$\mathcal{N} = 4$ Super Yang–Mills Theory on the Coulomb Branch

John H. Schwarz

California Institute of Technology

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Introduction

The goal of this lecture is to derive the world-volume action of a probe D3-brane in an $AdS_5 \times S^5$ type IIB background and to interpret it as an effective action for $\mathcal{N} = 4$ SYM theory on the Coulomb branch. First, I will explain exactly what this sentence means and why it is interesting. This lecture is based on arXiv:1311.0305.

The second lecture, which will discuss soliton solutions of the D3-brane action, is based on arXiv:1405.7444.

N = 4 Super Yang–Mills Theory

"Supersymmetric Yang–Mills Theories" (L. Brink, J. Scherk, and JHS), Nucl. Phys. **B121** (1977) 77.

$\mathcal{N} = 4$ is the maximal susy that is possible for a nongravitational QFT in 4d. $\mathcal{N} = 4$ SYM exists for any gauge group, but I will focus on $U(N)$, so the fields are $N \times N$ hermitian matrices.

It was derived from 10d SYM, which contains a vector A_M and a MW spinor Ψ, by dimensional reduction. This gives a 4d theory containing a vector A_μ, six scalars ϕ^I, and four Weyl spinors ψ^A.

The theory has an $SU(4) \sim SO(6)$ symmetry, which corresponds to rotations of the six extra dimensions. Because it also rotates the four supercharges Q^A, it is called an R-symmetry.

$\mathcal{N} = 4$ SYM is conformally invariant (and hence UV finite). Combining Lorentz invariance, translation invariance, supersymmetry, scale invariance, conformal invariance, R-symmetry, and conformal supersymmetry gives the supergroup $PSU(2,2|4)$. Its bosonic subgroup is $SU(2,2) \times SU(4)$.

The parameters of $\mathcal{N} = 4$ SYM are a YM coupling constant g, a vacuum angle θ, and the rank N. For large N and fixed

$$\lambda = g^2 N$$

the theory has a $1/N$ expansion with a nice topological interpretation.

The leading term (the planar approximation) has even more symmetry (dual conformal invariance) and is completely integrable. It is not known whether this extends to the complete theory.

Defining

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2},$$

the theory has an $SL(2, \mathbb{Z})$ duality symmetry under which

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers satisfying $ad - bc = 1$.

In the special case $\tau \to -1/\tau$ and $\theta = 0$, this gives $g \to 4\pi/g$, which is called S-duality. It is an exact nonabelian electric-magnetic symmetry.

Type IIB Superstring Theory

"Supersymmetrical String Theories" (M. B. Green and JHS), Phys. Lett. **109B** (1982) 444

This 10d theory has two supersymmetries of the same chirality, for a total of 32 conserved supercharges. Its massless bosonic fields are

- g_{MN} the 10d metric
- σ the dilaton
- B_{MN} the NS-NS two-form
- C, C_{MN}, C_{MNPQ} the RR zero, two, and four forms

The four-form has a self-dual field strength $F_5 = dC_4$.

The value of $\exp(\sigma)$ is the string coupling constant g_s and the value of C is called χ. In terms of these one can form

$$\tau = \chi + i/g_s.$$

Type IIB superstring theory has an exact $SL(2, \mathbb{Z})$ symmetry under which

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

as before.

Type IIB superstring theory has a few solutions that preserve all of the supersymmetry. The most obvious one is 10d Minkowski spacetime, *i.e.*, $g_{MN} = \eta_{MN}$. σ and C are constants and the other fields vanish.

A less obvious maximally supersymmetric solution has a metric describing the geometry $AdS_5 \times S^5$ – I'll give the formula later. Also σ and C are constants and $F_5 \sim N[\mathrm{vol}(AdS_5) + \mathrm{vol}(S^5)]$. N is the number of units of five-form flux threading the five-sphere, $\int_{S^5} F_5 \sim N$.

