Correlations of π-mesons in deformed analog of Bose gas model

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Outline

- Deformed oscillators
- μ-Bose gas model
  - Intercepts of momentum correlation functions
  - Asymptotics of the intercepts
- Conclusions

Quantum algebra $su_q(2)$ is realized by two modes of q-analogs of bosonic oscillator (Biedenharn*, Macfarlane**):

$$aa^\dagger - qa^\dagger a = q^{-N},$$

where q is deformation parameter.

Commutation relation of Arik-Coon deformed oscillator:

$$[a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1$$

$$[a^\pm, a^\pm] = 0, \quad [N, a^\pm] = \pm a$$

In q-analog of Fock space :

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{[N]_q!}}|0\rangle, \quad [N]_q = \frac{q^N - 1}{q - 1}, \quad N|n\rangle = n|n\rangle,$$

where $[N]_q! = [N]_q \cdot [N-1]_q \cdot ... \cdot [1]_q$, $[0]_q! = 1$.

*Biedenharn L.J., Phys. A: Math. Gen. **22** L873 (1989).

Mcfarlane A., J. Phys. A: Math. Gen. **22 4581 (1989).

The action of operators $a^\dagger$, a on the state n is defined by formula:

$$a|n\rangle = \sqrt{[n]_q}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle.$$

Hamiltonian: $H = \frac{1}{2}(aa^\dagger + a^\dagger a), \quad \hbar\omega = 1, \qquad H|n\rangle = E_n|n\rangle.$

The connection between $a^\dagger$, a and the operators $b^\dagger$, b of usual quantum oscillator:

$$a^\dagger = \sqrt{\frac{[N]_q}{N}}\, b^\dagger, \quad a = b\sqrt{\frac{[N]_q}{N}}, \quad \text{де} \quad [N]_q = \frac{1-q^N}{1-q}.$$

Coordinate representation of $a^\dagger$, a:

$$a = \frac{e^{-2i\alpha x} - e^{i\alpha d/dx} e^{-i\alpha x}}{-i\sqrt{1-e^{-2\alpha^2}}}, \quad a^\dagger = \frac{e^{2i\alpha x} - e^{i\alpha} e^{i\alpha d/dx}}{i\sqrt{1-e^{-2\alpha^2}}}, \quad \text{де} \quad \alpha = \sqrt{-\ln q/2}.$$

The corresponding operators of coordinate and momentum in q-deformed quantum mechanics:

$$\hat{x} = \frac{a+a^\dagger}{\sqrt{2}} = \sqrt{\frac{2}{1-e^{-2\alpha^2}}}\left(\sin(2\alpha x) - e^{\alpha^2/2}\sin\left(\alpha x + \frac{\alpha^2}{2}\, i\right) e^{i\alpha d/dx}\right),$$

$$\hat{p} = \frac{a-a^\dagger}{i\sqrt{2}} = \sqrt{\frac{2}{1-e^{-2\alpha^2}}}\left(\cos(2\alpha x) - e^{\alpha^2/2}\cos\left(\alpha x + \frac{\alpha^2}{2}\, i\right) e^{i\alpha d/dx}\right).$$

When studying deformed oscillators it is convenient to use the concept of structute function of deformation $\varphi(N)$:

$$a^\dagger a = \varphi(N), \qquad aa^\dagger = \varphi(N+1).$$

For the ordinary quantum oscillator: $a^\dagger a = N$, $aa^\dagger = N+1$.

Commutation relation for operators $a^\dagger$, a:

$$aa^\dagger - a^\dagger a = \varphi(N+1) - \varphi(N).$$

In the q-analog of Fock space:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(N)!}}|0\rangle, \quad N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n\rangle,$$

where $\varphi(N)! = \varphi(N)\cdot\varphi(N-1)\cdot\ldots\cdot\varphi(1), \quad \varphi(0)! = 1.$

