Analytical formulas, general properties, and calculation of transport coefficients in the hadron gas: shear and bulk viscosities

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O. Moroz, arXiv:1301.6670

Motivation for calculation of the shear (η) and bulk (ξ) viscosity coef.

- For evolution of a thermodynamic nonequilibrium system

- The η and ξ influence elliptic flow and expansion rate. Sensitivity of the ξ to the mass spectrum can be used to test m. s. hypotheses.
- The ξ is sensitive to particle number conservation/nonconservation and, hence, can be used for freeze-out criterion.
- To find possible new signatures of QGP.
- For study of the QCD phase transition:
  - The η/s is expected to have a minimum near the phase transition
  - The ξ/s (as well as the ξ) is expected to have a maximum near the phase transition and be dominant there

- The bound of the η/s

Conditions of applicability and approximations

- **Inelastic processes are neglected**
- **Boltzmann (classical) statistics is used**
- **Ideal gas equation of state is used**
- **Medium effects are neglected**

Good for T $\leq$ 140 MeV (zero chemical potential)

**At T = 140 MeV the upper error bound is up to a factor of 4.
The lower error bound is much smaller.**

**The Boltzmann equation is solved using variational method.
The errors do not exceed 10%.**

Comparison of calculations for the hadron gas

Previous calculations of the ξ

✓ **J. Noronha-Hostler, J. Noronha, C. Greiner, Phys. Rev. Lett. 103, 172302 (2009)**

Good agreement

Mass spectrum cut on 2 GeV, some special formula obtained using low energy theorem and an ansatz for the spectral density; quantitative accuracy is not clear. For 140 MeV<T<180 MeV the deviations are up a factor of 1.8. At T = 140 MeV they are equal to 4%.

✓ **A. S. Khvorostukhin, V. D. Toneev, D. N. Voskresensky, Nucl. Phys. A 845, 106 (2010)**

Worse agreement

Relaxation time approximation, not conserved particle numbers, medium effects, quantum statistics; *SHMC model and ideal gas results don't differ too much there.* The ξ is 8-58 times larger for 20 MeV<T<140 MeV (deviations grow as the temperature decreases)

The η doesn't differ considerably in previous works

- **M. I. Gorenstein, M. Hauer, O. N. Moroz, Phys. Rev. C77, 024911 (2008)**
- **A. S. Khvorostukhin, V. D. Toneev, D. N. Voskresensky, Nucl. Phys. A 845, 106 (2010)**
- **O. N. Moroz, arXiv:1112.0277**
- **…**

Numerical calculations at zero chemical potentials

UrQMD cross sections

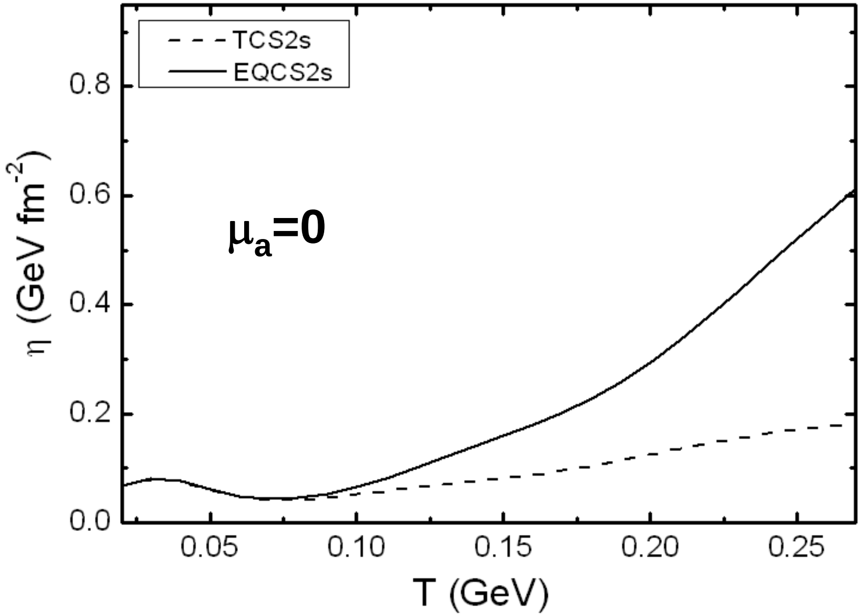
EQCS2 – elastic plus quasielastic cross section

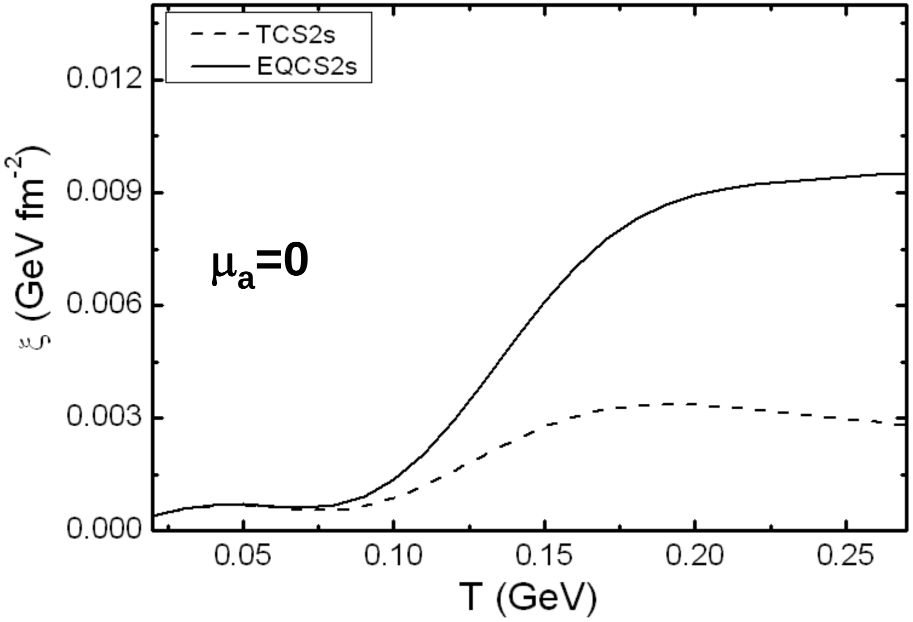
TCS2 – total cross section

“2” in the abbreviations designates taken into account decays of resonances

EQCS2 – for perturbative calculations

TCS2 – for qualitative calculations





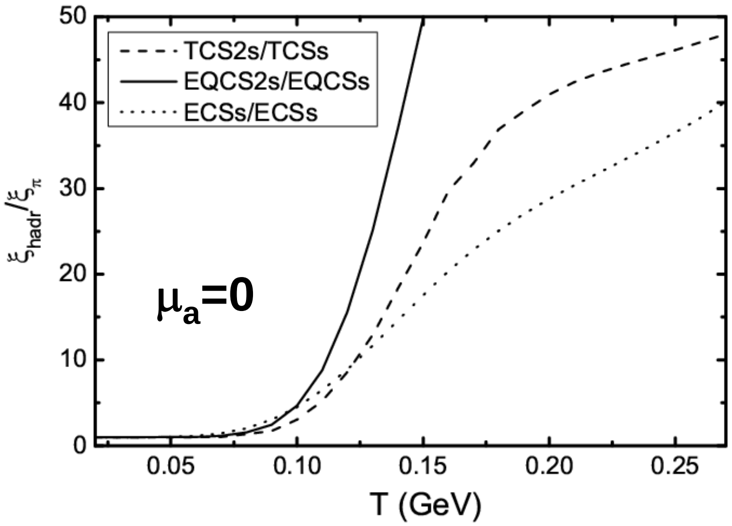
$(N' n)^2$ 12-D integrals;
N‘ - # of particle species,
n - # of used test-functions

322 particle species (including anti-particles) – hadrons with u, d, s quarks

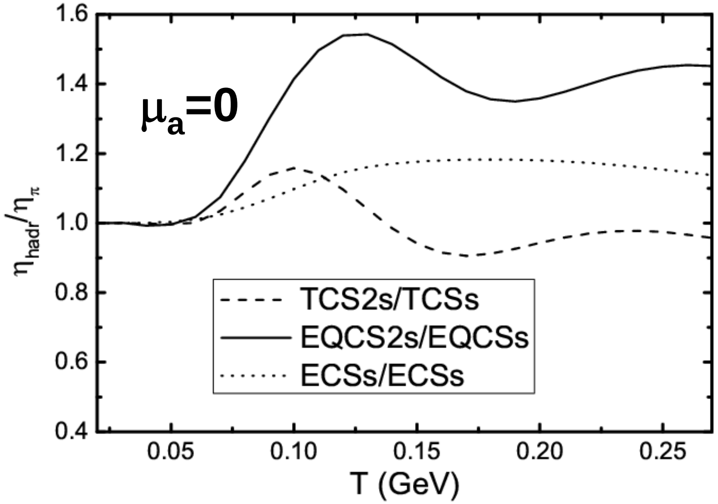
Calculations also above T_c because hadronic states survive above $T_c \approx$160 MeV:

Y. Hidaka and R. D. Pisarski,
Phys. Rev. D **78**, 071501 (2008)
A. Dumitru et al.,
Phys. Rev. D **83**, 034022 (2011)
C. Ratti et al.,
Phys. Rev. D **85**, 014004 (2012)
R. Bellwied,
PoS BORMIO **2012**, 045 (2012)

Mass spectrum dependence

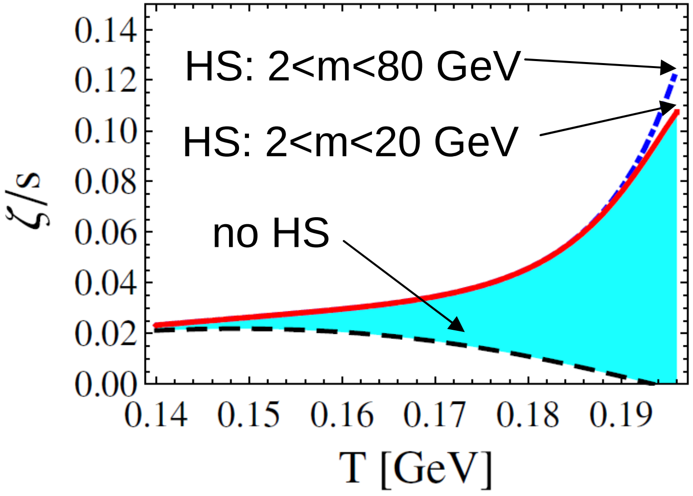


Comparison with pion gas



At T=120 (140) MeV the ξ_{hadr}/ξ_{pions}>8.6 (15)

The η_{hadr}/η_{pions} is not larger than 1.6.
The η_{hadr}/η_{pions} has smaller than 1 values for TCSs because hadronic TCSs are larger than pionic ones in average.



