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Some properties of the Iwasawa polynomials for irreducible symmetric spaces in supergravity

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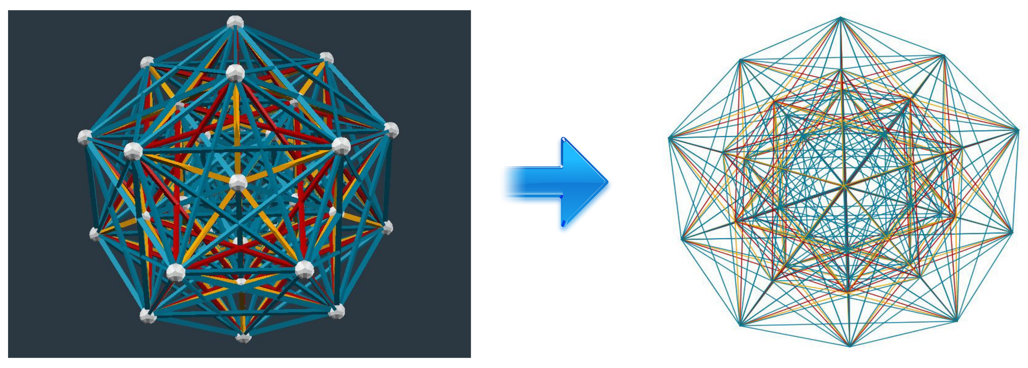
based on work with S. Cacciatori, S. Ferrara and A. Marrani, to appear soon



Abstract

We analyse the polynomial part of the Iwasawa realization of symmetric spaces appearing in supergravity. We first study the role of the principal $SL(2)_P$ subgroup and how it determines the structure of the nilpotent subalgebras. This allows us to compute the maximal degree of the polynomials for all the faithful representations of the algebra. In particular, we find the degree of the metric coefficients. Since the coset representative directly enters in the Lagrangian of the corresponding supergravity theory, this is important for the investigation of its features, and should be helpful e.g. in approaching such issues as its ultraviolet properties.

Plan

- The Iwasawa decomposition for symmetric spaces
- Konstant's analysis of the nilpotent subalgebra for the adjoint representation of the split form and the principal $\mathfrak{su}(2)_P$
- Generalization to all the faithful representations for all the non compact real forms
- Tables with summary of the results
- Applications to supergravity theories
- Conclusions and outlook

The Iwasawa decomposition for symmetric spaces

It can be applied only to a non compact group G_{nc}. Let's start by studying an irreducible representation of the algebra $\mathfrak{g}$.

1) Identification of the Lie algebra $\mathfrak{h}$ corresponding to the symmetrically embedded maximal compact subgroup H:
$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, with $\mathfrak{p}$ the vector space orthogonal to $\mathfrak{h}$ and $h = \dim(H)$.

2) Choice of a maximally non compact Cartan subalgebra $\mathfrak{a}$ as a pivot. It is generated by l commuting generators $\{c_1, \ldots, c_l\}$, where l is the rank of the group. Out of these, at most r can be chosen non compact, i.e. $\{c_1, \ldots, c_r\}$, where r is the rank of the coset G/H:
$\mathfrak{a} = \mathfrak{a}_\mathfrak{h} \oplus \mathfrak{a}_\mathfrak{p}$, with $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$, $\mathfrak{a}_\mathfrak{h} \subset \mathfrak{h}$, $\dim(\mathfrak{a}_\mathfrak{p}) = r$, $\dim(\mathfrak{a}_\mathfrak{h}) = s = l - r$.

3) Calculation of the corresponding system of positive roots $\{\alpha_i\}$ and of the corresponding eigenmatrices $\{\lambda_{\alpha_i}\}$, $i = 1, \ldots h - k$, with k the dimension of the normalizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{h}$.

