Logarithmic violation of scaling in strongly anisotropic turbulent transfer of a passive vector field.

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Outline

The task is divided into several steps:

- Definition of the model — stochastic differential equations;
- Field theoretic formulation, diagrammatic technique;
- Renormalization and fixed point, which defining the inertial range asymptotic behaviour;
- Renormalization of the operators $F_{N,\ p}$, critical dimension matrix;
- Asymptotic behaviour of the correlation function, OPE.

The step with surprising result — diagonalization of the matrix of critical dimensions.

Definitions and aims

- The field $\boldsymbol{\theta}$ is a passive vector field, e. g. a field of impurity;
- we work within fully developed turbulence, in the *inertial range*. It has two scales: large scale L, from which energy enter to the system, and small scale l, in which viscosity is important;
- we are interested in the inertial-range asymptotic of the scalar operators

$$F_{N,\ p} = (\theta_i\theta_i)^p\ (n_s\theta_s)^{2m}, \quad N = 2(p+m);$$

- the measurable quantities are some correlators like

$$G_{12} = \langle F_1,\ F_2\rangle\ .$$

To calculate their asymptotic we need to use OPE and know asymptotic of still operator F.

Description of the model

The stochastic equation for advection of the passive field is

$$\partial_t \theta_i + \partial_k \left(v_k \theta_i - \mathcal{A}_0 \; v_i \theta_k \right) + \partial \mathcal{P} = \nu_0 \; (\partial_\perp^2 + f_0 \partial_\parallel^2) \; \theta_i + f_i ,$$

where $\mathcal{P}$ is pressure term and f_i is random foreign force with zero mean and preassigned correlator

$$D_f = \left\langle f_i(t, \; \mathbf{x}) \; f_k(t', \; \mathbf{x}') \right\rangle = \delta(t - t') \; C_{ik}(r/L).$$

Description of the model

The velocity field possesses defined direction **n**:

$$\mathbf{v}(t,\ \mathbf{x}) = \mathbf{n}\cdot v(t,\ \mathbf{x}_\perp).$$

We neglect the feedback of the *passive* field θ to the velocity v and will consider a simplified model, where

$$\left\langle v_i(t,\ \mathbf{x})\ v_k(t',\ \mathbf{x}')\right\rangle = n_i n_k\ \left\langle v(t,\ \mathbf{x}_\perp)\ v(t',\ \mathbf{x}'_\perp)\right\rangle;$$

$$\left\langle v(t,\ \mathbf{x}_\perp)\ v(t',\ \mathbf{x}'_\perp)\right\rangle = \delta(t-t')\ \int_m^\infty \frac{d\mathbf{k}}{(2\pi)^d}\ e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\ D_v(k),$$

and for $D_v(k)$ we choose

$$D_v(k) = 2\pi\delta(k_\parallel)\cdot D_0\cdot\frac{1}{k_\perp^{d-1+\xi}}.$$

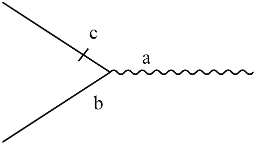
Field Theoretic Formulation

This stochastic problem is equivalent to the field theoretic model of the extended set of three felds $\Phi = \{\boldsymbol{\theta'}, \boldsymbol{\theta}, \boldsymbol{v}\}$ with action functional

$$S(\Phi) = \frac{1}{2}\theta'_i D_f \theta'_k - \frac{1}{2} v_i D_v^{-1} v_k +$$

$$+\theta'_k \left[-\partial_t \theta_k - (v_i \partial_i)\theta_k + \mathcal{A}_0(\theta_i \partial_i) v_k + \nu_0(\partial_\perp^2 + f_0 \partial_\parallel^2)\theta_k \right],$$

triple vertex

$V_{cab} =$  $= i\delta_{bc}\ p_a^\theta - i\mathcal{A}\delta_{ac}\ p_b^\theta$

Field Theoretic Formulation: propagators

and propagators

$$\langle v_i v_j \rangle_0 =$$

$$= n_i n_j \cdot 2\pi\delta(k_{\|}) \cdot D_0 \cdot \frac{1}{k_\perp^{d-1+\xi}}$$