J. Noronha-Hostler, J. Noronha, C. Greiner, Phys. Rev. Lett. **103**, 172302 (2009)

Experimental data supports Hagedorn states up to m≈2 GeV

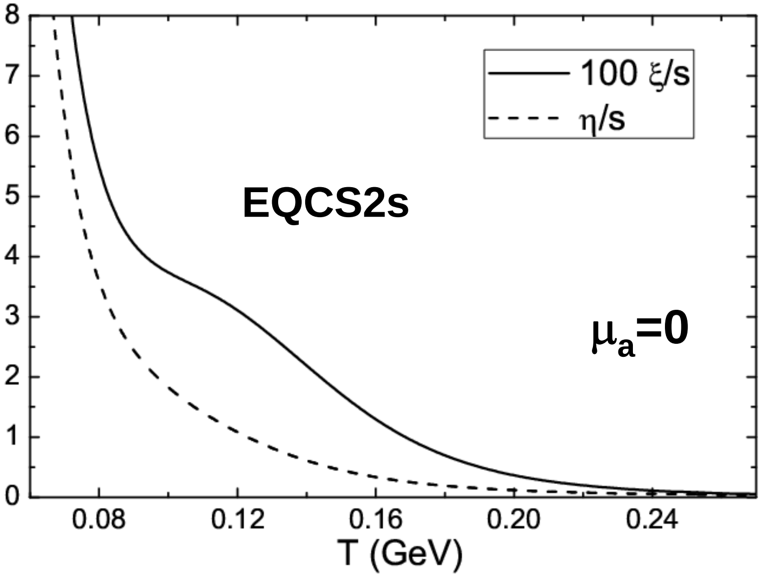
W. Broniowski, W. Florkowski and L. Y. Glozman, Phys. Rev. D **70**, 117503 (2004)

Quite large contribution from HS. Cumulative test

$$\rho_{HS}(m) = A\frac{e^{m/T_H}}{(m^2+m_0^2)^{5/4}},$$

A=0.5 GeV$^{3/2}$, m$_0$=0.5 GeV, T$_H$=196 MeV

Other calculations, elliptic flow and perspectives



Calculations of the elliptic flow (can be experimentally measured):

$\eta/s = 0.08$, $\xi/s = 0.04$ (in 2+1 D) P. Bozek, J. Phys. G **38**, 124043 (2011)

$\eta/s = 0.16$, $\xi/s = 0.04$, $\xi/s \lesssim 0.05$ at freeze-out

K. Dusling and T. Schafer, Phys. Rev. C **85**, 044909 (2012)

Rather in agreement

Perspectives

Viscosities differ very much in the hadron gas and in QGP.
This may lead to new QGP signatures (based on wave damping and shock waves formation): **D. A. Fogaca, F. S. Navarra and L. G. F. Filho, arXiv:1305.0798**

Single-component gas (1-st order)

J. L. Anderson, A. J. Kox, Physica, 1977, v. 89A, p. 408

$$z \equiv m/T$$

$$\eta = \frac{15}{64\pi}\frac{T}{r^2}\frac{z^2 K_2^2(z)\hat{h}^2}{(15z^2+2)K_2(2z)+(3z^3+49z)K_3(2z)}$$

incorrect value was 5

$$\gamma \equiv \frac{z^2+5\hat{h}-\hat{h}^2}{z^2+5\hat{h}-\hat{h}^2-1}$$

$$\hat{h} \equiv z\frac{K_3(z)}{K_2(z)}$$

$$\xi = \frac{1}{64\pi}\frac{T}{r^2}\frac{z^2 K_2^2(z)[(5-3\gamma)\hat{h}-3\gamma]^2}{2K_2(2z)+zK_3(2z)}$$

r – effective radius, $\sigma_{tot} = 4\pi\, r^2$, $K_n(z)$ – Bessel function

Corresponding distribution function in nonrelativistic and ultrarelativistic limits:

$$f \approx f^{(0)}(1+\varphi)$$

$$z \gg 1, \quad \varphi = \frac{5\pi e^{z-\hat{\mu}}}{32\sqrt{2}T^3 z^2 r^2}\left(-(\tau^2+2z\tau-z^2)\nabla_\mu U^\mu + 2\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\overset{\circ}{\overline{\nabla_\mu U_\nu}}\right)$$

$$z \ll 1, \quad \varphi = \frac{\pi e^{-\hat{\mu}}}{480 T^3 r^2}\left(-5z^2(\tau^2+8\tau-12)\nabla_\mu U^\mu + 36\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\overset{\circ}{\overline{\nabla_\mu U_\nu}}\right)$$

Binary mixture (1-st order)

r, s run over test-functions, K, l, m – over particle species

$$\xi = \frac{T}{\sigma(T)} \sum_k \sum_{r=0}^{n_1} x_k \alpha_k^r A_k^r, \qquad \eta = \frac{1}{10} \frac{T}{\sigma(T)} \sum_k \sum_{r=0}^{n_2} x_k \gamma_k^r C_k^r$$

α_k^r, γ_k^r – known functions, σ(T) – formal quantity

$$x_k \alpha_k^r = \sum_l \sum_{s=0}^{n_1} A_{lk}^{sr} A_l^s, \qquad x_k \gamma_k^r = \sum_l \sum_{s=0}^{n_2} C_{lk}^{sr} C_l^s$$

$$x_k \equiv n_k/n,$$

$$\pi_k^\mu \equiv p_k^\mu / T,$$

$$\tau_k \equiv \pi_k^\mu U_\mu,$$

$$A_{kl}^{rs} = x_k x_l [\tau^r, \tau_1^s]_{kl} + \delta_{kl} x_k \sum_m x_m [\tau^r, \tau^s]_{km}$$

$$C_{kl}^{rs} = x_k x_l [\tau^r \overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \tau_1^s \overset{\circ}{\overline{\pi_{1\mu} \pi_{1\nu}}}]_{kl} + \delta_{kl} x_k \sum_m x_m [\tau^r \overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \tau^s \overset{\circ}{\overline{\pi_\mu \pi_\nu}}]_{km}$$

$\overset{\circ}{\overline{\pi^\mu \pi^\nu}}$ **- traceless, symmetric**

Binary mixture:

1-component s. v.

$$z_2 \gg 1 \qquad \eta = \eta_{11} + \Delta\eta, \qquad \Delta\eta = z_2^{5/2} e^{-z_2} \frac{3 T g_2 e^{z_1 - \hat{\mu}_1 + \hat{\mu}_2}}{64\sqrt{2\pi}(3 + 3z_1 + z_1^2) g_1 \sigma_{12}^{cl}}$$

$$z_2 \gg 1 \qquad \xi = e^{-z_2} z_2^{5/2} \frac{g_2 T e^{-\hat{\mu}_1 + \hat{\mu}_2 + z_1} [2z_1^2 - 5 - 2\hat{h}_1^2 + 10\hat{h}_1]^2}{128\sqrt{2\pi} g_1 \sigma_{12}^{cl} (z_1^2 + 3z_1 + 3)[z_1^2 - 1 - \hat{h}_1^2 + 5\hat{h}_1]^2} + \ldots$$

Lowest order collision brackets

Introducing the assignments

$$\widetilde{K}_1 \equiv \frac{K_3(z_k+z_l)}{K_2(z_k)K_2(z_l)}, \quad \widetilde{K}_2 \equiv \frac{K_2(z_k+z_l)}{K_2(z_k)K_2(z_l)}, \quad \widetilde{K}_3 \equiv \frac{G(z_k+z_l)}{K_2(z_k)K_2(z_l)}$$

$$Z_{kl} \equiv z_k + z_l, \quad z_{kl} \equiv z_k - z_l$$

$$G(x) = x^{-3}\int_1^\infty du u^{-4} K_3(xu) = \frac{1}{32} G^{3,0}_{1,3}\left(\frac{x}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ -3, 0, 0 \end{matrix} \right)$$

Meijer function

the lowest order tensor collision brackets can be written as

$$[\overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \overset{\circ}{\overline{\pi_{1\mu}\pi_{1\nu}}}]_{kl} = \frac{\pi}{72 z_k^2 z_l^2 Z_{kl}^2}\left(P^{(0,0)}_{T11}\widetilde{K}_1 + P^{(0,0)}_{T12}\widetilde{K}_2 + P^{(0,0)}_{T13}\widetilde{K}_3\right)$$

$$P^{(0,0)}_{T11} = -2Z_{kl}[z_{kl}^6(5Z_{kl}^2-8)+24z_{kl}^4(Z_{kl}^2-16)-144z_{kl}^2Z_{kl}^2(Z_{kl}^2+8)+48Z_{kl}^4(Z_{kl}^2+72)]$$

$$P^{(0,0)}_{T12} = z_{kl}^6(5Z_{kl}^4-40Z_{kl}^2-64)-24z_{kl}^4(5Z_{kl}^4+8Z_{kl}^2+128)+576z_{kl}^2Z_{kl}^2(Z_{kl}^2-16)-192Z_{kl}^4(5Z_{kl}^2+16)$$

$$P^{(0,0)}_{T13} = -5z_{kl}^4Z_{kl}^6[z_{kl}^2(Z_{kl}^2-24)-24Z_{kl}^2]$$

Lowest order collision brackets

$$[\overset{\circ}{\overline{\pi^{\mu}\pi^{\nu}}}, \overset{\circ}{\overline{\pi_{\mu}\pi_{\nu}}}]_{kl} = \frac{\pi}{72 z_k^2 z_l^2 Z_{kl}^2}\left(P_{T21}^{(0,0)}\widetilde{K}_1 + P_{T22}^{(0,0)}\widetilde{K}_2 + P_{T23}^{(0,0)}\widetilde{K}_3\right)$$

$$\begin{aligned} P_{T21}^{(0,0)} &= 2Z_{kl}[z_{kl}^6(8-5Z_{kl}^2) + 72z_{kl}^4(3Z_{kl}^2-8) - 480z_{kl}^3 Z_{kl}(Z_{kl}^2-4) \\ &- 336z_{kl}^2 Z_{kl}^2(Z_{kl}^2+8) + 240z_{kl}Z_{kl}^3(Z_{kl}^2+8) + 192Z_{kl}^4(Z_{kl}^2+67)], \end{aligned}$$

$$\begin{aligned} P_{T22}^{(0,0)} &= z_{kl}^6(5Z_{kl}^4 - 40Z_{kl}^2 - 64) + 240z_{kl}^5 Z_{kl}^3 - 24z_{kl}^4(5Z_{kl}^4 + 48Z_{kl}^2 - 192) \\ &+ 1920z_{kl}^3 Z_{kl}(Z_{kl}^2-8) - 192z_{kl}^2 Z_{kl}^2(17Z_{kl}^2 - 112) + 1920z_{kl}Z_{kl}^3(Z_{kl}^2-8) \\ &+ 768Z_{kl}^4(5Z_{kl}^2+6), \end{aligned}$$

$$P_{T23}^{(0,0)} = -5z_{kl}^4 Z_{kl}^6[z_{kl}^2(Z_{kl}^2-24) + 48z_{kl}Z_{kl} - 24Z_{kl}^2]$$