4) Computation of a realization of a generic element g of the group by using the above fibration to construct a well-defined parametrization of the group:

$$g = HAN$$

with H the fiber,
A=Abelian subgroup generated by $\mathfrak{a}_p$,
N=nilpotent subgroup generated by the eigenmatrices $\{\lambda_{\alpha_i}\}$ of the adjoint action of A on the system of positive roots.

$$\Downarrow$$

A representative for the coset G/H can be written as:

$$\mathcal{M}[\vec{y}, \vec{x}] = \exp\left(\sum_{i=1}^{r} y^i c_i\right) \exp\left(\sum_{a=1}^{h-k} x^a \lambda_{\alpha_a}\right).$$

Konstant's analysis of the nilpotent subalgebra for the adjoint representation of the split form and the principal $\mathfrak{su}(2)_P$

[B. Kostant, Am. J. Math. **81**, 973–1032 (1959)]

Poincaré polynomial of G:

$$f_G(t) = \prod_{A=1}^{l} \left(1 + t^{2\delta_A - 1}\right) \text{ with } \sum_{A=1}^{l} (2\delta_A - 1) = \dim(G)$$

It determines the topology of the compact form G.

Invariant symmetric primitive polynomials:
l polynomials of order δ_A $(A = 1, \ldots, l)$ $\Rightarrow$
l primitive Racah-Casimir polynomials $\mathcal{C}_{\delta_A}$ of order δ_A and
l skewsymmetric invariant primitive tensors $\Omega^{(2\delta_A - 1)}$ of order $2\delta_A - 1$. $\Rightarrow$ Homology behaves as a product of l spheres $S^{(2\delta_A - 1)}$.

Principal $\mathfrak{su}(2)_P$: Unique $\mathfrak{su}(2)$ embedded in $\mathfrak{g}$ generally non-symmetrically (exception: $\mathfrak{su}(3)$) and generally maximally (exception: $\mathfrak{e}_6$) in such a way that $\delta_A = j_A + 1$, where:

$$\mathbf{Adj}(G) = \sum_{A=1}^{l} (\mathbf{2j_A + 1}) = \sum_{A=1}^{l} \mathbf{S}_{j_A}, \quad j_1 = 1 < \ldots < j_l$$

with $\mathbf{S}_{j_A}$ the $\mathfrak{su}(2)_P$-irrep. of spin $s = j_A$.

Nilpotent part of the coset representative $\mathcal{M}[\vec{x}, \vec{y}]$:

$$N(\vec{x}) = \exp\left(\sum_{a=1}^{h} x^a \lambda_{\alpha_a}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{a=1}^{h} x^a \lambda_{\alpha_a}\right)^n$$

$\Rightarrow$ Terms of degree n:

$$M_n = \frac{1}{n!} \sum_{d_1 + \ldots + d_h = n,\ d_i \geq 0} \frac{(x^1)^{d_1}}{d_1!} \cdots \frac{(x^h)^{d_h}}{d_h!} \sum_{\text{permutations } \sigma} \lambda_{\alpha_{\sigma_1}} \cdots \lambda_{\alpha_{\sigma_n}}$$

Problem: Determine the largest n so that $M_n \neq 0$ and $M_{n+1} = 0$.
Answer: Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, such that all other positive roots are linear combinations of these with non-negative integer coefficients.
Let $q = n_1 + \ldots + n_l$, with $n_1, \ldots, n_l$ the coefficients of the longest root $\alpha_L = n_1\alpha_1 + \ldots + n_l\alpha_l$. Notice that

$$q = j_l = C_G - 1$$

with C_G the Coxeter number of G, and j_l the maximal spin of the $\mathfrak{sl}(2)_P$-irreps. into which the adjoint irrep. $\mathbf{Adj}$ of G branches.

Then the nilpotency of the elements of N is $2q + 1$, i.e. the maximal degree is $2q$.

