BPS Soliton Solutions of a D3-Brane Action

John H. Schwarz

California Institute of Technology

Erice June-July 2014

Introduction

My first lecture described the construction of a world-volume action for a probe D3-brane in $AdS_5 \times S^5$. It was based on arXiv:1311.0305.

In this lecture, based on arXiv:1405.7444, I will derive supersymmetric soliton solutions of that theory. They will have surprising and tantalizing properties.

Before doing that, I will review a few results from the first lecture.

The Coulomb Branch

We discussed $\mathcal{N} = 4$ SYM theory, with gauge group $U(N+1)$, on the Coulomb branch. A scalar field vacuum value v breaks the gauge symmetry so that:

$$U(N+1) \rightarrow U(1) \times U(N).$$

The other $2N$ supermultiplets acquire mass gv.

In principle, one can integrate out the massive fields exactly. The resulting action for the $U(1)$ factor is called a *highly effective action* (HEA).

General Requirements for the HEA

- Field content is an abelian $\mathcal{N} = 4$ supermultiplet
- Global symmetries and dualities same as the original Coulomb branch theory
- Conformal symmetry spontaneously broken by vev of a massless scalar field
- The same BPS spectrum; the $SL(2, \mathbb{Z})$ multiplet containing the W particles and monopoles must be solitons of the HEA

The world-volume action for a D3-brane in $AdS_5 \times S^5$ was conjectured to be this HEA or else a useful approximation to it. I only described the bosonic truncation of the action. The fermions are not needed for the classical solutions that will be discussed.

The $SL(2, \mathbb{Z})$ symmetry of the type IIB background is an induced symmetry of the D3-brane action. The modular parameter is

$$\tau = \chi + \frac{i}{g_s} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

The D3-brane in $\mathbf{AdS_5 \times S^5}$

The ten-dimensional metric $ds^2 = g_{MN}(x)dx^M dx^N$ is

$$ds^2 = R^2 \left(\phi^2 dx \cdot dx + \phi^{-2} d\phi^2 + d\Omega_5^2 \right)$$

where ϕ is the radial coordinate of the AdS_5 metric.

We will be interested in a D3-brane at a fixed position on the S^5. In this case, the S^5 coordinates do not contribute to the D3-brane action.

The D3-brane action is $S = S_1 + S_2$. S_1 is a DBI functional of the embedding functions $x^M(\sigma^\alpha)$ and a world-volume $U(1)$ gauge field $A_\beta(\sigma^\alpha)$ with field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$:

$$S_1 = -T_{D3} \int \sqrt{-\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}\, d^4\sigma,$$

where $G_{\alpha\beta}$ is the induced 4d world-volume metric

$$G_{\alpha\beta} = g_{MN}(x)\partial_\alpha x^M \partial_\beta x^N.$$

S_2 is a Chern–Simons term.

General coordinate invariance allows one to choose the static gauge

$$x^M(\sigma) = \delta^M_\alpha \sigma^\alpha \quad \text{for} \quad M = 0, 1, 2, 3.$$

In this gauge $\phi^I(\sigma)$ and $A_\mu(\sigma)$ become functions of x^μ.

In static gauge we obtain

$$S = \frac{1}{\gamma^2} \int \phi^4 \left(1 - \sqrt{-\det M_{\mu\nu}}\right) d^4x + \frac{1}{4} g_s \chi \int F \wedge F,$$

where $\gamma = \sqrt{N/2\pi^2}$ and

$$M_{\mu\nu} = \eta_{\mu\nu} + \gamma^2 \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{\phi^4} + \gamma \frac{F_{\mu\nu}}{\phi^2}.$$

S duality

The $\tau \to -1/\tau$ duality of the $U(N+1)$ theory on the Coulomb branch has not been proved in the formulation with W fields, but it was proved for the D3-brane action in my first paper.

Recall that $\tau = \chi + i/g_s$, but that the action only depends on $g_s\chi$. When $\tau \to -1/\tau$, $g_s\chi \to -g_s\chi$. To show invariance under this sign change, we must also make a transformation that exchanges electric and magnetic fields.

The procedure is standard. It involves replacing fields by new ones such that the Bianchi identity

$$\partial_{[\mu} F_{\nu\rho]} = 0$$

becomes a field equation, and the field equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \right) = 0$$

becomes a Bianchi identity.

The nontrivial fact, which is proved by a straightforward calculation, is S duality: the new action, obtained by this field replacement and the sign change $g_s \chi \to -g_s \chi$, is identical to the original one.