Collision rates and mean free paths

The k’-l’ number of collisions per unit time per unit volume is

$$\widetilde{R}^{el}_{k'l'} \equiv \int \gamma_{k'l'} \boxed{g_{k'} \frac{d^3p_{k'}}{(2\pi)^3} f^{(0)}_{k'}} \boxed{g_{l'} \frac{d^3p_{1l'}}{(2\pi)^3} f^{(0)}_{1l'}} \gamma_{k'l'} \boxed{d^3p'_{k'} d^3p'_{1l'} \frac{W_{k'l'}}{p^0_{k'} p^0_{1l'} p'^0_{k'} p'^0_{1l'}}}$$

particle numbers

collision probability per unit time times unit volume (from analysis of collision integrals or from explicit expression through collision amplitudes)

$$\widetilde{R}^{el}_{k'l'} = g_{k'} g_{l'} \gamma_{k'l'} \frac{2\sigma^{cl}_{k'l'} T^6}{\pi^3} \left[(z_{k'} - z_{l'})^2 K_2(z_{k'} + z_{l'}) + z_{k'} z_{l'} (z_{k'} + z_{l'}) K_3(z_{k'} + z_{l'}) \right]$$

The k’-l’ collision rate *per particle of the k’-th species* is

$$R^{el}_{k'l'} \equiv g_{k'} g_{l'} \frac{\gamma_{k'l'}}{(2\pi)^6 n_{k'}} \int \frac{d^3p_{k'}}{p^0_{k'}} \frac{d^3p_{1l'}}{p^0_{1l'}} \frac{d^3p'_{k'}}{p'^0_{k'}} \frac{d^3p'_{1l'}}{p'^0_{1l'}} f^{(0)}_{k'} f^{(0)}_{1l'} W_{k'l'} = \frac{\widetilde{R}^{el}_{k'l'}}{\gamma_{k'l'} n_{k'}}$$

The mean free time of particle of the k’-th species is $\boxed{t^{el}_{k'} = \frac{1}{R^{el}_{k'}}}$ $\quad R^{el}_{k'} \equiv \sum_{l'} R^{el}_{k'l'}$

Collision rates and mean free paths

The mean free path of particle of the k'-th species is $\boxed{l^{el}_{k'} = \frac{\langle|\vec{v}_{k'}|\rangle}{R^{el}_{k'}}}$

$$\langle|\vec{v}_{k'}|\rangle = \frac{\int d^3p_{k'} \frac{|\vec{p}_{k'}|}{p^0_{k'}} f^{(0)}_{k'}(p_{k'})}{\int d^3p_{k'} f^{(0)}_{k'}(p_{k'})} = \frac{2e^{-z_{k'}}(1+z_{k'})}{z^2_{k'}K_2(z_{k'})} = \sqrt{\frac{8}{\pi z_{k'}}}\frac{K_{3/2}(z_{k'})}{K_2(z_{k'})}$$

The nonrelativistic limit of the $R^{el}_{k' l'}$ reproduces known expression in the nonrelativistic kinetic-molecular theory

$$n_{l'} 4\pi\sigma^{cl}_{k'l'}\langle|\vec{v}_{k'}|\rangle\sqrt{1+m_{k'}/m_{l'}}$$

In single-component gas the nonrelativistic limit of the mean free path

$$l^{el}_{1'} = \frac{\langle|\vec{v}_{1'}|\rangle}{R^{el}_{1'1'}} = \frac{\pi e^{-z_1}(z_1+1)}{g_1 4\sigma^{cl}_{11}T^3 z^3_1 K_3(2z_1)}$$

reproduces known expression for the Maxwell's mean free path (g_1=1)

$$l^{el}_1 = \frac{1}{4\pi\sigma^{cl}_{11}n_1\sqrt{2}} = \frac{1}{\sigma_{tot}n_1\sqrt{2}}$$

Conclusions

- The shear and the bulk viscosities of the hadron gas are calculated using UrQMD cross sections:
  - quantitatively at $T \leq 140$ MeV
  - qualitatively at higher temperatures
- The bulk viscosity of the hadron gas has strong mass spectrum dependence and at T = 120 MeV and z. c. p. it is larger in 8.6-15.6 times than the bulk viscosity of the pion gas. More accurate calculations are desirable to distinguish between different mass spectrum hypotheses.
- Rapid rise of the bulk viscosity can be used as a natural criterion for the chemical freeze-out. Numerical Kubo calculations, not involving approximation of conserved or not conserved particle numbers, are desirable for this.
- Analytical expressions are obtained:
  - correct (relativistic) shear viscosity in a single-component gas
  - nonequilibrium distribution function in the single-component gas
  - expressions for binary mixture
  - collision brackets for multi-component mixtures
  - collision rates and related quantities
- My calculations support there is a minimum of the η/s and maxima of the ξ and the ξ/s near the phase transition.

Thank you for your attention!

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Grazie per l’attenzione!

Definitions of the viscosity coefficients and Kubo formulas

Definitions:

$$\nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu, \quad \Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu$$

$$T^{(0)\mu\nu} = \epsilon U^\mu U^\nu - P\Delta^{\mu\nu}, \qquad T^{(1)\mu\nu} \equiv \eta \left(\Delta_\rho^\mu \Delta_\tau^\nu + \Delta_\rho^\nu \Delta_\tau^\mu - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\rho\tau} \right) \nabla^\rho U^\tau + \xi \Delta^{\mu\nu} \nabla_\rho U^\rho$$

shear v. c.

bulk v. c.

$$\Downarrow$$

$$P_{full} = P - \xi \nabla_\rho U^\rho$$

Kubo (or Green-Kubo) formulas can be derived within the linear response:

$$\xi = \lim_{\omega \to 0} \frac{1}{2\omega} \int d^4x e^{i\omega t} \langle [\overline{\mathcal{P}}(x), \overline{\mathcal{P}}(0)] \rangle_{eq} \qquad \overline{\mathcal{P}}(x) \equiv \frac{1}{3} T_i^i(x) - v_s^2 T_{00}(x)$$

$$\eta = \lim_{\omega \to 0} \frac{1}{20\omega} \int d^4x e^{i\omega t} \langle [\pi_{lm}(x), \pi^{lm}(0)] \rangle_{eq} \qquad \pi_{lm}(x) \equiv T_{lm}(x) - \frac{1}{3}\delta_{lm} T_i^i(x)$$

Zero frequency and zero momentum limits to avoid possible nonphysical contributions

L. P. Kadanoff and P. C. Martin, Ann. Phys. **24**, 1, 419 (1963)

Connection between Kubo formula and Boltzmann equation

Linear integral eqn. for unknown A(p). Needed for the bulk viscosity of the ϕ^4 field theory

S. Jeon, Phys. Rev. D **52**, 3591 (1995)
S. Jeon, L. G. Yaffe, Phys. Rev. D **53**, 5799 (1996)

Zero chemical potential μ=0

$$\frac{1}{3}\vec{p}^2 - v_s^2(\vec{p}^2+\tilde{m}^2)$$

$$= \frac{1}{4}\int\prod_{i=1}^{3}\frac{d^3p_i}{2E_i(2\pi)^3}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1+p_2-p_3-p)(1+f_1)(1+f_2)f_3f_p(A(p_1)+A(p_2)-A(p_3)-A(p))$$

$$+\text{"}2\leftrightarrow 4\text{"} \equiv (\hat{C}_{el}+\hat{C}_{inel})A \equiv \hat{C}A$$

$\tilde{m}^2 \equiv m_{th}^2 - T^2\frac{\partial m_{th}^2}{\partial T^2}$ - “subtracted” mass

$m_{th}^2 = m_0^2 + \delta m^2$

$\delta m^2 \propto \lambda T^2$ - perturbative leading contribution

$f_i \equiv f(p_i)$ - distribution function

$|\mathcal{M}|^2$ - the square of the dimensionless scattering amplitude

A similar equation for the shear viscosity was obtained too.

Similar calculations were done for gauge theories too:

J. S. Gagnon and S. Jeon, Phys. Rev. D **75**, 025014 (2007)

J. S. Gagnon and S. Jeon, Phys. Rev. D **76**, 105019 (2007)

Approximate zero modes – physical?

$|I\rangle = \hat{C}|A\rangle$ - the linear integral eqn. rewritten $\langle a|b\rangle \equiv \int \frac{d^3p}{(2\pi)^3} e^{-E/T} ab$

Inserting $1 = \sum_k |A_k\rangle\langle A_k|,$ A_k are eigenfunctions of the collision operator $\hat{C}$

one gets $\xi \propto \langle I|A\rangle = \langle I|\hat{C}^{-1}|I\rangle = \sum_k C_k^{-1}(\langle I|A_k\rangle)^2 \approx C_0^{-1}(\langle I|A_0\rangle)^2, \quad C_0 \ll C_i, \quad i \neq 0$

where $\hat{C}|A_0\rangle = \hat{C}_{inel}|A_0\rangle = C_0|A_0\rangle$ S. Jeon, Phys. Rev. D **52**, 3591 (1995)

And for ϕ^4 $\xi \propto e^{2m/T}, \quad T \ll m, \qquad C_0 \propto e^{-2m/T}$ - suppression from inelastic 2↔4

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S. Jeon, Phys. Rev. D **52**, 3591 (1995)

And for ϕ^4 $\xi \propto e^{2m/T},$ $T \ll m,$ $C_0 \propto e^{-2m/T}$ - suppression from inelastic 2↔4

How physical is this dependence?

Physical, without reservations, according to some literature

?

⋮

general issue

L. D. Landau and E.M. Lifshitz, Fluid Mechanics (1959)

⋮

E. Lu, G. D. Moore, Phys. Rev. C **83**, 044901 (2011)

⋮

Approximate zero modes – physical?

Arguments supporting the approximate zero modes (AZM) are physical:

- Needed for the chemical equilibration to restore full equilibrium
- Zero frequency limit in Kubo formula requires to consider anyhow long processes

S. Jeon, L. G. Yaffe, Phys. Rev. D **53**, 5799 (1996):

For applicability compare inelastic m. f. p. and the distance on which hydrodynamic description is considered.

My arguments and what I propose:

- No infinite timescale in the definition of the bulk viscosity
- Chemical equilibration is achieved through other perturbations
- Example when and how AZM are definitely nonphysical
- Gradients relaxation time (GRT) is an important timescale
- A physically motivated criterion of whether AZM are physical

There are also obvious limitations: mean free time of inelastic process should be smaller than the time of existence of the thermal part of a system.

Chemical equilibration

Boltzmann equation (BE): $p_k^\mu \partial_\mu f_k = C_k^{el}[f_k] + C_k^{inel}[f_k]$

$$f_k = f_k^{(0)}(1+\tilde{\varphi}_k)(1+\varphi_k) \approx f_k^{(0)}(1+\tilde{\varphi}_k+\varphi_k), \quad |\tilde{\varphi}_k| \ll 1, \quad |\varphi_k| \ll 1$$

φ_k - proportional to gradients, $\tilde{\varphi}_k$ - chemical perturbations, $f_k^{(0)}$ - equilibrium d. f.

linearized col. int.

