Supergravity as the square of Super Yang-Mills
A geometric approach

Silvia Nagy

Imperial College London,
based on work done in collaboration with:
L. Borsten, M. J. Duff and L. J. Hughes
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Clues for an unexpected relationship

A (super)quick intro to Supergravity and Super Yang-Mills

The Scalar cosets of Supergravity
N=1,2,4,8 Super Yang Mills over the division algebras

The Magic Square

Projective planes
Isometries of the projective planes
The Magic Square

A Magic Square of Supergravities

Tensoring the Multiplets
MAGIC!

Magic pyramid

Conclusions and future work
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This strange connection already found in...

Scattering Amplitudes
KLT Relations in String Theory
Supergravity Multiplets from Yang-Mills multiplets in 10d
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Supergravity

• Low energy limit of string theory
Supergravity

Low energy limit of string theory

General relativity + supersymmetry (local susy parameter)
Supergravity

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- Low energy limit of string theory
- General relativity + supersymmetry (local susy parameter)
- Field content- supergravity multiplets
Supergravity

- Low energy limit of string theory
- General relativity + supersymmetry (local susy parameter).
- Field content- supergravity multiplets.
- Characterised by scalar coset groups.
What are the scalar cosets?

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What are the scalar cosets?

Symmetries of theories obtained by reduction on various manifolds

Study symmetries of scalars

General form of transformation:

\[ V' = O V \Lambda \]  

\( V \) obtained by exponentiating scalars with Cartan Generator + positive root generators

\( \Lambda \) is the global symmetry transformation (G group)

\( O \) is the compensating transformation

Example: 2 torus reduction gives the scalar coset \( SL(2) \) \( SO(2) \)

Scalars determine symmetries of all fields!
What are the scalar cosets?

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Super Yang-Mills

\[ \mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{i}{2} \text{Tr}(\bar{\lambda}, \gamma^\mu D_\mu \lambda) \]  

\begin{itemize}
  \item \( \mathcal{N} = 1 \) Super YM Lagrangian:
\end{itemize}
Super Yang-Mills

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- SYM multiplets
Super Yang-Mills

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- SYM multiplets
- SYM theories are characterised by the R-symmetry, which describes transformations of different supercharges into each other.
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$\mathcal{N} = 1, 2, 4, 8$ Super Yang Mills over the division algebras

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In 3 dimensions, we can write the Lagrangian:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_{\mu} \phi^* A_{D \mu} \phi + i \bar{\lambda}_A \gamma^\mu D_{\mu} \lambda_A - \frac{1}{4} g^2 f_{BC \lambda} f_{DE \mu} \langle \phi^B | \phi^D \rangle \langle \phi^C | \phi^E \rangle + i g f_{BC \lambda} (\bar{\lambda}_A \phi^B \lambda_C - \bar{\lambda}_A (\phi^* B \lambda_C))$$
\[ N = 1, 2, 4, 8 \] Super Yang Mills over the division algebras

In 3 dimensions, we can write the Lagrangian:

\[ \mathcal{L} = - \frac{1}{4} F^A_{\mu \nu} F^{A \mu \nu} - \frac{1}{2} D_\mu \phi^A D^\mu \phi^A + i \bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \\
- \frac{1}{4} g^2 f_{BC}^A f_{DE}^A \langle \phi^B | \phi^D \rangle \langle \phi^C | \phi^E \rangle \\
+ \frac{i}{2} g f_{BC}^A \left( (\bar{\lambda}^A \phi^B) \lambda^C - \bar{\lambda}^A (\phi^B \lambda^C) \right) \]
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The projective plane

- A set of points and lines together with a relation between them, satisfying the following axioms:
  - For any two distinct points, there is a unique line on which they both lie.
  - For any two distinct lines, there is a unique point which lies on both of them.
  - There exist four points, no three of which lie on the same line.
  - The terms point and line are interchangeable in the above definition.
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The Fano Plane
A more intuitive definition

For any field $F$, the projective plane $FP^2$ is the set of equivalence classes of non-zero points in $F^3$, where the equivalence relation is given by:

$$(x, y, z) \equiv (rx, ry, rz)$$

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The real projective plane
Some simple Lie Algebras

- \( \mathfrak{so}(n) = \{x \in \mathbb{R}[n] : x^\dagger = -x, \text{tr}(x) = 0\} \)
Some simple Lie Algebras

- $so(n) = \{ x \in \mathbb{R}[n] : x^\dagger = -x, \text{tr}(x) = 0 \}$
- $su(n) = \{ x \in \mathbb{C}[n] : x^\dagger = -x, \text{tr}(x) = 0 \}$
Some simple Lie Algebras

- $\mathfrak{so}(n) = \{ x \in \mathbb{R}[n] : x^\dagger = -x, \text{tr}(x) = 0 \}$
- $\mathfrak{su}(n) = \{ x \in \mathbb{C}[n] : x^\dagger = -x, \text{tr}(x) = 0 \}$
- $\mathfrak{sp}(n) = \{ x \in \mathbb{H}[n] : x^\dagger = -x \}$
Isometries of projective planes
Isometries of projective planes