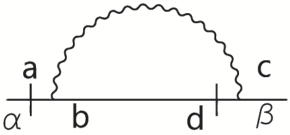
$$\left\langle \theta_i \theta'_j \right\rangle_0 =$$

$$= \frac{P_{ij}(\mathbf{p})}{-i\omega + \nu_0 p_\perp^2}.$$

Dyson equation

The only divergent diagram is the Self-Energy diagram:

$$\Sigma_{\alpha\beta} =$$



From Dyson equation it follows that:

- one *new* dimensionless constant u is needed;
- $\nu_0 = \nu, \ \mathcal{A}_0 = \mathcal{A}$ — no renormalization required;
- $f_0 = f \cdot Z_f$ with nontrivial constant Z_f;
- $u_0 = u \cdot Z_u$ with nontrivial constant Z_u.

Fixed points and asymptotic

From RG functions follows that system possesses fixed points

$$g^* = \frac{2(d-1)}{d-2+\mathcal{A}} \cdot \xi \quad \text{and} \quad u^* = \frac{(\mathcal{A}-1)^2}{d-2+\mathcal{A}}.$$

Therefore now we need to solve RG equation for the composite operator $F_{N,\ p} = (\theta_i \theta_i)^p \ (n_s \theta_s)^{2m}$, and the leading term of asymptotic is given by point

$$g = g^* \quad \text{and} \quad u = u^*.$$

Operator diagram

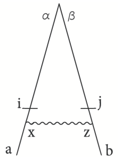
Our aim is to obtain asymptotic of composite operator, built solely from the fields itself:

$$F_{N,\ p} = (\theta_i \theta_i)^p \ (n_s \theta_s)^{2m}, \quad N = 2(p+m).$$

This operator can be renormalized multiplicatively,

$$F = Z_F \cdot F_R,$$

therefore we have to calculate one one-loop diagram



and then contract internal diagram block with external fields and operator vertex.

Operator diagram

The (exact!) result is

$$\frac{\delta^2 F_{N,p,m}}{\delta\theta_1 \cdot \delta\theta_2} \cdot I_{12}^{ab} \cdot \theta_a \theta_b =$$

$$= 2m(2m-1) \cdot F_{N,p+1,m-1} \; + \; (2p + 8pm - 2m(2m-1)) \cdot F_{N,p,m} \; +$$

$$+(4p(p-1) - 2p - 8pm) \cdot F_{N,p-1,m+1} \; - \; 4p(p-1) \cdot F_{N,p-2,m+2}.$$

There is mixing of operators $\Longrightarrow$ renormalization constant Z_F is a matrix!

The matrix Z_F

Let us consider the closed set of operators with fixed number of fields N:

$$\mathbf{F} = \begin{pmatrix} (\theta_i\theta_i)^N \\ (\theta_i\theta_i)^{N-2} \cdot (n_s\theta_s)^2 \\ \vdots \\ (n_s\theta_s)^N \end{pmatrix}.$$

Therefore the renormalozation matrix Z_F takes the form

The matrix Z_F: renormalization

$$Z_F = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \ddots & 0 \\ \vdots & 0 & a_{43} & \ddots & \ddots & a_{n-2n} \\ \vdots & & & \ddots & \ddots & a_{n-1n} \\ 0 & \dots & \dots & 0 & a_{nn-1} & a_{nn} \end{pmatrix},$$

and expression $F_i = Z_{ik} F_k^R$ (for example, at $N = 4$) in fact means that

$$\begin{pmatrix} \theta^4 \\ \theta^2 \theta_\parallel^2 \\ \theta_\parallel^4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^2 \tilde{\theta}_\parallel^2 \\ \tilde{\theta}_\parallel^4 \end{pmatrix}.$$