After linearization and simplification:

$$p_k^\mu (U_\mu D + \nabla_\mu) f_k^{(0)} + f_k^{(0)} p_k^\mu (U_\mu D + \nabla_\mu)\tilde{\varphi}_k \approx -f_k^{(0)} \mathcal{L}_k[\varphi_k] - f_k^{(0)} \mathcal{L}_k^{inel}[\tilde{\varphi}_k], \qquad D \equiv U^\mu \partial_\mu$$

The equation can be split as follows (φ_k and $\tilde{\varphi}_k$ are separated):

$$p_k^\mu U_\mu D \tilde{\varphi}_k + p_k^\mu \nabla_\mu \tilde{\varphi}_k \approx -\mathcal{L}_k^{inel}[\tilde{\varphi}_k], \qquad p_k^\mu U_\mu D f_k^{(0)} + p_k^\mu \nabla_\mu f_k^{(0)} \approx -f_k^{(0)} \mathcal{L}_k[\varphi_k]$$

or the linearized BE can be split within perturbation theory:

Also conditions $|\varphi_k| \ll |\tilde{\varphi}_k|$

$$p_k^\mu U_\mu D \tilde{\varphi}_k \approx -\mathcal{L}_k^{inel}[\tilde{\varphi}_k]$$

$$(p_k^\mu U_\mu D + p_k^\mu \nabla_\mu) f_k^{(0)} + \underline{f_k^{(0)} p_k^\mu \nabla_\mu \tilde{\varphi}_k} \approx -f_k^{(0)} \mathcal{L}_k[\varphi_k]$$

New in Chapman-Enskog procedure

Chemical reactions and illustrative example

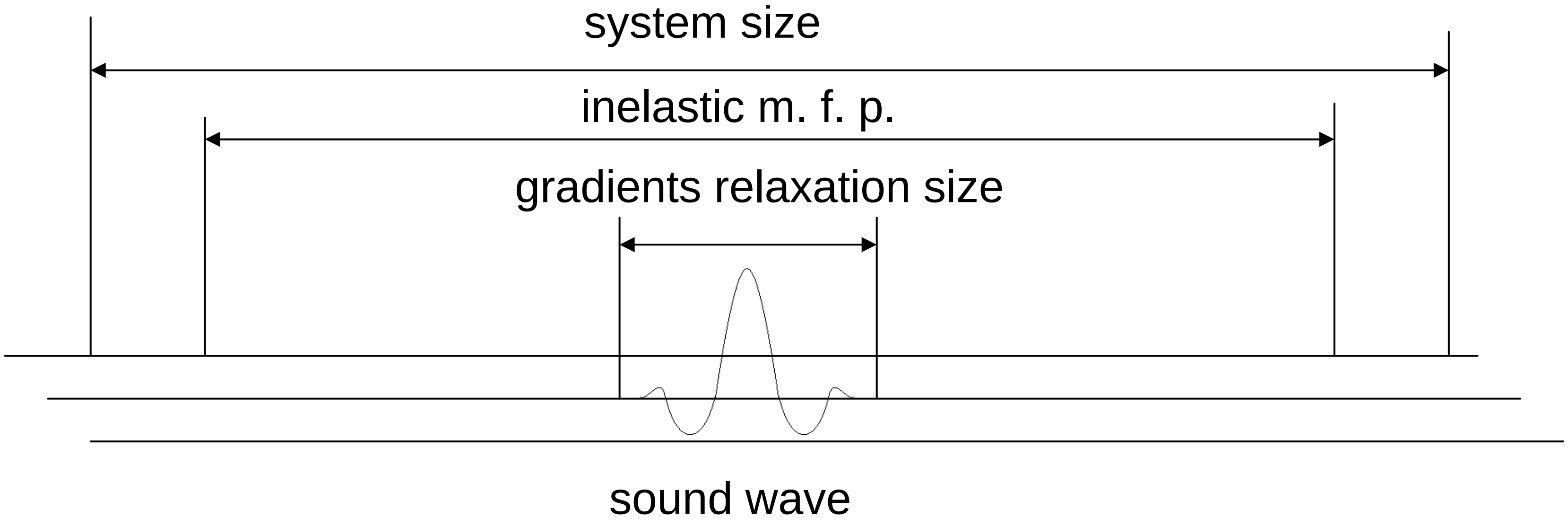
Also deviation from the chemical equilibrium itself is not a source of the bulk viscosity, as is stated in K. Paech and S. Pratt, Phys. Rev. C **74**, 014901 (2006)
If a nonzero value of the bulk viscosity is maintained only by a nonzero mass and there are inelastic processes, the bulk viscosity tends to zero if the mass tends to zero.

An illustrative example of what nonphysical contributions one can get from AZM

Recall $P_{full} = P - \xi \nabla_\rho U^\rho$

Consider infinitesimal $\nabla_\rho U^\rho$ and infinitely weak inelastic processes such that $\xi \nabla_\rho U^\rho$ is sizable and not large in compare to the P. Then we get a sizable correction from practically absent processes. Instead, the system is practically described solely by thermodynamic functions.

Gradients relaxation time



Sound wave damping depends on the bulk viscosity, $\nabla_\rho U^\rho \neq 0$

Necessary condition for the AZM to give physical contribution: the inelastic m. f. t. should be smaller than the GRT.

The GRT is a natural finite timescale which exists even in infinite systems considered during infinite time interval. The GRT is not defined for thermodynamic quantities, e. g., energy density, and there is no need in the GRT for them. This is the crucial difference.

Though one could talk about finite time corrections to the thermodynamic quantities.

Criterion and consequences

The gradients can relax solely by means of the elastic collisions (as usual).

The inelastic m. f. p. is cut by the elastic m. f. p. This limits transport of momentum from the inelastic processes, which is needed for the relaxation of the gradients. This suggests a criterion based on comparison of elastic and inelastic collision rates. Its optimal and universal form may be quite sophisticated.

Such a criterion was used for somewhat different purpose – to build a freeze-out line for the hadron gas: M. Bleicher and J. Aichelin, Phys. Lett. B **530**, 81 (2002). At zero chemical potentials the freeze-out temperature ≈ 160 MeV.

If AZMs are nonphysical ⇒ switch off inelastic process(es) ($\mu \neq 0$, another source term; exclude inelastic collision integral) ⇒ smaller bulk viscosity (quantitative smaller source terms). Physically: no fluctuations from nonconservation of particle numbers.

Estimation of this effect:

(by averaged source term)

Nonrelativistic limit

$$\sim \left(\frac{15}{4}(T/m)^4\right)^2$$

Ultrarelativistic limit

$$\sim 1/169$$

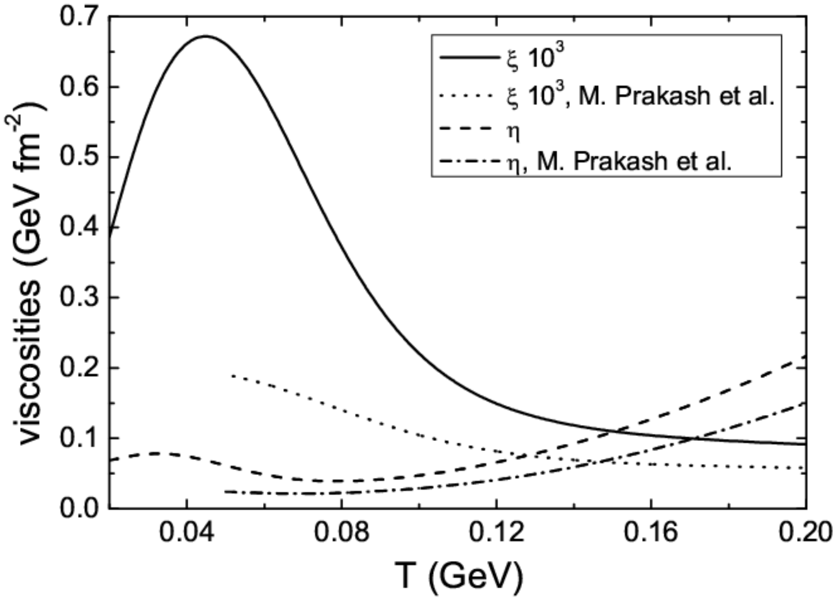
For the criterion in gluodynamics: gluon gg <-> gg and gg<->ggg collision rates close to each other (up to a factor of 2 or smaller) at $\alpha_s \sim 0.1$ ($T \sim 3\ T_c$)

Z. Xu and C. Greiner, Phys. Rev. Lett. **100**, 172301 (2008)
Z. Xu and C. Greiner, Phys. Rev. C **76**, 024911 (2007)

Don’t know for the full QGP

Comparison for the pion gas at $\mu = 0$

Technical difficulties ⇒ pion gas



✓ **M. Prakash et al.,**
Phys. Rep. 227, 321 (1993)

Same approximations except
for pion cross sections

✓ **A. Dobado et al.,**
Phys. Lett. B 702, 43 (2011):

The Inverse Amplitude Method, Bose
statistics; discrepancies up to a factor of 2.7

✓ **D. Fernandez-Fraile and A. G. Nicola, Phys. Rev. Lett. 102, 121601 (2009):**

Kubo formula (rough approximation), not conserved particle number, Bose
statistics; at T = 20 (60) MeV the ξ is 39 (8) times larger

✓ **E. Lu, G. D. Moore, Phys. Rev. C 83, 044901 (2011):**

Inelastic 2↔4 are left completely, Bose statistics; at T = 140 MeV the ξ is 57
times larger and diverges if T → 0

At T=180 MeV chem. rel. time of 2↔4 processes is 40 fm. This is much larger than the existence time of the thermal part of the hadronic fireball.

Some assignments

Multi-indices k,l,m,n will be used to denote particle species with certain spin states. Indexes k',l',m',n‘ will be used to denote particle species without regard to their spin states (and run from 1 to the number of particle species N') and a,b to denote conserved quantum numbers.

Spin interactions are neglected so that $\sum_k \ldots = \sum_{k'} g_{k'} \ldots$, where $g_{k'}$ is the spin degeneracy factor.

$$n \equiv \sum_k n_k \quad \equiv \quad \sum_{k'} n_{k'}, \quad n_a \equiv \sum_k q_{ak} n_k, \quad x_k \equiv \frac{n_k}{n}, \quad x_{k'} \equiv \frac{n_{k'}}{n}, \quad x_a \equiv \frac{n_a}{n},$$

$$\hat{\mu}_k \quad \equiv \quad \frac{\mu_k}{T}, \quad \hat{\mu}_a \equiv \frac{\mu_a}{T}, \quad z_k \equiv \frac{m_k}{T}, \quad \pi_k^\mu \equiv \frac{p_k^\mu}{T}, \quad \tau_k \equiv \frac{p_k^\mu U_\mu}{T}$$

where q_{ak} is conserved quantum numbers of the a-th kind of the k-th particle species.