Models of deformed oscillators

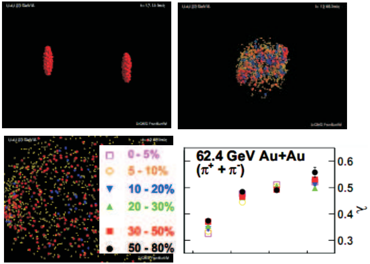
Structure function	Energy spectrum
$\varphi_n^{\mathsf{AK}} = \frac{q^n-1}{q-1}$	*Arik-Coon model* $E_n^{\mathsf{AK}} = \frac{1}{2}\left(\frac{q^{n+1}-1}{q-1} + \frac{q^n-1}{q-1}\right)$
$\varphi_n^{\mathsf{BM}} = \frac{q^n-q^{-n}}{q-q^{-1}}$	*Biedenharn-Macfarlane model* $E_n^{\mathsf{BM}} = \frac{1}{2}\left(\frac{q^{n+1}-q^{-(n+1)}}{q-q^{-1}} + \frac{q^n-q^{-n}}{q-q^{-1}}\right)$
$\varphi_n^{\mathsf{(p,q)}} = \frac{q^n-p^n}{q-p}$	*(p,q)-oscillator model* $E_n^{\mathsf{(p,q)}} = \frac{1}{2}\left(\frac{q^{n+1}-p^{n+1}}{q-p} + \frac{q^n-p^n}{q-p}\right)$
$\varphi_n^{\mathsf{TD}} = nq^{n-1}$	*Tamm-Dancoff model* $E_n^{\mathsf{TD}} = \frac{1}{2}\left((n+1)q^n + nq^{n-1}\right)$
$\varphi_n^{\mu} = \frac{n}{1+\mu n}$	*μ-oscillator* $E_n^{\mu} = \frac{1}{2}\left(\frac{n}{1+\mu n} + \frac{n+1}{1+\mu(n+1)}\right)$ *

* Jannussis A. J. Phys. A: Math. Gen. **26**, L233–L237, 1993

Deformed oscillators have application in different fields of physics:

- molecular and nuclear spectroscopy,
- mathematical physics and integrable systems theory,
- quantum optics,
- statistical mechanics,
- quantum algebras or algebras of deformed oscillator have effective application in phenomenological investigation of the properties of elementary particles and theoretical aspects of relativistic nuclear collisions.

Intercepts of correlation functions



In the experiments of relativistic heavy ion collisions, as the result of collisions, the secondary particles (π-mesons, K-mesons ect) are produced and then registered.

Two-particle momentum correlation function:

$$C^{(2)}(k_1, k_2) = \gamma \frac{P_2(k_1, k_2)}{P_1(k_1) P_1(k_2)},$$

can be rewritten in variables $Q = k_1 - k_2$, $K = (k_1 + k_2)/2$:

$$C^{(2)}(Q, K) \xrightarrow{k_1 = k_2} C^{(2)}(Q = 0, K) = 1 + \lambda^{(2)}(m, \mathbf{K}),$$

$\lambda^{(2)}$ - intercept of two-particle correlation function.

If assume that the particle are bosons then $\lambda^{(2)} = 1$.

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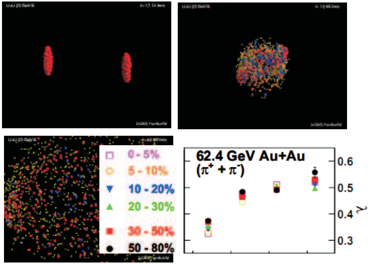

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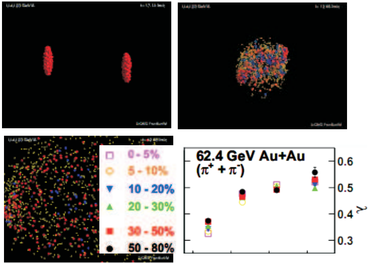

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On the set of μ-oscillators we develop the respective deformed analog of Bose gas model (μ-Bose gas).

The physical meaning of deformation parameter μ

- can be connected with the compositeness of particles (their substructure) or interaction between them;
- There are the models where both compositeness and interaction can be effectively taken into account.

  A.M. Gavrilik, Yu.A. Mishchenko, Ukr.J.Phys., 2013, **58**, 1171-1177.

Intercept can be rewritten in terms of operators $a^\dagger$, a:

$$\lambda^{(2)}(K) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} - 1 = \frac{\langle [N]_\mu [N-1]_\mu\rangle}{\langle [N]_\mu\rangle^2} - 1, \qquad a^\dagger a = [N]_\mu = \frac{N}{1+\mu N}.$$

$$a f(N) = f(N+1) a, \qquad a^\dagger f(N) = f(N-1) a^\dagger$$

Statistical average for a system with Hamiltonian H:

$$\langle N\rangle = \frac{Tr N e^{-\beta\sum_k H_k}}{Tr e^{-\beta\sum_k H_k}} = \frac{\sum_n \langle n|N e^{-\beta\sum_k H_k}|n\rangle}{\sum_n \langle n|e^{-\beta\sum_k H_k}|n\rangle} = \frac{\sum_n n e^{-\beta\varepsilon n}}{\sum_n e^{-\beta\varepsilon n}} = \frac{1}{e^{\beta\varepsilon} - 1},$$

where $\beta = \frac{1}{T}$, $k = 1$. Analogously one can obtain $\langle N^r\rangle$, $r \geq 2$.

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where $\beta = \frac{1}{T}$, $k = 1$. Analogously one can obtain $\langle N^r\rangle$, $r \geq 2$.