The isometry of this solution is given by the supergroup $PSU(2,2|4)$.

AdS/CFT Duality

Maldacena (1997): $\mathcal{N} = 4$ SYM with $U(N)$ gauge group is exactly equivalent ("dual") to type IIB superstring theory in an $AdS_5 \times S^5$ background with N units of five-form flux.

Evidence: Both have $PSU(2,2|4)$ symmetry, $SL(2,\mathbb{Z})$ duality, and much, much more. The evidence is overwhelming.

A complete proof is not possible, because we lack a nonperturbative definition of type IIB superstring theory (other than the one given by AdS/CFT duality).

D3-branes

Superstring theories contain various supersymmetric (and hence stable) extended objects that arise as non-perturbative solutions. Ones with p spatial dimensions are called p-branes. Some p branes, on which strings can end, are called Dp-branes. (D = Dirichlet.)

In type IIB superstring theory supersymmetric Dp-branes exist for $p = 1, 3, 5, 7$. Only the D3-brane is invariant under $SL(2, \mathbb{Z})$ transformations.

A stack of N coincident flat Dp-branes has fields that are localized on its $(p+1)$-dimensional world volume. They define a "world-volume theory," which is maximally supersymmetric. In particular, the world-volume theory of N coincident D3-branes is $\mathcal{N}=4$ SYM with a $U(N)$ gauge group.

The branes also act as sources of gravitational and other fields. Maldacena was led to his conjecture by realizing that N coincident D3-branes give a "black brane," whose near-horizon geometry is $AdS_5 \times S^5$ with N units of flux. In this way the branes are replaced by a 10d geometry with a horizon.

The Coulomb Branch

Consider N coincident flat D3-branes and pulling them apart along one of the orthogonal axes into clumps $N_1 + N_2 + \ldots + N_k = N$ at positions x_1, x_2, ..., x_k. Then the gauge symmetry is broken spontaneously,

$$U(N) \to U(N_1) \times U(N_2) \times \ldots \times U(N_k).$$

The off-diagonal fields acquire mass

$$m_{ij} = |x_i - x_j| T,$$

where T is the fundamental string tension. This is similar to the Higgs mechanism.

Let's discuss a specific example: $\mathcal{N} = 4$, $d = 4$ super Yang–Mills theory with $U(2)$ gauge group.

For most purposes one can say that a free $U(1)$ multiplet decouples leaving an $SU(2)$ theory. However, this 'decoupled' $U(1)$ is needed to get the full $SL(2, \mathbb{Z})$ duality group rather than a subgroup.

The $SU(2)$ theory on the Coulomb branch has unbroken $U(1)$ gauge symmetry. Let us call the massless supermultiplet the "photon" and the massive ones $W^{\pm}$.

The Probe Approximation

Consider a D3-brane embedded in the 10d spacetime.

- The probe approximation involves neglecting the back reaction of the brane on the geometry and the other background fields. Since the brane is a source for one unit of flux, this requires that N is large.
- D-brane actions include a DBI term involving a $U(1)$ field strength, $F_{\alpha\beta}$, on the brane. Since, no derivatives of F are included in the formula, F is required to vary sufficiently slowly so that its derivatives can be neglected.

Despite these approximations, the probe D3-brane action has some beautiful exact properties.

- It precisely realizes the isometry of the background as a world-volume superconformal symmetry $PSU(2,2|4)$. (Only the bosonic truncation is considered in the subsequent discussion.)
- The brane action also has the duality symmetry of the background: $SL(2,\mathbb{Z})$ for the D3-brane example.
- The D3-brane world-volume action has local symmetries: general coordinate invariance and a fermionic symmetry called kappa symmetry.

In principle, one can integrate out the massive fields exactly, thereby producing a very complicated formula in terms of the massless photon supermultiplet only.

$$\exp(iS_{\text{eff}}) = \int DW^+ DW^- \exp(iS)$$

The resulting effective action captures the entire theory and is valid (on the Coulomb branch) at all energies. This defines a *highly effective action* (HEA).

