basis

$$\mathbb{1} \otimes H \otimes H|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|001\rangle + |010\rangle - |100\rangle + |111\rangle\right)$$

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Conclusion

With an entangled resource we can win the game with certainty:

At the tables quantum players beat their poor classical cousins!

[Pan et al:2000]

Different kinds of entanglement

Comparing entanglement properties

- Using the totally entangled state GHZ state

$$|\Psi\rangle_{GHZ} = \frac{1}{\sqrt{2}} \left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right)$$

With a single run we can rule out local realist hidden model theories

- However, playing the same game this totally entangled state

$$|\Psi\rangle_{W} = \frac{1}{\sqrt{2}}(|001\rangle + |010\rangle + |100\rangle)$$

we only win with $p_{win}(W) = 7/8$. Cannot win with certainty [LB:2013]

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- How are we to mathematically distinguish the properties of W and GHZ?

Measuring and classifying entanglement

Entanglement classes

- Two n-qubit states have the same entanglement iff they are related by the equivalence group of *Stochastic Local Operations and Classical Communication* (SLOCC)

$$[\mathrm{SL}(2, \mathbb{C})]^{\otimes n}$$

- "gauge" group of n-qubit entanglement
- The space of entanglement classes:

$$\frac{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \mathbb{C}^2}{\mathrm{SL}_1(2, \mathbb{C}) \times \mathrm{SL}_2(2, \mathbb{C}) \times \ldots \mathrm{SL}_n(2, \mathbb{C})}$$

[Bennett et al:1999, Dur et al:2000]

2-qubit entanglement classification

$$|\psi\rangle = a_{AB}|AB\rangle$$

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$$|\psi\rangle = a_{AB}|AB\rangle$$

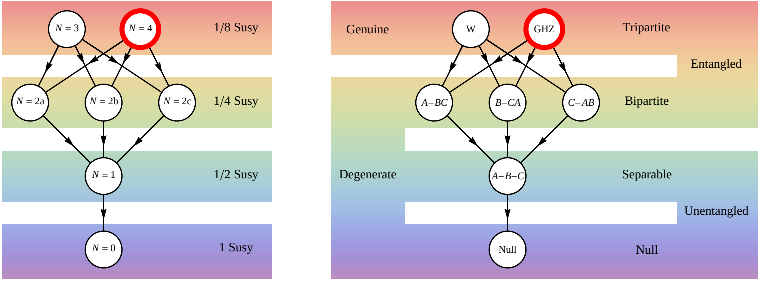
Just two entanglement classes separated by the $\mathrm{SL}_A(2,\mathbb{C}) \times \mathrm{SL}_A(2,\mathbb{C})$-invariant

$$\det a_{AB}$$

1. Separable A-B: $\det a_{AB} = 0$
2. Entangled EPR: $\det a_{AB} \neq 0$

3-qubit entanglement classification

$$|\psi\rangle = a_{ABC}|ABC\rangle$$



- There are 6 entanglement classes [Dur-Vidal:2000]
- SLOCC→ Freudenthal Triple Systems and "Groups of type E_7": $\mathbb{HA}/\mathbb{AH}$ row/column the magic square, cf. MJ Duff's lectures [Ferrara-Gunaydin:1997;LB-Dahanayake-Duff-Ebrahim-Rubens:2008, 2009]

3-qubit entanglement classification

Six entanglement classes are separated by $[\mathrm{SL}(2,\mathbb{C})]^3$-covariants [LB-Dahanayake-Duff-Ebrahim-Rubens:2009]:

1. Separable A-B-C: $\gamma^A = \gamma^B = \gamma^C = 0$

$$(\gamma^A)_{A_1 A_2} := a_{A_1}{}^{BC} a_{A_2 BC}, \quad (\gamma^B)_{B_1 B_2} := a^A{}_{B_1}{}^C a_{A B_2 C}, \quad (\gamma^C)_{C_1 C_2} := a^{AB}{}_{C_1} a_{ABC_2},$$

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3. Totally entangled W states: $T_{ABC} \neq 0$ and $\mathrm{Det}\, a_{ABC} = 0$

   $$T_{ABC} := (\gamma^A)_{AA'} a^{A'}{}_{BC}$$

   and $\mathrm{Det}\, a$ is Cayley's Hyperdeterminant [Cayley:1845]

   $$\mathrm{Det}\, a_{ABC} := -\det \gamma^A = -\det \gamma^B = -\det \gamma^C$$

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$\operatorname{Det} a = 0 \Rightarrow$ no strategy that wins the 3-player game with certainty [LB:2013]