Solitons

For spherically symmetrical static solutions, centered at $r = 0$, we require that $\vec{E}$ and $\vec{B}$ only have radial components, denoted E and B, and that E, B, and ϕ are functions of r only. It then follows that

$$-\det(M_{\mu\nu}) = -\det\left(\eta_{\mu\nu} + \gamma^2 \frac{\partial_\mu\phi\partial_\nu\phi}{\phi^4} + \gamma\frac{F_{\mu\nu}}{\phi^2}\right)$$

$$= \left(1 + \gamma^2\frac{(\phi')^2 - E^2}{\phi^4}\right)\left(1 + \gamma^2\frac{B^2}{\phi^4}\right).$$

This results in the Lagrangian density

$$\mathcal{L} = \frac{1}{\gamma^2}\phi^4 \left(1 - \sqrt{\left(1 + \gamma^2 \frac{(\phi')^2 - E^2}{\phi^4}\right)\left(1 + \gamma^2 \frac{B^2}{\phi^4}\right)} \right).$$

The equation of motion for A_0 is $\frac{\partial}{\partial r}(r^2 D) = 0$, where

$$D = \frac{\partial \mathcal{L}}{\partial E} = E\sqrt{\frac{1 + \gamma^2 B^2/\phi^4}{1 + \gamma^2[(\phi')^2 - E^2]/\phi^4}}.$$

The $F \wedge F$ term, which is not included here, is included in the manuscript.

For a soliton centered at $r = 0$, with p units of electric charge g and q units of magnetic charge g_m, where $g_m = 4\pi/g$, we have

$$D = \frac{pg}{4\pi r^2} \quad \text{and} \quad B = \frac{qg_m}{4\pi r^2}.$$

Thus, $D^2 + B^2 = Q^2/r^4$, where (including the $F \wedge F$ term)

$$Q = \frac{g}{4\pi}|p + q\tau|,$$

and

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}.$$

Eliminating E in favor of D gives the Hamiltonian $H = \int \mathcal{H} d^3x = 4\pi \int \mathcal{H} r^2 dr$. $\mathcal{H} = DE - \mathcal{L}$ is

$$\mathcal{H} = \frac{1}{\gamma^2}\left(\sqrt{(\phi^4 + \gamma^2(\phi')^2)(\phi^4 + \gamma^2 X^2)} - \phi^4\right),$$

and

$$X = \sqrt{D^2 + B^2} = Q/r^2.$$

We want to find functions $\phi(r)$ that give BPS extrema of H with $\phi \to v$ as $r \to \infty$. The BPS condition turns out to require that the two factors inside the square root are equal, which implies that $\mathcal{H} = (\phi')^2$.

The proof goes as follows. One first writes the formula for $\mathcal{H}$ in the form

$$(\gamma^2\mathcal{H} + \phi^4)^2 = (\gamma^2 X|\phi'| + \phi^4)^2 + \gamma^2\phi^4(X - |\phi'|)^2,$$

Thus,

$$(\gamma^2\mathcal{H} + \phi^4)^2 \geq (\gamma^2 X|\phi'| + \phi^4)^2$$

which implies $\mathcal{H} \geq X|\phi'|$. Saturation of the BPS bound is achieved for $|\phi'| = X$ and then

$$\mathcal{H} = X^2 = (\phi')^2 = Q^2/r^4.$$

The equation $(\phi')^2 = Q^2/r^4$, together with the B. C. $\phi \to v$ as $r \to \infty$, has two BPS solutions

$$\phi_\pm = v \pm Q/r,$$

where (as before) $Q = \frac{g}{4\pi}|p + q\tau|$.

The ϕ_+ solution is similar to the flat space ($\mathbb{R}^{9,1}$) case studied by Callan and Maldacena in 1997. It describes a funnel-shaped protrusion of the D3-brane extending to the boundary of AdS at $\phi = +\infty$. It gives infinite mass (proportional to $\int dr/r^2$), but it is not the solution I am after.

The ϕ_- solution is different. $\phi = 0$ corresponds to the horizon of the Poincaré patch of AdS_5. Thus the ϕ_- solution must be cut off at

$$r_0 = \frac{Q}{v}.$$

Thus, the masses of BPS solitons are given by

$$M = 4\pi \int_{r_0}^{\infty} \mathcal{H} r^2 dr = \frac{4\pi Q^2}{r_0} = vg|p + q\tau|.$$

As expected, for $N = 1$, the $(p, q) = (\pm 1, 0)$ solitons are $W^{\pm}$ with mass vg and the $(p, q) = (0, \pm 1)$ solitons are magnetic monopoles with mass $4\pi v/g$ (for $\theta = 0$).

Interpretation

The charge of the ϕ_- solution is uniformly spread on the sphere $r = r_0$, which we call a soliton bubble. The interior of the bubble should not contribute to the mass of the soliton. So, how should we think about the interior of the bubble in the QFT?

From the 10d viewpoint, the bubble is on the horizon of the Poincaré patch of AdS_5, where it intersects the boundary of global AdS.

Black Hole Analogy

One has $M = 4\pi v^2 r_0$ for all (p, q). For comparison, the radius of the horizon of an extremal Reissner–Nordstrom black hole in four dimensions is $r_0 = MG$. Thus, v is the analog of the Planck mass. One also has $Mr_0 \sim Q^2$ in both cases.