Generalization to all the faithful representations for all the non compact real forms

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l and $\mathfrak{g}^T$ be its real form corresponding to a given symmetric space of type T and rank l. Consider its Satake diagram in the classification by [Araki, Journal of Mathematics, Osaka City University, **13** (1962) 1] and let $\vec{\varepsilon}^T$ be the column vector in $\mathbb{R}^l$ with entry 1 if the corresponding simple root in the diagram is associated to a white dot, i.e. if it belongs to the quotient, and zero otherwise (*Satake vector*).
Let $\{\vec{e}_i\}_{i=1}^{l}$ the canonical basis of $\mathbb{R}^l$. Then the nilpotent $N(\vec{x})$ in the i-th fundamental representation is a polynomial of degree

$$d^i_{\mathfrak{g}^T} = 2j^i_{\mathfrak{g}^T}, \text{ where } j^i_{\mathfrak{g}^T} = \vec{e}_i \cdot \boldsymbol{C}^{-1}\vec{\varepsilon}^T.$$

Here $\boldsymbol{C}$ is the Cartan matrix of $\mathfrak{g}$.

For reducible representations: the degree is determined by the maximal spin among all sub representations,
for semisimple groups: by the maximal degree among the simple factors.

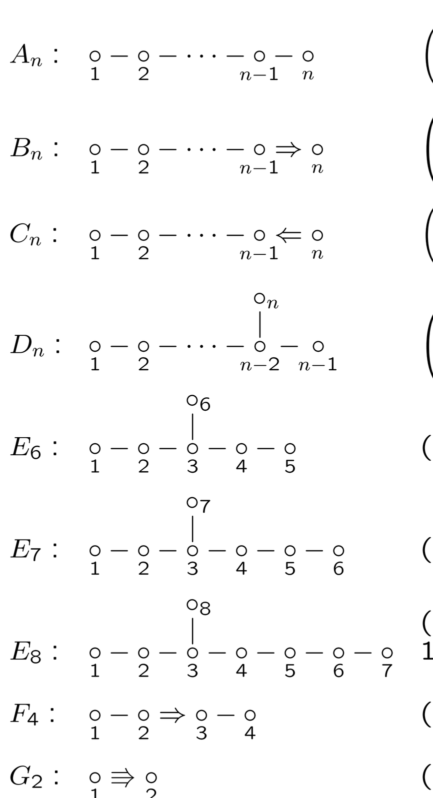
Tables with summary of the results

The first table shows for the simple Lie algebras the corresponding Dynkin diagram and the dimensions of the fundamental representations associated to each of the nodes, by applying the Weyl character formula (see e.g. [Bourbaki, *"Groupes et algèbres de Lie"*, Chap. VIII.13, Herman, Paris, 1975]).

In the second table the main ingredients necessary for the Iwasawa construction of the non-compact, irreducible, Riemannian, globally symmetric spaces $T = G_{nc}/H$ are listed (see also [Cacciatori, Dalla Piazza, Scotti, arXiv:1207.1262 [math.GR]]), where
T is the coset of the non compact real form G_{nc} of the Lie group G with respect to its maximal compact subgroup H in the classification from [Araki, Journal of Mathematics, Osaka City University, **13** (1962) 1; Helgason, *"Differential Geometry, Lie Groups and Symmetric Spaces"*, Academic Press, New York, 1978)].
$\Lambda_{G/H}$ specifies the type of the root system,
$(n_1, \ldots, n_r)$ are the coefficients of the longest root α_L,
$\vec{m}_\lambda$, $\vec{m}_{2\lambda}$ the multiplicities for the roots and the double roots.

The third table lists the rank of the group, the vector of the degrees of the polynomials in the fundamental representations, and the degree of the polynomial part in the biinvariant metric.