Critical dimension matrix

The critical dimension matrix of operator $F_{N,p}$ is

$$\Delta_{Np,\ Np'} = -2(p+m) + \gamma^*_{Np,\ Np'},$$

where $2(p+m) = N$ is it's canonical dimension and $\gamma^*_{Np,\ Np'}$ is a matrix with the following terms:

$$\gamma^*_{N,\ p+1} = \eta \cdot 2m(2m-1) \cdot \xi;$$

$$\gamma^*_{N,\ p} = \eta \cdot (2p + 8pm - 2m(2m-1)) \cdot \xi;$$

$$\gamma^*_{N,\ p-1} = \eta \cdot (4p(p-1) - 2p - 8pm) \cdot \xi;$$

$$\gamma^*_{N,\ p-2} = \eta \cdot (-4p(p-1)) \cdot \xi.$$

RG equation

The basic RG statement is the following: if the quantity is multiplicatively renormalizable, so

$$F = Z_F \cdot F_R,$$

we can use RG equation:

$$[\mathcal{D}_{RG} + \gamma_F] \, F_R = 0,$$

where $\mathcal{D}_{RG}$ is some differential operator, γ_F – anomalous dimension of F. It is Euler equation and provided exponential law, moreover, the leading term of the asymptotic is at fixed point $g = g^*$, so

$$F \cong const \cdot x^{\Delta_F}.$$

Therefore the aim is to calculate critical dimension Δ_F of the operator F.

The necessity of the diagonalization

If Z is a matrix Z_{ik}, i.e., if there is a mixing of operators,

$$F_i = Z_{ik} F_k^R ,$$

we need to diagonalize our system to solve the RG equations. So our aim is to find the eigenvalues of the matrix

$$\Delta_{Np,\ Np'} = -2(p+m) + \gamma^*_{Np,\ Np'} ,$$

or, equivalently, to find the diagonal matrix

$$\widetilde{\Delta}_F = U_F^{-1} \Delta_F U_F .$$

Jordan Form

It is proved, that for any dimension N

$$\lambda_1 = \ \dots \ = \lambda_{N/2+1} = -2(p+m) = -N,$$

and the critical dimension marix is not diagonalizable, but have a Jordan form!

$$\widetilde{\Delta}_F = \begin{pmatrix} -2(p+m) & 1 & 0 & \dots & 0 \\ 0 & -2(p+m) & 1 & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & & 0 & -2(p+m) \end{pmatrix}.$$

Asymptotic behaviour of the operator $F_{N,p}$

According to the RG equation, at $r \gg l$

$$\left\langle \widetilde{\mathbf{F}}^R \right\rangle \propto \nu_0^{d_F^\omega} \cdot M^{-N} \cdot (Mr)^{\widetilde{\Delta}_F} \cdot \Phi\left(\frac{f}{M^\xi}\right) \cdot \mathbf{C}_0,$$

where Φ is some unknown function, $\mathbf{C}_0$ – some constant vector, $\widetilde{\Delta}_F$ – the critical dimension (Jordan!) matrix.

$$(Mr)^{\widetilde{\Delta}_F} = \begin{pmatrix} (Mr)^\lambda & (Mr)^\lambda \cdot \ln(Mr) & \dots & \frac{(Mr)^\lambda \cdot (\ln(Mr))^{n-1}}{(n-1)!} \\ 0 & (Mr)^\lambda & & \vdots \\ \vdots & & \ddots & (Mr)^\lambda \cdot \ln(Mr) \\ 0 & \dots & 0 & (Mr)^\lambda \end{pmatrix}.$$

RG equation for correlation function G

The main aim is to find the asymptotic behaviour of the correlator

$$G = \langle F_{N_1,p_1} \; F_{N_2,p_2} \rangle$$

with arbitrary N_1 and N_2.
RG equation takes form

$$\mathcal{D}_{RG} \; G_{ik} = \Delta_{is} G_{sk} + \Delta_{ks} G_{is} ,$$

where $G_{ij} = \langle F_i F_j \rangle$, Δ_{ij} is the critical dimension of G_{ij}.