Introduction

Superqubits

Super entanglement classification

The superqubit

The superqubit [LB-Dahanayake-Duff-Rubens: 2009]:

$$\begin{aligned}|\Psi\rangle &= |X\rangle a_X \\ &= |A\rangle a_A + |\bullet\rangle a_\bullet\end{aligned}$$

where a_A $(A = 0, 1)$ and $a_\bullet$ are commuting/anticommuting supernumbers

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- Even element of a *super Hilbert space* [DeWitt:1984, Rogers:1980, Rudolph:2000]:

$$^\ddagger : \mathcal{H} \to \mathcal{H}^\ddagger; \quad |\psi\rangle \mapsto (|\psi\rangle)^\ddagger := \langle\psi|.$$

1. ‡ sends pure bosonic (fermionic) supervectors in $\mathcal{H}$ into bosonic (fermionic) supervectors in $\mathcal{H}^\ddagger$.
2. ‡ is linear
$$(|\psi\rangle + |\phi\rangle)^\ddagger = \langle\psi| + \langle\phi|.$$
3. For pure even/odd α and $|\psi\rangle$
$$(|\psi\rangle\alpha)^\ddagger = (-)^{\alpha\psi}\alpha^\#\langle\psi|, \quad (\alpha\langle\psi|)^\ddagger = (-)^{\psi+\alpha\psi}|\psi\rangle\alpha^\#,$$
where $^\#$ is the superconjugate.

Superqubit norm

The even Grassmann valued norm

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2} a^{\#}_{A_1} a_{A_2} - a^{\#}_{\bullet} a_{\bullet},$$

is invariant under the *supergroup* of local unitaries

$$\mathrm{SU}(2) \longrightarrow \mathrm{UOSp}(2|1)$$

[Berezin-Tolstoy:1981]

- A normalized superqubit $|\Psi\rangle$ may be regarded as an element of the projective space [Landi:1999]

$$S^{2|2} = \mathrm{UOSp}(1|2)/\,\mathrm{U}(0|1)$$

  known as the *supersphere*

$$|\Psi\rangle = Z(\eta,\alpha,\beta)|0\rangle, \quad \text{where} \quad Z \in UOsp(2|1)$$

$$Z(\eta,\alpha,\beta) = S(\eta)U(\alpha,\beta) = \begin{pmatrix} 1+\frac{1}{4}\eta\eta^{\#} & -\frac{\eta}{2} & \frac{\eta^{\#}}{2} \\ -\frac{\eta^{\#}}{2} & 1-\frac{1}{8}\eta\eta^{\#} & 0 \\ -\frac{\eta}{2} & 0 & 1-\frac{1}{8}\eta\eta^{\#} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\beta^{\#} \\ 0 & \beta & \alpha^{\#} \end{pmatrix}$$

Superqubits

2-superqubit

$$|\Psi\rangle = |AB\rangle a_{AB} + |A\bullet\rangle a_{A\bullet} + |\bullet B\rangle a_{\bullet B} + |\bullet\bullet\rangle a_{\bullet\bullet}$$

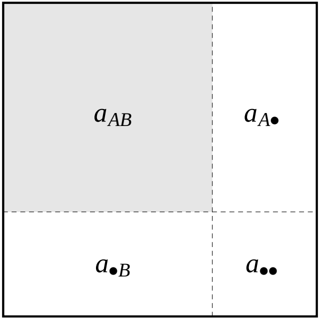

Figure : The 3×3 square supermatrix

Superqubits

3-superqubit

$$|\Psi\rangle = |ABC\rangle a_{ABC} + |AB\bullet\rangle a_{AB\bullet} + |A\bullet C\rangle a_{A\bullet C} + |\bullet BC\rangle a_{\bullet BC}$$
$$+|A\bullet\bullet\rangle a_{A\bullet\bullet} + |\bullet B\bullet\rangle a_{\bullet B\bullet} + |\bullet\bullet C\rangle a_{\bullet\bullet C}$$
$$+|\bullet\bullet\bullet\rangle a_{\bullet\bullet\bullet}$$

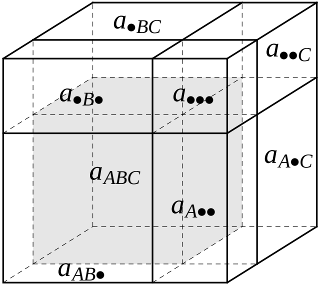

Figure : The $3 \times 3 \times 3$ cubic superhypermatrix

Introduction

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Super entanglement classification

Super entanglement classification

The super SLOCC group

$$\boxed{\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{OSp}(2|1)}$$

[Berezin-Tolstoy:1981]

Super entanglement classification

The super SLOCC group

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2 superqubits: the superdeterminant

$$\mathrm{SL}_A(2,\mathbb{C}) \times \mathrm{SL}_B(2,\mathbb{C}) \longrightarrow \mathrm{OSp}_A(2|1) \times \mathrm{OSp}_B(2|1)$$