Everywhere the particle number densities are summed the spin degeneracy factor $g_{k'}$ appears and then gets absorbed into the $n_{k'}$ or the $x_{k'}$ by definition. All other quantities with primed and unprimed indexes *don't differ*, except for collision rates, mean free times and mean free paths, the γ_{kl}, the coefficients $A_{k'l'}^{rs}$, $C_{k'l'}^{rs}$ and, of course, quantities whose free indexes set indexes of the particle number densities n_k.

Also the assignment $\int \frac{d^3 p_k}{p_k^0} \equiv \int_{p_k}$ **is used for compactness somewhere.**

Details of calculations

One more assignment for projector: $\Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu$

Particle 4-flows and energy-momentum tensor:

$$N_k^\mu = \int \frac{d^3p_k}{(2\pi)^3 p_k^0} p_k^\mu f_k, \qquad T^{\mu\nu} = \sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^0} p_k^\mu p_k^\nu f_k$$

Local-equilibrium DF: $f_k^{(0)} = e^{(\mu_k - p_k^\mu U_\mu)/T}$

Deviation from the chemical equilibrium can be taken into account by the γ factors

J. Cleymans et al., Phys. Rev. C73, 034905 (2006); A. Andronic et al., Nucl. Phys. A772, 167-199 (2006)

From variation of the

$$U[f] = \sum_k \int \frac{d^3p_k d^3x}{(2\pi)^3} f_k(1 - \ln f_k) - \sum_k \int \frac{d^3p_k d^3x}{(2\pi)^3} \beta p_k^0 f_k - \sum_{a,k} \lambda_a q_{ak} \int \frac{d^3p_k d^3x}{(2\pi)^3} f_k$$

one reproduces $f^{(0)}_k$ with $U^\mu=(1,0,0,0)$ and $\mu_k=\sum_a q_{ak}\mu_a$, where the Lagrange coefficients $\beta=1/T$, $\lambda_a=\mu_a$

After substitution of the $f^{(0)}_k$ into the N^μ_k and the $T^{\mu\nu}$ one finds

$$N_k^{(0)\mu} = n_k U^\mu, \quad T^{(0)\mu\nu} = \epsilon U^\mu U^\nu - P\Delta^{\mu\nu}$$

where

$$n_k = U_\mu N_k^{(0)\mu} = \frac{1}{2\pi^2} T^3 z_k^2 K_2(z_k) e^{\hat{\mu}_k}$$

Details of calculations

$$\epsilon = U_\mu U_\nu T^{(0)\mu\nu} = \sum_k \int \frac{d^3p_k}{(2\pi)^3} p_k^0 f_k^{(0)} = \sum_k n_k e_k, \quad e_k \equiv m_k \frac{K_3(z_k)}{K_2(z_k)} - T$$

$$P = -\frac{1}{3} T^{(0)\mu\nu}\Delta_{\mu\nu} = \sum_k \frac{1}{3} \int \frac{d^3p_k}{(2\pi)^3 p_k^0} \vec{p_k}^{\,2} f_k^{(0)} = \sum_k n_k T = nT$$

Also the following assignments are used

$$e \equiv \frac{\epsilon}{n} = \sum_k x_k e_k, \quad h_k \equiv e_k + T, \quad h \equiv \frac{\epsilon + P}{n} = \sum_k x_k h_k,$$

$$\hat{e}_k \equiv \frac{e_k}{T} = z_k \frac{K_3(z_k)}{K_2(z_k)} - 1, \quad \hat{e} \equiv \frac{e}{T}, \quad \hat{h}_k \equiv \frac{h_k}{T} = z_k \frac{K_3(z_k)}{K_2(z_k)}, \quad \hat{h} \equiv \frac{h}{T}$$

Landau-Lifshitz condition: $(T^{\mu\nu} - T^{(0)\mu\nu})U_\mu = 0, \quad (N_a^\mu - N_a^{(0)\mu})U_\mu = 0$

The ∂_μ can be split covariantly: $\partial_\mu = U_\mu U^\nu \partial_\nu + \Delta_\mu^\nu \partial_\nu \equiv U_\mu D + \nabla_\mu$

$$T^{(1)\mu\nu} \equiv 2\eta \overset{\circ}{\overline{\nabla^\mu U^\nu}} + \xi \Delta^{\mu\nu}\nabla_\rho U^\rho = \eta \left(\Delta_\rho^\mu \Delta_\tau^\nu + \Delta_\rho^\nu \Delta_\tau^\mu - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\rho\tau} \right) \nabla^\rho U^\tau + \xi \Delta^{\mu\nu}\nabla_\rho U^\rho$$

$$\Downarrow$$

$$\overset{\circ}{\overline{a_{\mu\nu}}} \equiv \left(\frac{\Delta_{\mu\rho}\Delta_{\nu\tau} + \Delta_{\nu\rho}\Delta_{\mu\tau}}{2} - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\rho\tau} \right) a^{\rho\tau} \equiv \Delta_{\mu\nu\rho\tau} a^{\rho\tau}$$

$$P_{full} = P - \xi \nabla_\rho U^\rho$$

The Boltzmann Equation

elastic collision term

neglected

$$p_k^\mu \partial_\mu f_k = (p_k^\mu U_\mu D + p_k^\mu \nabla_\mu) f_k = C_k^{el}[f_k] + C_k^{inel}[f_k]$$

$$\begin{aligned} C_k^{el}[f_k] \quad = \quad & \sum_l \gamma_{kl} \frac{1}{2} \int \frac{d^3 p_{1l}}{(2\pi)^3 2p_{1l}^0} \frac{d^3 p'_k}{(2\pi)^3 2p'^0_k} \frac{d^3 p'_{1l}}{(2\pi)^3 2p'^0_{1l}} (f'_k f'_{1l} - f_k f_{1l}) \\ \times \quad & |\mathcal{M}_{kl}|^2 (2\pi)^4 \delta^4(p'_k + p'_{1l} - p_k - p_{1l}) \end{aligned}$$

p_k and p_{1k} are different variables

$|\mathcal{M}_{kl}(p'_k, p'_{1l}; p_k, p_{1l})|^2 \equiv |\mathcal{M}_{kl}|^2$ **- the square of dimensionless elastic scattering amplitude, averaged over the initial spin states and summed over the final ones**

$$W_{kl} \equiv \frac{|\mathcal{M}_{kl}|^2}{64\pi^2} \delta^4(p'_k + p'_{1l} - p_k - p_{1l})$$

$$C_k^{el}[f_k] = (2\pi)^3 \sum_l \gamma_{kl} \int_{p_{1l}, p'{}_k, p'{}_{1l}} \left(\frac{f'_k}{(2\pi)^3} \frac{f'_{1l}}{(2\pi)^3} - \frac{f_k}{(2\pi)^3} \frac{f_{1l}}{(2\pi)^3} \right) W_{kl}$$

Connection of the W_{kl} and the differential cross section σ_{kl}: $\quad W_{kl} = s \sigma_{kl} \delta^4(p'_k + p'_{1l} - p_k - p_{1l})$

$$W_{kl}(p'_k, p'_{1l}; p_k, p_{1l}) = W_{kl}(p_k, p_{1l}; p'_k, p'_{1l}) = W_{lk}(p'_{1l}, p'_k; p_{1l}, p_k) \qquad s = (p_k + p_{1l})^2$$

$$\int \frac{d^3 p_k}{(2\pi)^3 p_k^0} C_k^{el}[f_k] = 0 \qquad \sum_k \int \frac{d^3 p_k}{(2\pi)^3 p_k^0} p_k C_k^{el}[f_k] = 0 \qquad C_k^{el}[f_k^{(0)}] = 0$$

Towards approximate solution of the Boltzmann equation

Nonequilibrium DF is sought in the form: $f_k \approx f_k^{(0)} + f_k^{(1)} \equiv f_k^{(0)}(1+\varphi_k), \quad |\varphi_k| \ll 1$

Hydrodynamics

Integration over $\dfrac{d^3p_k}{(2\pi)^3 p_k^0}$ **with $f_k=f^{(0)}_k$ in the l.h.s. and the inelastic collision integrals retained:**

$$\partial_\mu N_k^{(0)\mu} = Dn_k + n_k \nabla_\mu U^\mu = I_k, \quad \sum_k q_{ak} I_k = 0$$

Integration over $\dfrac{d^3p_k}{(2\pi)^3 p_k^0} p_k^\mu$ **with $f_k=f^{(0)}_k$ in the l.h.s. and the inelastic collision integrals retained:**

$$\partial_\rho T^{(0)\rho\nu} = \partial_\rho(\epsilon U^\rho U^\nu - P\Delta^{\rho\nu}) = 0$$

$\longleftarrow$ **IG EoS**

$$\Downarrow$$

$$DU^\mu = \frac{1}{\epsilon+P}\nabla^\mu P = \frac{1}{hn}\nabla^\mu P, \qquad D\epsilon = -(\epsilon+P)\nabla_\mu U^\mu = -hn\nabla_\mu U^\mu$$

Using μ_a and T as independent thermodynamic variables one gets

$$Df_k^{(0)} = \sum_a \frac{\partial f_k^{(0)}}{\partial \mu_a} D\mu_a + \frac{\partial f_k^{(0)}}{\partial T} DT + \frac{\partial f_k^{(0)}}{\partial U^\mu} DU^\mu$$

$$Dn_a = \sum_b \frac{\partial n_a}{\partial \mu_b} D\mu_b + \frac{\partial n_a}{\partial T} DT = -n_a \nabla_\mu U^\mu, \qquad D\epsilon = \frac{\partial \epsilon}{\partial T} DT + \sum_a \frac{\partial \epsilon}{\partial \mu_a} D\mu_a = -hn\nabla_\mu U^\mu$$

Details of calculations

Solving the linear system of equations over Dμ_a and DT one gets

$$DT = -RT\nabla_\mu U^\mu, \quad D\mu_a = T\sum_b \tilde{A}^{-1}_{ab}(RB_b - x_b)\nabla_\mu U^\mu$$

where

$$\frac{\partial n_a}{\partial \mu_b} \equiv \frac{n}{T}\tilde{A}_{ab}, \quad \frac{\partial n_a}{\partial T} \equiv \frac{n}{T}B_a, \quad \frac{\partial \epsilon}{\partial T} \equiv nC_{\{\mu\}}, \quad \frac{\partial \epsilon}{\partial \mu_a} \equiv nE_a, \quad R \equiv \frac{\hat{h} - \sum_{a,b} E_a \tilde{A}^{-1}_{ab} x_b}{C_{\{\mu\}} - \sum_{a,b} E_a \tilde{A}^{-1}_{ab} B_b}$$

Using IG formulas one gets

$$\begin{aligned} \tilde{A}_{ab} &= \sum_k q_{ak} q_{bk} x_k, \quad E_a = \sum_k q_{ak} x_k \hat{e}_k, \quad B_a = E_a - \sum_b \tilde{A}_{ab}\hat{\mu}_b, \\ C_{\{\mu\}} &= \sum_k x_k(3\hat{h}_k + z_k^2 - \hat{\mu}_k \hat{e}_k) = \sum_k x_k(3\hat{h}_k + z_k^2) - \sum_a E_a \hat{\mu}_a. \end{aligned}$$

$$D\hat{\mu}_a = T\sum_b \tilde{A}^{-1}_{ab}(RE_b - x_b)\nabla_\mu U^\mu \quad \Longrightarrow \quad D\hat{\mu}_a = 0$$ **if m$_k$=0**