Obviously, we cannot carry out such an exact computation. However, in some cases we know many of the properties that the HEA should possess.

General Requirements for an HEA

- It should have all of the unbroken and spontaneously broken global symmetries of the original Coulomb branch theory. Conformal symmetry is spontaneously broken as a consequence of assigning a vev to a massless scalar field that has a flat potential.

- It should have the same duality properties as the Coulomb branch theory containing explicit W fields.

- It should have the same BPS spectrum. The $W^{\pm}$ multiplets must reappear in the HEA as solitons.

I conjecture that the probe D3-brane action, with $N = 1$, is the HEA for the $\mathcal{N} = 4$, $d = 4$ SYM theory with $U(2)$ gauge symmetry on the Coulomb branch, despite the fact that it is only an approximate solution of a different problem.

I am somewhat torn about this. On the one hand, it seems to be too simple to be the exact answer for such a complicated path integral. On the other hand, the evidence is compelling that it has all of the properties of the HEA. There is no known argument for the existence of a second such action.

The AdS Poincaré patch

AdS_{p+2} with unit radius is given by $y \cdot y - uv = -1$, where $y \cdot y = -(y^0)^2 + \sum_1^p (y^i)^2$.

The Poincaré-patch metric of radius R is

$$ds^2 = R^2(dy \cdot dy - dudv).$$

Defining $x^\mu = y^\mu / v$ and eliminating $u = v^{-1} + vx \cdot x$,

$$ds^2 = R^2(v^2 dx \cdot dx + v^{-2} dv^2).$$

The D3-brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$

The ten-dimensional metric is

$$ds^2 = R^2 \left(v^2 dx \cdot dx + v^{-2} dv^2 + d\Omega_5^2 \right)$$
$$= R^2 \left(v^2 dx \cdot dx + v^{-2} dv \cdot dv \right).$$

v is now the length of the six-vector $v^I = c\phi^I$.

According to the AdS/CFT dictionary

$$R^4 = 4\pi g_{\mathrm{s}} N l_{\mathrm{s}}^4 \quad \text{and} \quad \int_{S^5} F_5 \sim N.$$

The D3-brane action has two terms: $S = S_1 + S_2$.

S_1 is a DBI functional of the embedding functions $x^M(\sigma^\alpha)$ and a world-volume $U(1)$ gauge field $A_\beta(\sigma^\alpha)$ with field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$:

$$S_1 = -T_{D3} \int \sqrt{-\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}\, d^4\sigma,$$

where $G_{\alpha\beta}$ is the induced 4d world-volume metric

$$G_{\alpha\beta} = g_{MN}(x)\partial_\alpha x^M \partial_\beta x^N.$$

As usual, $\alpha' = l_\mathrm{s}^2$ and the D3-brane tension is

$$T_{D3} = \frac{2\pi}{g_\mathrm{s}(2\pi l_\mathrm{s})^4}.$$

Only dimensionless combinations occur in the brane action:

$$R^4 T_{D3} = \frac{N}{2\pi^2} \quad \text{and} \quad 2\pi\alpha'/R^2 = \sqrt{\pi/g_\mathrm{s}N}.$$

General coordinate invariance allows one to choose the static gauge

$$x^\mu(\sigma) = \delta^\mu_\alpha \sigma^\alpha.$$

In this gauge ϕ^I and A_μ become functions of x^μ.

The complete answer (aside from fermions) in terms of canonically normalized fields is

$$S = \frac{1}{\gamma^2} \int \phi^4 \left(1 - \sqrt{-\det M_{\mu\nu}}\right) d^4x + \frac{1}{4} g_s \chi \int F \wedge F,$$

where $\gamma = \sqrt{N/2\pi^2}$ and

$$M_{\mu\nu} = \eta_{\mu\nu} + \gamma^2 \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{\phi^4} + \gamma \frac{F_{\mu\nu}}{\phi^2}.$$

It is a stunning fact that g_s and χ only appear (as a product) in the last term. Rescaling all fields by γ shows that the loop expansion of this theory is a $1/N$ expansion. An important prediction of our conjecture is that all this should be the case for the HEA.