$\mathrm{OSp}_A(2|1) \times \mathrm{OSp}_B(2|1)$-invariant superdeterminant

$$\begin{aligned}\operatorname{sdet} a_{XY} &= \frac{1}{2}(a^{AB}a_{AB} - a^{A\bullet}a_{A\bullet} - a^{\bullet B}a_{\bullet B} - a^{\bullet\bullet}a_{\bullet\bullet}) \\ &= (a_{00}a_{11} - a_{01}a_{10} + a_{0\bullet}a_{1\bullet} + a_{\bullet 0}a_{\bullet 1}) - \frac{1}{2}{a_{\bullet\bullet}}^2,\end{aligned}$$

Super entanglement classification

The super SLOCC group

$$\boxed{\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{OSp}(2|1)}$$

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Not the Berezinian, but quite natural:

$$\det a_{AB} = \frac{1}{2}\mathrm{tr}[(a\varepsilon)^t \varepsilon a] \quad \longrightarrow \quad \mathrm{sdet}\, a_{XY} = \frac{1}{2}\mathrm{str}[(aE)^{st} E a]$$

3 superqubits: the superhyperdeterminant

Recall, for 3 qubits we had $[\mathrm{SL}(2,\mathbb{C})]^3$-covariants

$$a_{ABC}$$

$$\gamma^{A}_{A_1 A_2}$$

$$\gamma^{B}_{B_1 B_2}$$

$$\gamma^{C}_{C_1 C_2}$$

$$T_{ABC}$$

$$\mathrm{Det}\, a_{ABC}$$

3 superqubits: the superhyperdeterminant

Need $[\mathrm{OSp}(2|1)]^3$-supercovariants:

1. Quadratic $(\mathbf{5},\mathbf{1},\mathbf{1})$

$$\Gamma^{A}_{X_1X_2} := \left(\begin{array}{c|c}\gamma_{A_1A_2} & \gamma_{A_1\bullet}\\ \hline \gamma_{\bullet A_2} & \gamma_{\bullet\bullet}\end{array}\right) = \left(\begin{array}{c|c}\gamma_{A_1A_2} & \gamma_{A_1\bullet}\\ \hline \gamma_{A_2\bullet} & 0\end{array}\right),$$

$$\gamma_{A_1A_2} := a_{A_1}{}^{BC}a_{A_2BC} - a_{A_1}{}^{B\bullet}a_{A_2B\bullet} - a_{A_1}{}^{\bullet C}a_{A_2\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{A_2\bullet\bullet}$$

$$\gamma_{A_1\bullet} := a_{A_1}{}^{BC}a_{\bullet BC} + a_{A_1}{}^{B\bullet}a_{\bullet B\bullet} + a_{A_1}{}^{\bullet C}a_{\bullet\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{\bullet\bullet\bullet}$$

2. Cubic triple product $(\mathbf{3},\mathbf{3},\mathbf{3})$

$$T_{XYZ} = \Gamma^{A}_{XX'}a^{X'}{}_{YZ}.$$

3. Quartic superhyperdeterminant $(\mathbf{1},\mathbf{1},\mathbf{1})$

$$\mathrm{sDet}\, a_{XYZ} = -\,\mathrm{sdet}\,\Gamma^{A} = -\,\mathrm{sdet}\,\Gamma^{B} = -\,\mathrm{sdet}\,\Gamma^{C}$$

This is the unique quartic invariant first found in [Castellani-Grassi-Sommovigo:2010] in the context of $\mathcal{N}=2$ sugra.

Summary

3-superqubit entanglment

- All 3-qubit entanglement covariants may supersymmetrized in a very nature manner:

$$\begin{array}{ccc} a_{ABC} & \to & a_{XYZ} \\ \gamma_A & \to & \Gamma_X \\ \gamma_B & \to & \Gamma_Y \\ \gamma_C & \to & \Gamma_Z \\ T_{ABC} & \to & T_{XYZ} \\ \mathrm{Det}\, a_{ABC} & \to & \mathrm{sDet}\, a_{XYZ} \end{array}$$

  such that

$$\mathrm{sDet}\, a_{XYZ} = -\,\mathrm{sdet}\,\Gamma_X = -\,\mathrm{sdet}\,\Gamma_Y = -\,\mathrm{sdet}\,\Gamma_Z.$$

  cf.

$$\mathrm{Det}\, a_{ABC} = -\det\Gamma_A = -\det\Gamma_B = -\det\Gamma_C.$$

- Cayley's Hyperdeterminant may be supersymmtrized! The Superhyperdeterminant is invariant under $\mathrm{OSp}(2|1) \times \mathrm{OSp}(2|1) \times \mathrm{OSp}(2|1)$

Conclusions

- Here we proposed a supersymmetric generalization of the qubit, the *superqubit*.
- Not HEP susy:
  - Not a representation of the super-Poincaré group
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- In particular, super-Bell and super-GHZ states are characterized, respectively, by nonvanishing superdeterminant (distinct from the Berezinian) and superhyperdeterminant
- This mathematical construction seems a very natural one. From a physical point of view, it makes contact with various condensed-matter systems: supersymmetric t-J [Wiegmann:1988, Sarkar:1991, Mavromatos:1999] quantum Hall effect [Hasebe], supersymmetric valence-bond solid states [Hasebe-Totsuka].