In the case of v. c. p.: $$DT|_{\mu_a=0} = -\frac{h}{C_{\{\mu\}}}\nabla_\mu U^\mu, \quad D\mu_a|_{\mu_a=0} = 0$$

and μ_a can be excluded from the DFs if one doesn't consider heat conductivity and diffusion

Details of calculations

In case of only elastic collisions ($q_{ak}=\delta_{ak}$) one gets

$$\tilde{A}_{kl} = \delta_{kl} x_k, \quad B_k = x_k(\hat{e}_k - \hat{\mu}_k), \quad E_k = \hat{e}_k x_k, \quad R = \frac{1}{c_\upsilon}$$

$$C_{\{\mu\}} - \sum_{a,b} E_a \tilde{A}^{-1}_{ab} B_b = \sum_k x_k(-\hat{h}_k^2 + 5\hat{h}_k + z_k^2 - 1) \equiv \sum_k x_k c_{\upsilon,k} \equiv c_\upsilon .$$

$$D\mu_k = \left(\frac{T}{c_\upsilon}(\hat{e}_k - \hat{\mu}_k) - T \right) \nabla_\mu U^\mu$$

⟹ **$D\mu_k \neq 0$, if $\mu_k = 0$**

As long as the heat conductivity and diffusion are not considered their gradients are taken equal to zero:

$$\nabla_\nu P = \nabla_\nu T = \nabla_\nu \mu_a = 0$$

Now one can transform the l.h.s. of the Boltzmann equation:

$$(p_k^\mu U_\mu D + p_k^\mu \nabla_\mu) f_k^{(0)} = -T f_k^{(0)} \pi_k^\mu \pi_k^\nu \overset{\circ}{\overline{\nabla_\mu U_\nu}} + T f_k^{(0)} \hat{Q}_k \nabla_\rho U^\rho$$

where

charge conservation laws, suppression

$$\hat{Q}_k \equiv \tau_k^2 \left(\frac{1}{3} - R \right) + \tau_k \sum_{a,b} q_{ak} \tilde{A}^{-1}_{ab} (R E_b - x_b) - \frac{1}{3} z_k^2$$

Useful formula: $\pi_k^\mu \pi_k^\nu \overset{\circ}{\overline{\nabla_\mu U_\nu}} = \overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}} \overset{\circ}{\overline{\nabla_\mu U_\nu}}$

Details of calculations

In case of only elastic collisions the $\hat{Q}_k$ simplifies to

$$\hat{Q}_k = \left(\frac{4}{3} - \gamma\right) \tau_k^2 + \tau_k((\gamma - 1)\hat{h}_k - \gamma) - \frac{1}{3} z_k^2$$

where $\gamma \equiv \frac{1}{c_v} + 1$

Introducing the assignments $(F, G)_k \equiv \frac{1}{4\pi z_k^2 K_2(z_k) T^2} \int_{p_k} F(p_k) G(p_k) e^{-\tau_k}$

$$\alpha_k^r \equiv (\hat{Q}_k, (\tau_k)^r), \quad \gamma_k^r \equiv ((\tau_k)^r \overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}, \overset{\circ}{\overline{\pi_{k\mu} \pi_{k\nu}}}), \quad a_k^r \equiv (1, (\tau_k)^r)_k$$

one can show that

$$\alpha_k^0 = 1 + \sum_{a,b} q_{ak} \tilde{A}_{ab}^{-1} (RB_b - x_b) - (\hat{e}_k - \hat{\mu}_k) R$$

$$\alpha_k^1 = \hat{h}_k + \sum_{a,b} \hat{e}_k q_{ak} \tilde{A}_{ab}^{-1} (RB_b - x_b) - (3\hat{h}_k + z_k^2 + \hat{\mu}_k(1 - \hat{h}_k)) R$$

$\sum_k q_{ak} x_k \alpha_k^0 = 0, \quad \sum_k x_k \alpha_k^1 = 0$ **- consequence of conservation laws**

Details of calculations

The next step is to transform the r.h.s.

$$C_k^{el}[f_k] \approx -f_k^{(0)} \sum_l \mathcal{L}_{kl}^{el}[\varphi_k]$$

where

$$\mathcal{L}_{kl}^{el}[\varphi_k] \equiv \frac{\gamma_{kl}}{(2\pi)^3} \int_{p_{1l},{p'}_k,{p'}_{1l}} f_{1l}^{(0)} (\varphi_k + \varphi_{1l} - \varphi'_k - \varphi'_{1l}) W_{kl}$$

The φ_k is sought in the form

expansion/compression rate

$$\varphi_k = \frac{1}{n\sigma(T)} \left(-A_k(p_k)\nabla_\mu U^\mu + C_k(p_k)\overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}\,\overset{\circ}{\overline{\nabla_\mu U_\nu}} \right)$$

Then one gets

$$\hat{Q}_k = \sum_l x_l L_{kl}^{el}[A_k]$$

$$\overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}} = \sum_l x_l L_{kl}^{el}[C_k \overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}]$$

Equations of this form with thermal masses and cross sections were obtained by Jeon in field theories and should be recoverable at low T and n, where the Boltzmann equation should be valid for any short-range interactions.

where

$$L_{kl}^{el}[\chi_k] = \frac{1}{n_l T \sigma(T)} \frac{\gamma_{kl}}{(2\pi)^3} \int_{p_{1l},{p'}_k,{p'}_{1l}} f_{1l}^{(0)} (\chi_k + \chi_{1l} - \chi'_k - \chi'_{1l}) W_{kl}$$

Transport coefficients and their properties

After substitution of the f$^{(1)}_k$ into the e.-m. tensor and comparison with its expansion over the gradients one finds

$$\xi = -\frac{1}{3}\frac{T}{\sigma(T)}\sum_k x_k(\Delta^{\mu\nu}\pi_{\mu k}\pi_{\nu k}, A_k)_k$$

$$\eta = \frac{1}{10}\frac{T}{\sigma(T)}\sum_k x_k(\overset{\circ}{\overline{\pi_k^{\mu}\pi_k^{\nu}}}, C_k\overset{\circ}{\overline{\pi_{k\mu}\pi_{k\nu}}})_k$$

Conditions of fit for the energy density and the net charge densities:

$$\sum_k q_{ak}\int\frac{d^3p_k}{(2\pi)^3p_k^0}p_k^{\mu}U_{\mu}f_k^{(0)}\varphi_k = 0, \quad \sum_k\int\frac{d^3p_k}{(2\pi)^3p_k^0}p_k^{\mu}p_k^{\nu}U_{\nu}f_k^{(0)}\varphi_k = 0$$

Conditions of fit rewritten for the bulk viscosity (3-vector part is satisfied automatically)

$$\sum_k q_{ak}x_k(\tau_k, A_k)_k = 0, \quad \sum_k x_k(\tau_k^2, A_k)_k = 0$$

CoF exclude zero modes $A_k = \sum_a C_a q_{ak} + C\tau_k$

For the shear viscosity CoF are satisfied automatically

Transport coefficients and their properties

Using CoF and $\Delta^{\mu\nu}\pi_{\mu,k}\pi_{\nu,k} = z_k^2 - \tau_k^2$ **the ξ can be rewritten in the form**

$$\xi = \frac{T}{\sigma(T)}\sum_k x_k(\hat{Q}_k, A_k)_k = \frac{T}{\sigma(T)}\sum_k x_k\left(\sum_l x_l L^{el}_{kl}[A_k], A_k\right)_k = \frac{T}{\sigma(T)}[\{A\},\{A\}]$$

where for sets of equal length $\{F\} = (F_1, ..., F_k, ...), \quad \{G\} = (G_1, ..., G_k, ...)$

$$[\{F\},\{G\}] \equiv \frac{1}{n^2\sigma(T)}\sum_{k,l}\frac{\gamma_{kl}}{(2\pi)^6}\int_{p_k,p_{1l},p'_k,p'_{1l}} f_k^{(0)} f_{1l}^{(0)}(F_k + F_{1l} - F'_k - F'_{1l})G_k W_{kl}$$

Under the integration

$$(F_k + F_{1l} - F'_k - F'_{1l})G_k = \frac{1}{4}(F_k + F_{1l} - F'_k - F'_{1l})(G_k + G_{1l} - G'_k - G'_{1l})$$

so that one gets $[\{F\},\{G\}] = [\{G\},\{F\}], \quad [\{F\},\{F\}] \geq 0$

Similarly one gets

$$\eta = \frac{1}{10}\frac{T}{\sigma(T)}\sum_k x_k\left(\overset{\circ}{\overline{\pi_k^\mu\pi_k^\nu}}, C_k\overset{\circ}{\overline{\pi_{k\mu}\pi_{k\nu}}}\right)_k = \frac{1}{10}\frac{T}{\sigma(T)}\sum_k x_k\left(\sum_l x_l L^{el}_{kl}[C_k\overset{\circ}{\overline{\pi_k^\mu\pi_k^\nu}}], C_k\overset{\circ}{\overline{\pi_{k\mu}\pi_{k\nu}}}\right)_k$$

$$= \frac{1}{10}\frac{T}{\sigma(T)}[\{C\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\},\{C\overset{\circ}{\overline{\pi_\mu\pi_\nu}}\}].$$

Variational method

The generalized functional is

$$F[\chi] = \sum_k x_k (S_k^{\mu\ldots\nu}, \chi_{k\mu\ldots\nu})_k - \frac{1}{2}[\{\chi^{\mu\ldots\nu}\}, \{\chi_{\mu\ldots\nu}\}]$$

$$S_k^{\mu\ldots\nu} = \hat{Q}_k, \quad \chi_{k\mu\ldots\nu} = A_k$$ **- for the bulk viscosity**

$$S_k^{\mu\ldots\nu} = \overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}, \quad \chi_k^{\mu\ldots\nu} = C_k \overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}$$ **- for the shear viscosity**

Equating to zero the first variance of the functional one gets

$$\sum_k x_k (S_k^{\mu\ldots\nu}, \delta\chi_{k\mu\ldots\nu})_k - [\{\chi^{\mu\ldots\nu}\}, \{\delta\chi_{\mu\ldots\nu}\}] = 0$$

Because $\delta\chi_{k\mu\ldots\nu}$ **are independent one gets the generalized integral equations**

$$S_k^{\mu\ldots\nu} = \sum_l x_l L_{kl}^{el}[\chi_{k\mu\ldots\nu}]$$

Also $\delta^2 F[\chi] = -[\{\delta\chi^{\mu\ldots\nu}\}, \{\delta\chi_{\mu\ldots\nu}\}] \leq 0$