Fundamental Representations of the Exceptional Lie groups



- A_n: $\left(\left\{\binom{n+1}{i}\right\}_{i=1}^{n}\right)$
- B_n: $\left(\left\{\binom{2n+1}{i}\right\}_{i=1}^{n-1}, 2^n\right)$
- C_n: $\left(\left\{\binom{2n}{i} - \binom{2n}{i-2}\right\}_{i=1}^{n}\right)$
- D_n: $\left(\left\{\binom{2n}{i}\right\}_{i=1}^{n-2}, 2^{n-1}, 2^{n-1}\right)$
- E_6: $(27, 351, 2925, 351, 27, 78)$
- E_7: $(133, 8645, 365750, 27664, 1539, 56, 912)$
- E_8: $(3875, 6696000, 6899079264, 146325270, 2450240, 30380, 248, 147250)$
- F_4: $(52, 1274, 273, 26)$
- G_2: $(14, 7)$

Non compact Lie groups with the list of data necessary for the analysis

T	G_{nc}	H	$\Lambda_{G/H}$	$(n_1, ..., n_r)$	$\vec{m}_\lambda$, $\vec{m}_{2\lambda}$
AI	$SL(n+1,\mathbb{R})$	$SO(n+1)$	$A_n\,(n \geq 1)$	(1,1,. . . ,1)	(1), (0)
AII	$SU^*(2k)$	$USp(2k)$	$A_{k-1}\,(k > 1)$	(1,1,. . . ,1)	(4), (0)
AIII$_a$	$SU(p,q)$	$S(U(p) \times U(q))$	$B_p\,(1 < p < q)$	(2,2,. . . ,2)	$2(1, q-p), (0, 1)$
AIII$_b$	$SU(p,p)$	$S(U(p) \times U(p))$	$C_p\,(p > 1)$	$(2, 2, \ldots, 2, 1)$	(1,2), (0,0)
AIV	$SU(1,n)$	$S(U(1) \times U(n))$	A_1	(2)	(2n-2), (1)
BI$_a$	$SO(n,n+1)$	$SO(n) \times SO(n+1)$	$B_n\,(n \geq 2)$	(1,2,. . . ,2)	(1,1), (0,0)
BI$_b$	$SO(p,q)$	$SO(p) \times SO(q)$	$B_p\,(1 < p < n)$	(1,2,. . . ,2)	(1,2(n-p)+1), (0,0)
BII	$SO(1,2n)$	$SO(2n)$	A_1	(1)	(2n-1), (0)
CI	$Sp(2n,\mathbb{R})$	$U(n)$	$C_n\,(n \geq 3)$	(2,2,. . . ,2,1)	(1,1), (0,0)
CII_a	$USp(2p,2q)$	$USp(2p) \times USp(2q)$	$B_p\,(1 \leq p \leq (n-1)/2)$	(2,2,. . . ,2)	(4,4n-8p), (0,3)
CII_b	$USp(2k,2k)$	$USp(2k) \times USp(2k)$	C_k	$(2, 2, \ldots, 2, 1)$	(3,4), (0,0)
DI_a	$SO(n,n)$	$SO(n) \times SO(n)$	$D_n\,(n > 3)$	(1,2,. . . ,2,1,1)	(1), (0)
DI_b	$SO(n-1,n+1)$	$SO(n-1) \times SO(n+1)$	$B_{n-1}\,(n > 2)$	(1,2,. . . ,2)	(1,2), (0,0)
DI_c	$SO(p,q)$	$SO(p) \times SO(q)$	$B_p\,(1 < p < n-1)$	(1,2,. . . ,2)	(1,2(n-p)), (0,0)
DII	$SO(1,2n-1)$	$SO(2n-1)$	A_1	(1)	(2n-2), (0)
$DIII_a$	$SO^*(4k+2)$	$U(2k+1)$	$B_k\,(k \geq 2)$	(2,2,. . . ,2)	(4,4), (0,1)
$DIII_b$	$SO^*(4k)$	$U(2k)$	$C_k\,(k \geq 2)$	(2,2,. . . ,2,1)	(1,4), (0,0)
G	$G_{2(2)}$	$SO(4)/\mathbb{Z}_2$	G_2	(3,2)	(1,1), (0,0)
FI	$F_{4(4)}$	$USp(6) \times USp(2)$	F_4	(2,3,4,2)	(1,1), (0,0)
FII	$F_{4(-20)}$	$SO(9)$	A_1	(2)	(8), (7)
EI	$E_{6(6)}$	$USp(8)/\mathbb{Z}_2$	E_6	(1,2,2,3,2,1)	(1),(0)
EII	$E_{6(2)}$	$(USp(2) \times SU(6))/\mathbb{Z}_2$	F_4	(2,3,4,2)	(1,2), (0,0)
EIII	$E_{6(-14)}$	$(U(1) \times SO(10))/\mathbb{Z}_4$	B_2	(2,2)	(6,8), (0,1)
EIV	$E_{6(-26)}$	F_4	A_2	(1,1)	(8), (0)
EV	$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$	E_7	(2,2,3,4,3,2,1)	(1),(0)
EVI	$E_{7(-5)}$	$(SU(2) \times SO(12))/\mathbb{Z}_2$	F_4	(2,3,4,2)	(1,4), (0,0)
EVII	$E_{7(-25)}$	$(U(1) \times E_6)/\mathbb{Z}_3$	C_3	(2,2,1)	(1,8), (0,0)
EVIII	$E_{8(8)}$	$S_s(16)$	E_8	(2,3,4,6,5,4,3,2)	(1), (0)
EIX	$E_{8(-24)}$	$(SU(2) \times E_7)/\mathbb{Z}_2$	F_4	(2,3,4,2)	(1,8), (0,0)