- \( \text{isom}(\mathbb{R}P^2) \cong \text{so}(3) \)
Isometries of projective planes

- $\text{isom}(\mathbb{RP}^2) \cong so(3)$
- $\text{isom}(\mathbb{CP}^2) \cong su(3)$
Isometries of projective planes

- $\text{isom}(\mathbb{R}P^2) \cong \text{so}(3)$
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Isometries of projective planes

\begin{itemize}
  \item \text{isom}(\mathbb{RP}^2) \cong so(3)
  \item \text{isom}(\mathbb{CP}^2) \cong su(3)
  \item \text{isom}(\mathbb{HP}^2) \cong sp(3)
  \item \text{isom}(\mathbb{OP}^2) \cong f_4
\end{itemize}
What about the exceptional groups?
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- \( \text{isom}(\mathbb{C} \otimes \mathbb{O} \mathbb{P}^2) \cong e_6 \)
What about the exceptional groups?

- $\text{isom}(\mathbb{C} \otimes \mathbb{O}) \mathbb{P}^2 \cong e_6$
- $\text{isom}(\mathbb{H} \otimes \mathbb{O}) \mathbb{P}^2 \cong e_7$
What about the exceptional groups?

- \(\text{isom}((\mathbb{C} \otimes \mathbb{O}) \mathbb{P}^2) \cong e_6\)
- \(\text{isom}((\mathbb{H} \otimes \mathbb{O}) \mathbb{P}^2) \cong e_7\)
- \(\text{isom}((\mathbb{O} \otimes \mathbb{O}) \mathbb{P}^2) \cong e_8\)
The magic square

- Define the magic square by:

\[ M(A_1, A_2) = isom((A_1 \otimes A_2)P^2) \] (5)
The magic square

- Define the magic square by:

\[ M(A_1, A_2) = \text{isom}( (A_1 \otimes A_2) \mathbb{P}^2 ) \]  

<table>
<thead>
<tr>
<th>( A_L / A_R )</th>
<th>( \mathbb{R} )</th>
<th>( \mathbb{C} )</th>
<th>( \mathbb{H} )</th>
<th>( \mathbb{O} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>( \text{SL}(2, \mathbb{R}) )</td>
<td>( \text{SU}(2, 1) )</td>
<td>( \text{USp}(4, 2) )</td>
<td>( F_4(-20) )</td>
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<tr>
<td>( \mathbb{C} )</td>
<td>( \text{SU}(2, 1) )</td>
<td>( \text{SU}(2, 1) \times \text{SU}(2, 1) )</td>
<td>( \text{SU}(4, 2) )</td>
<td>( E_6(-14) )</td>
</tr>
<tr>
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<td>( \text{USp}(4, 2) )</td>
<td>( \text{SU}(4, 2) )</td>
<td>( \text{SO}(8, 4) )</td>
<td>( E_7(-5) )</td>
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Tensoring the Multiplets

- In 3 dimensions, we tensor together left and right multiplets of Super YM, for $\mathcal{N} = 1, 2, 4, 8$
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$$\mathcal{N}_L(SYM) + \mathcal{N}_R(SYM) = \mathcal{N}_{SuGra}$$
Tensoring the Multiplets

- In 3 dimensions, we tensor together left and right multiplets of Super YM, for $\mathcal{N} = 1, 2, 4, 8$
  \[ \mathcal{N}_L(SYM) + \mathcal{N}_R(SYM) = \mathcal{N}_{SuGra} \] (6)

- We get the supergravity magic square:
## Magic Square of Supergravities

<table>
<thead>
<tr>
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<th>R</th>
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<th>H</th>
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<td>R</td>
<td>SL(2, R)</td>
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SYM in various dimensions

- Remember the lagrangian:

\[ \mathcal{L} = -\frac{1}{4} Tr(F_{\mu\nu}, F^{\mu\nu}) - \frac{i}{2} Tr(\bar{\lambda}, \gamma^\mu D_\mu \lambda) \quad (7) \]
SYM in various dimensions

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- We want its SUSY variation to vanish
**SYM in various dimensions**

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(7)

- We want its SUSY variation to vanish
- We get a term of the form:

\[ \text{Tr}(\lambda, \gamma^{\mu} [(\epsilon \gamma^\mu \lambda), \lambda]) \]  

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- We want its SUSY variation to vanish
- We get a term of the form:
  \[ \text{Tr}(\lambda, \gamma^\mu [(\epsilon \gamma^\mu \lambda), \lambda]) \]  

- Only vanishes in 3, 4, 6 and 10 dimensions!
What have we learnt so far?

• Extended Super YM theories characterised by division algebras in 3D.
• Tensor them to get supergravities, whose global symmetry groups are given by the magic square construction.
• Pure super-Yang-Mills theories only exist in 3, 4, 6, and 10 dimensions!
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- Division algebras provide further evidence for the idea that Supergravity is, in a sense, the square of a gauge theory.
Conclusions and future work

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- Supergravity pyramid, Lagrangian.
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- Symmetries of SuGra from symmetries of SYM.