This reduces the problem to the problem of finding of the maximum of the functional F

Variational method

The maximal value of the functional F:

$$F_{max}[\chi] = \frac{1}{2}[\{\chi^{\mu\ldots\nu}\}, \{\chi_{\mu\ldots\nu}\}]|_{\chi=\chi_{\max}}$$

Transport coefficients can be expressed through it:

$$\xi = 2\frac{T}{\sigma(T)}F_{max}\Big|_{S_k^{\mu\ldots\nu}=\hat{Q}_k,\, \chi_{k\mu\ldots\nu}=A_k}, \quad \eta = \frac{1}{5}\frac{T}{\sigma(T)}F_{max}\Big|_{S_k^{\mu\ldots\nu}=\overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}},\, \chi_k^{\mu\ldots\nu}=C_k\overset{\circ}{\overline{\pi_k^\mu \pi_k^\nu}}}$$

Further unknown functions A_k and C_k in the φ_k are written as a linear combination of test-functions τ_k^r with coefficients being varied:

$$A_k = \sum_{r=0}^{n_1} A_k^r \tau_k^r, \quad C_k = \sum_{r=0}^{n_2} C_k^r \tau_k^r$$

and viscosities become of the form:

$$\xi = \frac{T}{\sigma(T)} \sum_{k'=1}^{N'} \sum_{r=0}^{n_1} x_{k'} \alpha_{k'}^r A_{k'}^r,$$

$$\eta = \frac{1}{10}\frac{T}{\sigma(T)} \sum_{k'=1}^{N'} \sum_{r=0}^{n_2} x_{k'} \gamma_{k'}^r C_{k'}^r$$

Variational method

Variation of coefficients $A_{k'}^{r}$ and $C_{k'}^{r}$ results in the matrix equations

$$x_{k'}\alpha^{r}_{k'} = \sum_{l'=1}^{N'}\sum_{s=0}^{n_1} A^{sr}_{l'k'}A^{s}_{l'}, \qquad x_{k'}\gamma^{r}_{k'} = \sum_{l'=1}^{N'}\sum_{s=0}^{n_2} C^{sr}_{l'k'}C^{s}_{l'}$$

where

$$A^{rs}_{k'l'} = x_{k'}x_{l'}[\tau^{r},\tau^{s}_{1}]_{k'l'} + \delta_{k'l'}x_{k'}\sum_{m'} x_{m'}[\tau^{r},\tau^{s}]_{k'm'}$$

$$C^{rs}_{k'l'} = x_{k'}x_{l'}[\tau^{r}\overset{\circ}{\overline{\pi^{\mu}\pi^{\nu}}},\tau^{s}_{1}\overset{\circ}{\overline{\pi_{1\mu}\pi_{1\nu}}}]_{k'l'} + \delta_{k'l'}x_{k'}\sum_{m'} x_{m'}[\tau^{r}\overset{\circ}{\overline{\pi^{\mu}\pi^{\nu}}},\tau^{s}\overset{\circ}{\overline{\pi_{\mu}\pi_{\nu}}}]_{k'm'}$$

$$[F,G_1]_{kl} \equiv \frac{\gamma_{kl}}{T^{6}(4\pi)^{2}z_k^{2}z_l^{2}K_2(z_k)K_2(z_l)\sigma(T)}\int_{p_k,p_{1l},{p'}_k,{p'}_{1l}} e^{-\tau_k-\tau_{1l}}(F_k-{F'}_k)G_{1l}W_{kl}$$

$$[F,G]_{kl} \equiv \frac{\gamma_{kl}}{T^{6}(4\pi)^{2}z_k^{2}z_l^{2}K_2(z_k)K_2(z_l)\sigma(T)}\int_{p_k,p_{1l},{p'}_k,{p'}_{1l}} e^{-\tau_k-\tau_{1l}}(F_k-{F'}_k)G_{k}W_{kl}$$

Properties: $[\tau^{r},\tau^{s}]_{kl} > 0$

$$[F,G_1]_{kl} = [G,F_1]_{lk}, \quad [F,G]_{kl} = [G,F]_{kl} \quad \Rightarrow \quad A^{rs}_{k'l'} = A^{sr}_{l'k'}, \quad C^{rs}_{k'l'} = C^{sr}_{l'k'}$$

$\alpha^{0}_{k} = 0$ $\quad A^{0s}_{k'l'} = 0$ **(elastic collisions)** $\quad \sum_{k'} A^{1s}_{k'l'} = 0$

Variational method & some transformations

$$\sum_{k'} A^{1s}_{k'l'} = 0, \quad \sum_{k'} x_{k'}\alpha^{1}_{k'} = 0 \quad \Longrightarrow$$

rank of the matrix A^{rs}_{kl} reduces on 1 and there is one independent coefficient, let it be $A^{1}_{1'}$

Using $\sum_{k'} A^{1s}_{k'l'} = 0$ one gets (one equation with r=1 and some k' is omitted)

$$x_{k'}\alpha^{r}_{k'} = \sum_{l'=2}^{N'} A^{1r}_{l'k'}(A^{1}_{l'} - A^{1}_{1'}) + \sum_{l'=1}^{N'}\sum_{s=2}^{n_1} A^{sr}_{l'k'}A^{s}_{l'}$$

can be eliminated by shift

and

$$\xi = \frac{T}{\sigma(T)}\sum_{k'=2}^{N'} x_{k'}\alpha^{1}_{k'}(A^{1}_{k'} - A^{1}_{1'}) + \frac{T}{\sigma(T)}\sum_{k'=1}^{N'}\sum_{r=2}^{n_1} x_{k'}\alpha^{r}_{k'}A^{r}_{k'}$$

present only in mixtures

Coefficients $A^{1}_{1'}$ and $A^{0}_{k'}$ are used to satisfy CoF implicitly

And test-functions 1 and $\tau_{k'}$ are important in providing of the coefficients $A^{1}_{1'}$ and $A^{0}_{k'}$.

12-dimensional integrals

The main part of the work lies in calculation of the following 12-D integrals:

$$J_{kl}^{(a,b,d,e,f|q,r)} \equiv \frac{\gamma_{kl}}{T^6(4\pi)^2 z_k^2 z_l^2 K_2(z_k)K_2(z_l)\sigma(T)} \int_{p_k,p_{1l},p'_k,p'_{1l}} e^{-P\cdot U/T}(1+\alpha_{kl})^q$$

$$\times(1-\alpha_{kl})^r \left(\frac{P^2}{T^2}\right)^a \left(\frac{P\cdot U}{T}\right)^b \left(\frac{Q\cdot U}{T}\right)^d \left(\frac{Q'\cdot U}{T}\right)^e \left(\frac{-Q\cdot Q'}{T^2}\right)^f W_{kl}.$$

where

$$\alpha_{kl} \equiv \frac{m_k^2 - m_l^2}{P^2}, \quad P^\mu \equiv p_k^\mu + p_{1l}^\mu = p'^\mu_k + p'^\mu_{1l}$$

$$Q^\mu \equiv \Delta_P^{\mu\nu}(p_{k\nu} - p_{1l\nu}), \quad Q'^\mu \equiv \Delta_P^{\mu\nu}(p'_{k\nu} - p'_{1l\nu}), \quad \Delta_P^{\mu\nu} \equiv g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}$$

and dot "·" denotes convolution of 4-vectors

The collisions brackets $[F,G_1]_{kl}$ and $[F,G]_{kl}$ and collision rates can be expressed through the J integrals

Single-component gas (1-st order)

$$\eta = \frac{1}{10}\frac{T}{\sigma(T)}\frac{(\gamma^0)^2}{C^{00}}, \qquad \xi = \frac{T}{\sigma(T)}\frac{(\alpha^2)^2}{A^{22}}$$

J. L. Anderson, A. J. Kox, Physica, 1977, v. 89A, p. 408

$$\eta = \frac{15}{64\pi}\frac{T}{r^2}\frac{z^2K_2^2(z)\hat{h}^2}{(\boxed{15}z^2+2)K_2(2z)+(3z^3+49z)K_3(2z)}$$

incorrect value was 5

$$\xi = \frac{1}{64\pi}\frac{T}{r^2}\frac{z^2K_2^2(z)[(5-3\gamma)\hat{h}-3\gamma]^2}{2K_2(2z)+zK_3(2z)}$$

$$\gamma = \frac{1}{c_v}+1 = \frac{z^2+5\hat{h}-\hat{h}^2}{z^2+5\hat{h}-\hat{h}^2-1}$$

$z \gg 1$ $\qquad \eta = \frac{5}{64\sqrt{\pi}}\frac{T}{r^2}\sqrt{z}\left(1+\frac{25}{16}z^{-1}+...\right), \qquad \xi = \frac{25}{256\sqrt{\pi}}\frac{T}{r^2}z^{-3/2}\left(1-\frac{183}{16}z^{-1}+...\right)$

$z \ll 1$ $\qquad \eta = \frac{3}{10\pi}\frac{T}{r^2}\left(1+\frac{1}{20}z^2+...\right), \qquad \xi = \frac{1}{288\pi}\frac{T}{r^2}z^4\left(1+\left(\frac{49}{12}-6\ln 2+6\gamma_E\right)z^2+6z^2\ln z+...\right)$

Single-component gas (1-st order)

For the φ one gets $\varphi = \dfrac{1}{n\sigma(T)}\left(-(A^0 + A^1\tau + A^2\tau^2)\nabla_\mu U^\mu + C^0 \overset{\circ}{\overline{\pi^\mu\pi^\nu}}\,\overset{\circ}{\overline{\nabla_\mu U_\nu}}\right)$

$$C^0 = \frac{15}{64\pi}\frac{\sigma(T)}{r^2}\frac{z^2K_2^2(z)\hat{h}}{(15z^2+2)K_2(2z)+(3z^3+49z)K_3(2z)} \qquad A^2 = \frac{1}{64\pi}\frac{\sigma(T)}{r^2}\frac{z^2K_2^2(z)[(5-3\gamma)\hat{h}-3\gamma]}{2K_2(2z)+zK_3(2z)}$$

$$A^0 = A^2\frac{a^2a^4-(a^3)^2}{\Delta_A}, \quad A^1 = A^2\frac{a^2a^3-a^1a^4}{\Delta_A}, \quad \Delta_A \equiv a^1a^3-(a^2)^2$$

$$z \gg 1, \quad \varphi = \frac{5\pi e^{z-\hat{\mu}}}{32\sqrt{2}T^3z^2r^2}\left(-(\tau^2+2z\tau-z^2)\nabla_\mu U^\mu + 2\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\,\overset{\circ}{\overline{\nabla_\mu U_\nu}}\right)$$

$$z \ll 1, \quad \varphi = \frac{\pi e^{-\hat{\mu}}}{480T^3r^2}\left(-5z^2(\tau^2+8\tau-12)\nabla_\mu U^\mu + 36\overset{\circ}{\overline{\pi^\mu\pi^\nu}}\,\overset{\circ}{\overline{\nabla_\mu U_\nu}}\right)$$

How large can be influence of inelastic reactions?