Summary of the analysis: fundamental degrees and degree of the metric

T	l	$(d_1, \ldots, d_l)$	$d_\mathfrak{g}$
AI	n	$(\{i(n+1-i)\}_{i=1}^{n})$	$2(n-1)$
AII	$2k-1$	$(\{2(k-i)(i-1)+k-1, 2i(k-i)\}_{i=1}^{k-1}, k-1)$	$2k-4$
AIII$_a$	$p+q-1=2n-1$	$(\{i(2p+1-i)\}_{i=1}^{p-1}, \{p(p+1)\}_{i=p}^{q}, \{(2n-i)(i+p-q+1)\}_{i=q+1}^{2n-1})$	$4p-2$
AIII$_b$	2p-1	$(\{i(2p-i)\}_{i=1}^{2p-1})$	$4p-4$
AIV	n	(2,. . . ,2)	2
BI$_a$	n	$(\{i(2n+1-i)\}_{i=1}^{n-1}, n(n+1)/2)$	$4n-4$
BI$_b$	$n=(p+q-1)/2$	$(\{i(2p+1-i)\}_{i=1}^{p}, \{p(p+1)\}_{i=p+1}^{n})$	$4p-4$
BII	n	$(2, \ldots, 2, 1)$	0
CI	n	$(\{i(2n-i)\}_{i=1}^{n})$	$4n-4$
CII_a	$p+q=n=2k$	$(\{2(2i-1)(p+1)-2i^2, 2i(2p-i+1)\}_{i=1}^{p}, \{2p(p+1)\}_{i=p+1}^{n})$	$4p-2$
CII_b	$2k$	$(\{2(2i-1)(2k-1)-2i^2+2i-1, 2i(2k-i)\}_{i=1}^{k})$	$4k-4$
DI_a	n	$(\{i(2n-1-i)\}_{i=1}^{n-2}, n(n-1)/2, n(n-1)/2)$	$4n-8$
DI_b	n	$(\{i(2n-1-i)\}_{i=1}^{n-2}, n(n-1)/2, n(n-1)/2)$	$4n-8$
DI_c	$n=(p+q)/2$	$(\{2pi-i^2+i\}_{i=1}^{p-1}, \{p^2+p\}_{i=p}^{n})$	$4p-4$
DII	n	(2,. . . ,2,1,1)	0
$DIII_a$	$2k+1$	$(\{2ki-2\lfloor\frac{i}{2}\rfloor^2\}_{i=1}^{2k-1}, k^2+k-1, k^2+k-1)$	$4k-2$
$DIII_b$	$2k$	$(\{2ki-2\lfloor\frac{i}{2}\rfloor^2-i\}_{i=1}^{2k-2}, k^2-1, k^2)$	$4k-4$
G	2	$(10, 6)$	8
FI	4	$(22, 42, 30, 16)$	20
FII	4	$(4, 8, 6, 4)$	2
EI	6	$(16, 30, 42, 30, 16, 22)$	20
EII	6	$(16, 30, 42, 30, 16, 22)$	20
EIII	6	$(6, 10, 14, 10, 6, 8)$	6
EIV	6	$(4, 6, 8, 6, 4, 4)$	2
EV	7	$(34, 66, 96, 75, 52, 27, 49)$	32
EVI	7	$(22, 42, 60, 46, 32, 16, 30)$	20
EVII	7	$(10, 18, 26, 21, 16, 9, 13)$	8
EVIII	8	$(92, 182, 270, 220, 168, 114, 58, 136)$	56
EIX	8	$(32, 62, 92, 76, 60, 42, 22, 46)$	20