Solution of the RG equation for correlator G

Solution of the RG equation gives us

$$G^R \propto (\Lambda r)^{N_1+N_2} \cdot P_{(N_1+N_2)/2}\left[\ln \Lambda r\right] \cdot \Phi\left(Mr,\ mr,\ fr^{\xi}\right).$$

Operator Product Expansion

This representation describe the behaviour of the correlation function for $\Lambda r \gg 1$ and any fixed value of Mr. The inertial range $L \gg r \gg l$ corresponds to the additional condition $Mr \to 0$, which is studied using OPE. According to the OPE

$$F_1(x')F_2(x'') = \sum_{\widetilde{F}} C_{\tilde{F}}(\mathbf{r})\widetilde{F}(t, \mathbf{x}),$$

where $\widetilde{F}$ are all possible operators. After averaging with the weight $\exp S_R$ the sought-for asymptotic takes the form

$$\langle \widetilde{F}_\alpha \rangle \propto (Mr)^{\widetilde{\Delta}_\alpha},$$

where $\widetilde{\Delta}_\alpha$ is a Jordan matrix.

Asymptotic behaviour of the correlator G

So using RG and OPE together one can obtain the desired asymptotic behaviour:

$$G(r) = \langle F_{N_1,p_1}\ F_{N_2,p_2} \rangle \propto \nu^{d_G^{\omega}} \cdot [\ln \Lambda r]^{(N_1+N_2)/2} \cdot [\ln Mr]^{(N_1+N_2)/2} \cdot \widetilde{\Phi}\left(f r^{\xi} \right).$$

One more interesting question

In this work we deal with composite operators

$$F_{N,\ p} = (\theta_i\theta_i)^p\ (n_s\theta_s)^{2m}, \quad N = 2(p+m).$$

At some point we have to diagonalize the critical dimension matrix

$$\Delta_{Np,\ Np'} = -2(p+m) + \gamma^*_{Np,\ Np'},$$

where $N = -2(p+m)$ is the canonical dimension of the operator $F_{N,\ p}$, and $\gamma^*_{Np,\ Np'}$ is a matrix, which elements depends of the p and m.

For this reason we have to introduce a matrix U:

$$\widetilde{\Delta}_F = U_F^{-1}\Delta_F U_F.$$

One more interesting question

For $N = 10$ 6×6 matrix γ^* and diagonalizing matrix U are

$$\gamma^* = \begin{pmatrix} 10 & 70 & -80 & & & \\ 2 & 38 & 8 & -48 & & \\ & 12 & 42 & -30 & -24 & \\ & & 30 & 22 & -44 & -8 \\ & & & 56 & -22 & -34 \\ & & & & 90 & -90 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 5/90 & 10/90 \cdot 56 & 10/90 \cdot 56 \cdot 30 & 5/\text{prev} \cdot 12 & 1/\text{prev} \cdot 2 \\ 1 & 4/90 & 6/90 \cdot 56 & 4/90 \cdot 56 \cdot 30 & 1/\text{prev} \cdot 12 & \\ 1 & 3/90 & 3/90 \cdot 56 & 1/90 \cdot 56 \cdot 30 & & \\ 1 & 2/90 & 1/90 \cdot 56 & & & \\ 1 & 1/90 & & & & \\ 1 & & & & & \end{pmatrix}.$$

Conclusion

- We applied the field theoretic renormalization group and the operator product expansion to the analysis of the inertial-range asymptotic behavior of a divergence-free vector field, passively advected by strongly anisotropic random flow;
- All multiloop diagram for this model are equal to zero, i.e., the model is solved exactly;
- The anomalous scaling, which is typical for such models, is violated;
- The key point is that the matrices of scaling dimensions of the relevant families of composite fields (operators) appear nilpotent and cannot be diagonalized and can only be brought to Jordan form; hence the logarithms.

For details see arXiv:1406.3808

Thank you for attention!