$$z \gg 1, \quad \frac{\alpha^2|_{q_{11}=1}}{\alpha^2|_{q_{11}=0}} = \frac{4z^4}{15} + \ldots$$

divergence of the ξ at least as $O(Tz^{13/2}) \propto O(T^{-11/2})$

$$z \ll 1, \quad \frac{\alpha^2|_{q_{11}=1}}{\alpha^2|_{q_{11}=0}} = 13 + \ldots$$

Phenomenological formulas: $\eta_{ph} \propto ln < |\vec{p}| >$

$\xi_{ph} \propto (\alpha^2)^2$ $\qquad \eta_{ph} = \dfrac{5}{64\sqrt{\pi}}\dfrac{\sqrt{mT}}{r^2}\dfrac{K_{5/2}(m/T)}{K_2(m/T)}$

Binary mixture (1-st order)

$$\eta = \frac{T}{10\sigma(T)}\frac{1}{\Delta_\eta}[(x_{1'}\gamma_1^0)^2 C^{00}_{2'2'} - 2x_{1'}x_{2'}\gamma_1^0\gamma_2^0 C^{00}_{1'2'} + (x_{2'}\gamma_2^0)^2 C^{00}_{1'1'}]$$

$$\Delta_\eta = C^{00}_{1'1'}C^{00}_{2'2'} - (C^{00}_{1'2'})^2$$

1-2 collisions dominate here

$z_2 \gg 1$

$$\Delta\eta = z_2^{5/2} e^{-z_2} \frac{3Tg_2 e^{z_1-\hat{\mu}_1+\hat{\mu}_2}}{64\sqrt{2\pi}(3+3z_1+z_1^2)g_1\sigma^{cl}_{12}}$$

“Mixture” approximation:

$$\xi = \frac{T}{\sigma(T)}\frac{(x_{2'}\alpha_2^1)^2}{A^{11}_{2'2'}} = \frac{T}{\sigma(T)}\frac{x_{1'}x_{2'}\alpha_1^1\alpha_2^1}{A^{11}_{1'2'}}$$

$$[\tau^r, \tau^s]_{kl} > 0$$

$$\xi = \frac{T}{\sigma(T)}\frac{x_{2'}(\alpha_2^1)^2}{x_{1'}[\tau,\tau]_{12}} = \frac{T}{\sigma(T)}\frac{x_{1'}(\alpha_1^1)^2}{x_{2'}[\tau,\tau]_{12}} > 0$$

$z_2 \gg 1$

$$\xi = e^{-z_2} z_2^{5/2} \frac{g_2 T e^{-\hat{\mu}_1+\hat{\mu}_2+z_1}[2z_1^2 - 5 - 2\hat{h}_1^2 + 10\hat{h}_1]^2}{128\sqrt{2\pi}g_1\sigma^{cl}_{12}(z_1^2+3z_1+3)[z_1^2 - 1 - \hat{h}_1^2 + 5\hat{h}_1]^2} + \ldots$$

Lowest order collision brackets

Introducing the assignments

$$\widetilde{K}_1 \equiv \frac{K_3(z_k+z_l)}{K_2(z_k)K_2(z_l)}, \quad \widetilde{K}_2 \equiv \frac{K_2(z_k+z_l)}{K_2(z_k)K_2(z_l)}, \quad \widetilde{K}_3 \equiv \frac{G(z_k+z_l)}{K_2(z_k)K_2(z_l)}$$

$$Z_{kl} \equiv z_k + z_l, \quad z_{kl} \equiv z_k - z_l$$

$$G(x) = x^{-3}\int_1^\infty du u^{-4} K_3(xu) = \frac{1}{32} G^{3,0}_{1,3}\left(\frac{x}{2}, \frac{1}{2} \middle| \begin{array}{c} 1 \\ -3, 0, 0 \end{array} \right)$$

Meijer function

the lowest order tensor collision brackets can be written as

$$[\overset{\circ}{\overline{\pi^\mu \pi^\nu}}, \overset{\circ}{\overline{\pi_{1\mu}\pi_{1\nu}}}]_{kl} = \frac{\pi}{72 z_k^2 z_l^2 Z_{kl}^2}\left(P^{(0,0)}_{T11}\widetilde{K}_1 + P^{(0,0)}_{T12}\widetilde{K}_2 + P^{(0,0)}_{T13}\widetilde{K}_3\right)$$

$$P^{(0,0)}_{T11} = -2Z_{kl}[z_{kl}^6(5Z_{kl}^2-8)+24z_{kl}^4(Z_{kl}^2-16)-144z_{kl}^2Z_{kl}^2(Z_{kl}^2+8)+48Z_{kl}^4(Z_{kl}^2+72)]$$

$$P^{(0,0)}_{T12} = z_{kl}^6(5Z_{kl}^4-40Z_{kl}^2-64)-24z_{kl}^4(5Z_{kl}^4+8Z_{kl}^2+128)+576z_{kl}^2Z_{kl}^2(Z_{kl}^2-16)-192Z_{kl}^4(5Z_{kl}^2+16)$$

$$P^{(0,0)}_{T13} = -5z_{kl}^4 Z_{kl}^6[z_{kl}^2(Z_{kl}^2-24)-24Z_{kl}^2]$$

Lowest order collision brackets

$$[\overset{\circ}{\overline{\pi^{\mu}\pi^{\nu}}}, \overset{\circ}{\overline{\pi_{\mu}\pi_{\nu}}}]_{kl} = \frac{\pi}{72 z_k^2 z_l^2 Z_{kl}^2} (P_{T21}^{(0,0)} \widetilde{K}_1 + P_{T22}^{(0,0)} \widetilde{K}_2 + P_{T23}^{(0,0)} \widetilde{K}_3)$$

$$\begin{aligned}
P_{T21}^{(0,0)} &= 2Z_{kl}[z_{kl}^6(8 - 5Z_{kl}^2) + 72 z_{kl}^4(3Z_{kl}^2 - 8) - 480 z_{kl}^3 Z_{kl}(Z_{kl}^2 - 4) \\
&- 336 z_{kl}^2 Z_{kl}^2 (Z_{kl}^2 + 8) + 240 z_{kl} Z_{kl}^3 (Z_{kl}^2 + 8) + 192 Z_{kl}^4 (Z_{kl}^2 + 67)],
\end{aligned}$$

$$\begin{aligned}
P_{T22}^{(0,0)} &= z_{kl}^6 (5Z_{kl}^4 - 40 Z_{kl}^2 - 64) + 240 z_{kl}^5 Z_{kl}^3 - 24 z_{kl}^4 (5 Z_{kl}^4 + 48 Z_{kl}^2 - 192) \\
&+ 1920 z_{kl}^3 Z_{kl} (Z_{kl}^2 - 8) - 192 z_{kl}^2 Z_{kl}^2 (17 Z_{kl}^2 - 112) + 1920 z_{kl} Z_{kl}^3 (Z_{kl}^2 - 8) \\
&+ 768 Z_{kl}^4 (5 Z_{kl}^2 + 6),
\end{aligned}$$

$$P_{T23}^{(0,0)} = -5 z_{kl}^4 Z_{kl}^6 [z_{kl}^2 (Z_{kl}^2 - 24) + 48 z_{kl} Z_{kl} - 24 Z_{kl}^2]$$

Collision rates and mean free paths

The k’-l’ number of collisions per unit time per unit volume is

$$\widetilde{R}^{el}_{k'l'} \equiv \int \gamma_{k'l'} \boxed{g_{k'} \frac{d^3p_{k'}}{(2\pi)^3} f^{(0)}_{k'}}\boxed{g_{l'} \frac{d^3p_{1l'}}{(2\pi)^3} f^{(0)}_{1l'}} \gamma_{k'l'} \boxed{d^3p'_{k'}\, d^3p'_{1l'} \frac{W_{k'l'}}{p^0_{k'} p^0_{1l'} p'^0_{k'} p'^0_{1l'}}}$$

particle numbers

collision probability per unit time times unit volume (from analysis of collision integrals or from explicit expression through collision amplitudes)

$$\widetilde{R}^{el}_{k'l'} = g_{k'} g_{l'} \gamma_{k'l'} \frac{2\sigma^{cl}_{k'l'} T^6}{\pi^3} [(z_{k'} - z_{l'})^2 K_2(z_{k'} + z_{l'}) + z_{k'} z_{l'} (z_{k'} + z_{l'}) K_3(z_{k'} + z_{l'})]$$

The k’-l’ collision rate *per particle of the k’-th species* is

$$R^{el}_{k'l'} \equiv g_{k'} g_{l'} \frac{\gamma_{k'l'}}{(2\pi)^6 n_{k'}} \int \frac{d^3p_{k'}}{p^0_{k'}} \frac{d^3p_{1l'}}{p^0_{1l'}} \frac{d^3p'_{k'}}{p'^0_{k'}} \frac{d^3p'_{1l'}}{p'^0_{1l'}} f^{(0)}_{k'} f^{(0)}_{1l'} W_{k'l'} = \frac{\widetilde{R}^{el}_{k'l'}}{\gamma_{k'l'} n_{k'}}$$

The mean free time of particle of the k’-th species is $\boxed{t^{el}_{k'} = \frac{1}{R^{el}_{k'}}}$ $\quad R^{el}_{k'} \equiv \sum_{l'} R^{el}_{k'l'}$

Collision rates and mean free paths

The mean free path of particle of the k'-th species is

$$\boxed{l^{el}_{k'} = \frac{\langle|\vec{v}_{k'}|\rangle}{R^{el}_{k'}}}$$

$$\langle|\vec{v}_{k'}|\rangle = \frac{\int d^3p_{k'} \frac{|\vec{p}_{k'}|}{p^0_{k'}} f^{(0)}_{k'}(p_{k'})}{\int d^3p_{k'} f^{(0)}_{k'}(p_{k'})} = \frac{2e^{-z_{k'}}(1+z_{k'})}{z^2_{k'}K_2(z_{k'})} = \sqrt{\frac{8}{\pi z_{k'}}}\frac{K_{3/2}(z_{k'})}{K_2(z_{k'})}$$

The nonrelativistic limit of the $R^{el}_{k'l'}$ reproduces known expression in the nonrelativistic kinetic-molecular theory

$$n_{l'}4\pi\sigma^{cl}_{k'l'}\langle|\vec{v}_{k'}|\rangle\sqrt{1+m_{k'}/m_{l'}}$$

In single-component gas the nonrelativistic limit of the mean free path

$$l^{el}_{1'} = \frac{\langle|\vec{v}_{1'}|\rangle}{R^{el}_{1'1'}} = \frac{\pi e^{-z_1}(z_1+1)}{g_1 4\sigma^{cl}_{11}T^3z^3_1K_3(2z_1)}$$

reproduces known expression for the Maxwell's mean free path (g_1=1)

$$l^{el}_1 = \frac{1}{4\pi\sigma^{cl}_{11}n_1\sqrt{2}} = \frac{1}{\sigma_{tot}n_1\sqrt{2}}$$