If one tries to take this analogy seriously, a natural question is: What, if anything, does Q^2 have to do with entropy? Perhaps it is the entanglement entropy between the inside and outside of the soliton bubble.

Clearly, these solitons are not black holes. The only sensible interpretation is that the gauge theory is in the ground state of the *conformal phase* of $U(N+1)$ inside the sphere. This implies that the bubble is a phase boundary.

This interpretation has the advantage that the parameter τ is required to describe the $U(N+1)$ theory in the conformal phase. This would explain where g and θ come from when $N = 1$.

For comparison, the single monopole solution of the nonabelian $SU(2)$ gauge theory has a triplet of scalar fields

$$\phi^a(\vec{x}) = \frac{x^a}{r}\phi(r)$$

with

$$\phi(r) = v(\coth y - 1/y),$$

where $y = gvr$ (for $\theta = 0$).

This differs from $\phi_-(r) = v(1 - 1/y)$, the D3-brane theory result, by a series of terms of the form $\exp(-2nM_W r)$, where $M_W = gv$ and n is a positive integer.

Yet, $\phi(r)$ is strictly positive for $r > 0$, and ϕ^a is nonsingular at the origin. Thus, there is no sign of a bubble in the nonabelian description. So, which formula correctly describes what is happening?

At least for $\mathcal{N} = 4$, the effect of integrating out the fields of mass M_W should be to cancel the exponential terms for $y > 1$ and to give $\phi = 0$ for $y < 1$. After all, the $U(1)$ HEA is supposed to incorporate all of those contributions. Hence the bubble is real.

Multi-soliton Solutions

It is easy to guess (and to derive) the generalization to the case of n solitons of equal charge. The obvious guess is

$$\phi(\vec{x}) = v - Q \sum_{k=1}^{n} \frac{1}{|\vec{x} - \vec{x}_k|}.$$

The surfaces of the bubbles are given by $\phi(\vec{x}) = 0$. The fields $\vec{D}$ and $\vec{B}$ are then proportional to $\vec{\nabla}\phi$, with coefficients determined by the charges.

This is much simpler than the usual multi-monopole analysis! I claim it is also more accurate.

Magnetic Bags

By considering multi-monopole solutions of large magnetic charge in the nonabelian gauge theory on the Coulomb branch, Bolognesi (hep-th/0512133) deduced the existence of “magnetic bags” with properties that are very close to those of the soliton bubbles that were found here. He also pointed out the analogy to black holes.

The analysis in terms of nonabelian gauge fields is much more complicated, subtle, and approximate than the analysis of the abelian effective action.

Problems for the Future

- Understand the extent to which symmetry and other general considerations should determine the HEA and *whether* the world-volume theory of a brane probe should give an HEA
- Explore soliton solutions of the other p-brane actions
- Incorporate fermions
- Explore tree approximation scattering amplitudes
- Generalize the analysis to higher-rank gauge theories on the Coulomb branch

Conclusion

The action of a probe D3-brane in $AdS_5 \times S^5$ is a candidate for the HEA of a $U(1)$ factor on the Coulomb branch. It incorporates all the required symmetries and dualities, and it gives the expected BPS soliton solutions.

Even so, it might only be an approximation that has all of the required symmetries and is sufficient for computing susy-protected quantities. It is important to settle this question.

There is much more to be explored.

EXTRA SLIDES

$\mathbf{U(N+1) \to U(1) \times U(N)}$ with $\mathbf{N > 1}$

In this case the basic solitons (Ws and monopoles) should transform as $U(N)$ (and dual $U(N)$) fundamental irreps, and thus source nonabelian gauge fields.

The formulas I have presented are only applicable for solitons that are singlets of $U(N)$ (and its dual). Thus, for $N > 1$ one cannot avoid confronting nonabelian gauge fields.

Solitons of the 6d (2,0) Theory

The M5-brane action in $AdS_7 \times S^4$ is the candidate HEA. We expect half-BPS solutions describing infinitely-extended strings with a self-dual charge.

For strings in the x^5 direction with transverse positions given by four-vectors $\vec{x}_k$ the formula should be

$$\phi(\vec{x}) = v^2 - \sum_{k=1}^{n} \frac{1}{|\vec{x} - \vec{x}_k|^2}.$$

The locus $\phi(\vec{x}) = 0$ describes soliton bubbles whose interiors should be in the conformal phase. Also, the self-dual three-form field should take the form $H_{0i5} \sim \partial_i \phi$.

If all the centers coincide the formula becomes

$$\phi(r) = v^2 - n/r^2, \tag{1}$$

which vanishes for $r_0 = \sqrt{n}/v$. The string tension is $T = nv^2$. These formulas match an extremal black string of charge n in 6d. The analogy suggests that $n^{3/2}v$ might correspond to entanglement entropy per unit length.

A curious fact about the (2,0) theory is that there is an action for an abelian factor in the Coulomb phase even though it is generally believed that the conformal phase has no Lagrangian description. The loop expansion parameter is $1/N^2$.