Applications to supergravity

From the results in tables 1–3:

1) Simplest case: 4-dim. $\mathcal{N} = 4$ pure supergravity and $\mathcal{N} = 2$ supergravity minimally coupled to one Abelian vector multiplet $\frac{G}{H} = \frac{SL(2,\mathbb{R})}{SO(2)}$ in the representation 2
$\Rightarrow$ Result: d=1

2) The t^3 model: $\frac{G}{H} = \frac{SL(2,\mathbb{R})}{SO(2)}$ in the rep. $4 =$ weight $3\lambda_1$
$\Rightarrow$ Result: d=3
This agrees with the findings of [Ceresole, Ferrara, Gnecchi, Marrani, arXiv:1210.5983 [hep-th]].
It is an example of a representation which does not correspond to a fundamental weight.

3) $\mathcal{N} = 2$ Magic theories associated to the algebra $\mathcal{L}_3(\mathbb{A}_S, J_3(\mathbb{B}))$
[Günaydin, Sierra, Townsend, Phys. Lett. **133B** (1983) 72]

In this table G_{nc} describes the electric magnetic duality of the corresponding theory, which is defined in D dimensions. Each subsequent row can be obtained by dimensional reduction of the preceding one, each column from the following one by truncation.

$\mathbb{A}_S \setminus \mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	D	d	d_g
$\mathbb{C}_S$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$	5	4	2
$\mathbb{H}_S$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$	4	9	8
$\mathbb{O}_S$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$	3	22	20

Notice that the values of d, d_g depend only on the space-time dimension D. This is consistent with the Tits-Satake projection [Fré, Sorin, Trigiante, arXiv:1107.5986 [hep-th]].

Conclusions and outlook

- From the point of view of mathematics, we have extended the methods by Kostant [B. Kostant, Am. J. Math. **81**, 973–1032 (1959)] for studying the maximal nilpotency of subalgebras in the adjoint representation of the split form to all the faithful representations of the algebra for all the non compact forms. This technique is based on the embedding of the principal $sl(2)_P$, determining the dimensions of the Hopf spheres which enter in the cohomology of the group. Therefore, such an approach should lead to a deeper understanding of its cohomological structure.
- From the point of view of physics, it is well-known that irreducible symmetric spaces play a prominent role in supergravity theories (see e.g. [Ferrara, Marrani, arXiv:0808.3567 [hep-th]]). Then the Iwasawa parametrization provides a way to choose the fields so that they enter in a polynomial way in the Lagrangian. A knowledge of their structure could thus provide useful insights on the quantum properties of the theory, including the ultraviolet behaviour.