The Iwasawa decomposition for symmetric spaces

It can be applied only to a non compact group $G_{nc}$. Let’s start by studying an irreducible representation of the algebra $\mathfrak{g}$.

1) Identification of the Lie algebra $\mathfrak{g}$ corresponding to the symmetrically embedded maximal compact subgroup $H$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, with $\mathfrak{p}$ the vector space orthogonal to $\mathfrak{h}$ and $\mathfrak{h} = \text{dim}(H)$.

2) Choice of a maximal non compact Cartan subgroup $\mathfrak{a}$ as a pivot. It is generated by $i$ commuting generators $(a_1, ..., a_i)$, where $i$ is the rank of the group. Out of these, all but one can be chosen non compact, i.e. $(a_1, ..., a_{i-1})$, where $i$ is the rank of the group $G/D$.

3) Calculation of the corresponding system of positive roots $(\rho_i)$ and of the corresponding eigenmatrices $(\lambda_{\rho_i})$, $i = 1, ..., n$, of the dimension of the normalizer of $a_i$ in $H$.

4) Computation of a realization of a generic element $\gamma$ of the group $G$ by using the above fiberation to construct a well-defined parametrization of the group:

$$\gamma = HAN$$

with $H$ the fiber,

$\mathfrak{A}$Abelian subgroup generated by $\mathfrak{a}_i$,

$\mathfrak{N}$nilpotent subgroup generated by the eigenmatrices $(\lambda_{\rho_i})$, $i = 1, ..., n - 2$, $k = i$ with the dimension of the normalizer of $\mathfrak{a}_i$ in $\mathfrak{h}$.

Generalization to all the faithful representations for all the non compact real forms

Let $p$ be a complex simple Lie algebra of rank $l$ and $\mathfrak{g}$ be its real form corresponding to a given symmetric space of type $T$ and rank $l$. Consider its Satake diagram in the classification by Vinberg (Journal of Mathematics, Osaka City University, 13 (1962)). Let $E_i$ be the column vector in $\mathbb{R}^l$ with entry $1$ if the corresponding simple root in the diagram is associated to a white dot, i.e. if it belongs to the quotient, and zero otherwise (Satake vector). Let $|E_i|_2$ be the canonical basis of $E_i$. Then the nilpotent $\mathfrak{N}(E_i)$ in the $\mathfrak{g}$-$\mathfrak{h}$ fundamental representation is a polynomial degree $d_{2p} = 2d_{2l}$, where $d_{2l} = \mathfrak{g}^{\mathfrak{h}}$. Here $\mathfrak{g}$ is the Cartan matrix of $\mathfrak{g}$.

For reducible representations: the degree is determined by the maximal span among all subrepresentations.

For semisimple groups: by the maximal span among the simple factors.

Non compact Lie groups with the list of data necessary for the analysis

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\text{dim}(\mathfrak{g})$</th>
<th>$\mathfrak{a}$</th>
<th>$\text{dim}(\mathfrak{a})$</th>
<th>$\mathfrak{p}$</th>
<th>$\text{dim}(\mathfrak{p})$</th>
<th>$\mathfrak{N}$</th>
<th>$\text{dim}(\mathfrak{N})$</th>
<th>$\mathfrak{H}$</th>
<th>$\text{dim}(\mathfrak{H})$</th>
</tr>
</thead>
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<tr>
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<td>C</td>
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<td>$\mathfrak{p}$</td>
<td>1</td>
<td>$\mathfrak{N}$</td>
<td>1</td>
<td>$\mathfrak{H}$</td>
<td>1</td>
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<tr>
<td>$\mathfrak{so}_2$</td>
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<td>$\mathfrak{p}$</td>
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<td>$\mathfrak{N}$</td>
<td>0</td>
<td>$\mathfrak{H}$</td>
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</tbody>
</table>

Applications to supergravity

From the results in tables 1–3:

1) Simplest case: 4-dim. $\mathfrak{N} = 4$ pure supergravity and $\mathfrak{N} = 2$ supergravity minimally coupled to one Abelian vector multiplet

$\mathfrak{g} = \mathfrak{so}(1,1)(\mathbb{C})$ in the representation 2.

Result: $\text{dim}(\mathfrak{g}) = 1$.

2) The $\mathfrak{N}$ model $\frac{\mathfrak{g}}{\mathfrak{h}}$ in the rep. 4 = weight $3\mathfrak{e}_1$.

This agrees with the findings of [Greene, Ferrara, Grossi, Mariani, Milanowski, 1201.0531; 1204.5175].

It is an example of a representation which does not correspond to a fundamental weight.

3) $\mathfrak{N} = 2$ Magic theories associated to the algebra $\mathfrak{E}(A_1, A_1(3,3), (3,3,3))$


In this table $\mathfrak{C}_{\mathfrak{N}}$ describes the electric magnetic duality of the corresponding theory, which is defined in $D$ dimensions. Each subsequent row can be obtained by dimensional reduction of the previous one, e.g., each column from the following one by truncation:

| $\mathfrak{N}$ | $\mathfrak{g}$ | $\text{dim}(\mathfrak{g})$ | $\mathfrak{a}$ | $\text{dim}(\mathfrak{a})$ | $\mathfrak{p}$ | $\text{dim}(\mathfrak{p})$ | $\mathfrak{N}$ | $\text{dim}(\mathfrak{N})$ | $\mathfrak{H}$ | $\text{dim}(\mathfrak{H})$ |
|---------------|------------------|---------------|------------------|---------------|------------------|---------------|------------------|---------------|------------------|
| 2 | $\mathfrak{so}(1,1)(\mathbb{C})$ | 1 | C | 0 | $\mathfrak{p}$ | 0 | $\mathfrak{N}$ | 0 | $\mathfrak{H}$ | 0 |
| 2 | $\mathfrak{su}(2)$ | 2 | C | 1 | $\mathfrak{p}$ | 1 | $\mathfrak{N}$ | 1 | $\mathfrak{H}$ | 1 |
| 2 | $\mathfrak{so}(2)$ | 1 | C | 0 | $\mathfrak{p}$ | 0 | $\mathfrak{N}$ | 0 | $\mathfrak{H}$ | 0 |

Notice that the values of $d_{2p}$ depend only on the space-time dimension $D$. This is consistent with the Tate-Satake projection [Fru, Sonn, Triegrar, arXiv:1107.5846 (hep-th)].