Superqubits and nonlocality

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- In principle Nature could do better

$$S(\text{no-sulu}) \leq 4$$

without violating the no superluminal signaling principle [Popescu-Rohrlich:1994]

Superqubits and nonlocality

Can superqubits break Tsirlson's bound? [LB-Brádler-Duff:2012]

- Super resource:

$$\Gamma_{AB} = \left(1 + \frac{1}{2}X + \frac{3}{8}X^2\right)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{p}{\sqrt{2}}\theta_A|\bullet 1\rangle + \frac{q}{\sqrt{2}}\theta_B|1\bullet\rangle\right)$$

- Local superunitaries:

$$Z_{iA} \otimes Z_{jB} = S(2r_i\theta_A)U(\alpha_i, \beta_i) \otimes S(2s_j\theta_B)U(\gamma_j, \delta_j).$$

- Born rule:

$$p_{\mathcal{G}}(\varphi, \psi) = \langle\varphi|\psi\rangle\left(\langle\varphi|\psi\rangle\right)^{\#}.$$

  The rationale behind this definition is clear: for ordinary (non-Grassmann) states we recover the usual Born rule.

- Grassmann norm:

$$|\tau|_R \overset{\mathrm{df}}{=} \int \prod_i^n e^{\theta_i\theta_i^{\#}} \tau \mathrm{d}^{2n}\theta,$$

Beats Tsirelson's bound - but at a cost: extended probabilities!

Super SLOCC algebras

Table : The action of the $\mathfrak{osp}(1|2)$ generators on the superqubit fields.

Generator	Field acted upon	
	a_{A_3}	$a_\bullet$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)}$	0
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_\bullet$	a_{A_1}

Table : The action of the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ generators on the 2-superqubit fields.

Generator	Field acted upon: Bosons		Fermions	
	$a_{A_3B_3}$	$a_{\bullet\bullet}$	$a_{A_3\bullet}$	$a_{\bullet B_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)B_3}$	0	$\varepsilon_{(A_1\|A_3}a_{\|A_2)\bullet}$	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\|B_3}a_{A_3\|B_2)}$	0	0	$\varepsilon_{(B_1\|B_3}a_{\bullet\|B_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3}$	$a_{A_1\bullet}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet}$	$a_{A_1B_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet}$	$-a_{\bullet B_1}$	$a_{A_3B_1}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet}$

Super SLOCC algebras

Table : The action of the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ generators on the 3-superqubit fields.

Generator	Bosons acted upon			
	$a_{A_3B_3C_3}$	$a_{A_3\bullet\bullet}$	$a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)B_3C_3}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)\bullet\bullet}$	0	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\vert B_3}a_{A_3\vert B_3)C_2}$	0	$\varepsilon_{(B_1\vert B_3}a_{\bullet\vert A_2)\bullet}$	0
$P_{C_1C_2}$	$\varepsilon_{(C_1\vert C_3}a_{A_3B_3\vert C_2)}$	0	0	$\varepsilon_{(C_1\vert C_3}a_{\bullet\bullet\vert C_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3C_3}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet\bullet}$	$a_{A_1B_3\bullet}$	$a_{A_1\bullet C_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet C_3}$	$a_{A_3B_1\bullet}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet\bullet}$	$-a_{\bullet B_1C_3}$
$2Q_{C_1}$	$\varepsilon_{C_1C_3}a_{A_3B_3\bullet}$	$-a_{A_3\bullet C_1}$	$-a_{\bullet B_3C_1}$	$\varepsilon_{C_1C_3}a_{\bullet\bullet\bullet}$
	Fermions acted upon			
	$a_{A_3B_3\bullet}$	$a_{A_3\bullet C_3}$	$a_{\bullet B_3C_3}$	$a_{\bullet\bullet\bullet}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)B_3\bullet}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)\bullet C_3}$	0	0
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$2Q_{C_1}$	$a_{A_3B_3C_1}$	$-\varepsilon_{C_1C_3}a_{A_3\bullet\bullet}$	$-\varepsilon_{C_1C_3}a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_1}$