Conclusion

We have conjectured that the world-volume action of a probe D3-brane in an $AdS_5 \times S^5$ background of type IIB superstring theory, with one unit of flux (N=1), can be reinterpreted as the HEA of $U(2)$ $\mathcal{N} = 4$ SYM theory on the Coulomb branch. An explicit formula for the bosonic part of the action has been presented.

It is likely that there is a generalization in which the formula with $N > 1$ plays a role in the Coulomb branch decomposition $U(N+1) \rightarrow U(N) \times U(1)$. However, certain issues still need to be clarified.

The evidence presented so far for the conjecture that the probe D3-brane action is an HEA is that the action incorporates all of the required symmetries and dualities: $PSU(2,2|4)$ superconformal symmetry (when fermions are included) and $SL(2,\mathbb{Z})$ duality.

The second lecture will describe soliton solutions of this action. This will give further support to the conjecture.

EXTRA SLIDES

This theory has a famous soliton solution: the 't Hooft–Polyakov monopole. This solution preserves half the supersymmetry. One says that it is "half BPS." This monopole is part of an infinite $SL(2, \mathbb{Z})$ multiplet of half-BPS (p, q) states, with p units of electric charge and q units of magnetic charges, where p and q are coprime.

The masses, determined by supersymmetry, are

$$M_{p,q} = vg|p + q\tau| = vg\sqrt{\left(p + \frac{\theta}{2\pi}q\right)^2 + \left(\frac{4\pi q}{g^2}\right)^2}$$

where $< \phi >= v$ is the vev of a massless scalar field. The W mass is $M_{1,0} = gv$ and the monopole mass is $M_{0,1} = 4\pi v/g$ for $\theta = 0$.

In a d-dimensional CFT, every term in the action must have have dimension d. On the Coulomb branch there is a scale, the vev of a scalar field. However, the full conformal symmetry is realized on the action. Only the choice of vacuum breaks the symmetry (and all choices are equivalent).

The effective action contains inverse powers of the scalar field. The vev ensures that these are not singular. Individual terms in the HEA can be arbitrarily complicated and still end up with dimension d by including an appropriate (inverse) power of the scalar field.

The Chern–Simons Term

$$S_2 \sim \int C_4 + \chi \int F \wedge F$$

The RR four-form potential C_4 has a self-dual field strength $F_5 = dC_4$.

$$F_5 \sim \mathrm{vol}(S^5) + \mathrm{vol}(AdS_5)$$

The constant χ is the value of the RR 0-form C_0. It is proportional to a theta angle.

S_1 contains a "potential" term $\int \phi^4 d^4x$, which should not appear, since there should be no net force acting on the brane. It is canceled by the $\int C_4$ term in S_2:

$$\int_{D3} C_4 = \int_M F_5 \sim \int_0^{\phi} \int_{D3'} \mathrm{vol}(AdS_5)'$$

$$\mathrm{vol}(AdS_5)' \sim (\phi')^3 d\phi' \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Thus,

$$\int_{D3} C_4 \sim \int \phi^4 d^4x.$$

The coefficients work perfectly.

Other Examples

In general, we consider the world-volume action of a probe p-brane in an $AdS_{p+2} \times M_q$ background with N units of flux, $\int_{M_q} F_q = N$. This lecture described a probe D3-brane in $AdS_5 \times S^5$. The manuscript also discusses

- M2-brane in $AdS_4 \times S^7/\mathbb{Z}_k$
- D2-brane in $AdS_4 \times CP^3$
- M5-brane in $AdS_7 \times S^4$