We choose the Hamiltonian in the form

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} N_{\mathbf{k}}$$

The energy of particles: $\omega = (m_\pi^2 + \mathbf{K}^2)^{1/2}$.

Intercept of two-particle correlation function:

$$\lambda_\mu^{(2)} = \Big\{X^{-1} - \Big(\frac{1}{\mu} + \frac{1}{\mu^2}\Big)\Phi(e^{-\beta}, 1, \mu^{-1}) - \Big(\frac{1}{\mu} - \frac{1}{\mu^2}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 1)\Big\} \times$$
$$\times \Big(X^{-1} - \mu^{-1}\Phi(e^{-\beta}, 1, \mu^{-1})\Big)^{-2} X^{-1} - 1, \quad (1 - e^{-\beta}) = X$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$.

Intercept of three-particle correlation function: $\lambda^{(3)}(K) = \frac{\langle a^\dagger a^\dagger a^\dagger aaa\rangle}{\langle a^\dagger a\rangle^3} - 1$

$$\lambda_\mu^{(3)} = X^{-2}\Big\{X^{-1} - \Big(\frac{1}{\mu} + \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1}) - \Big(\frac{1}{\mu} - \frac{1}{\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 1) -$$
$$- \Big(\frac{1}{\mu} - \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 2)\Big\} \cdot \Big(X^{-1} - \mu^{-1}\Phi(e^{-\beta}, 1, \mu^{-1})\Big)^{-3} - 1.$$

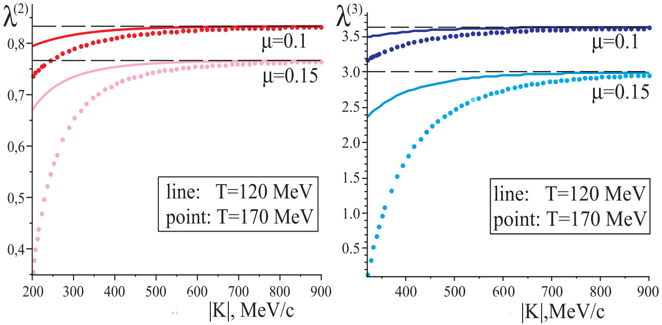


Fig.2. *Left*: Dependence of intercept $\lambda^{(2)}$ on the momentum. For curve μ=0.1 the asymptote $\lambda^{(2)}_{as.} = 0.83$, for curve μ=0.15 asymptote $\lambda^{(2)}_{as.} = 0.7664$.

Right: Dependence of intercept $\lambda^{(3)}$ on the momentum. For curve μ=0.1 the asymptote $\lambda^{(3)}_{as.} = 3.6365$, for curve μ=0.15 the asymptote $\lambda^{(3)}_{as.} = 2.9964$.

* Gavrilik A.M., Rebesh A.P., Eur.Phys.J.A 47:55, 8 pp. (2011).

Intercept of r-particle correlation function

$$\lambda^{(r)}(K) = \frac{\langle (a^\dagger)^r (a)^r \rangle}{\langle a^\dagger a \rangle^r} - 1 = \frac{\langle [N]_\mu [N-1]_\mu \cdot \ldots \cdot [N-r+1]_\mu \rangle}{\langle [N]_\mu \rangle^r} - 1.$$

$$\lambda_\mu^{(r)}(k) = \Big(1 + \mu^{-1}(1 - e^{-\beta\hbar\omega}) \sum_{l=0}^{r-1} A_l^{(r)}(\mu) \Phi(e^{-\beta\hbar\omega}, 1, \mu^{-1} - l)\Big) \times$$
$$\times \Big(1 + \mu^{-1}(1 - e^{-\beta\hbar\omega}) A_0^{(1)}(\mu) \Phi(e^{-\beta\hbar\omega}, 1, \mu^{-1})\Big)^{-r} - 1, \; r = 2, 3, \ldots, \; \mu > 0$$

The coefficients $A_l^{(r)}$:

$A_0^{(1)}(\mu) = -1;$

$A_0^{(2)}(\mu) = -1 - \frac{1}{\mu}, \quad A_1^{(2)}(\mu) = -1 + \frac{1}{\mu};$

$A_0^{(3)}(\mu) = -1 - \frac{3}{2\mu} - \frac{1}{2\mu^2}, \quad A_1^{(3)}(\mu) = -1 + \frac{1}{\mu^2}, \quad A_2^{(3)}(\mu) = -1 + \frac{3}{2\mu} - \frac{1}{2\mu^2};$

$\ldots\,.$

Asymptotics of the intercepts: $\beta\omega \longrightarrow \infty$

$$\lambda^{(r)}_{\mu,\,asympt} = \lim_{\omega\to\infty} \frac{\sum_{n=0}^{\infty} \frac{n}{1+\mu n} \cdot \ldots \cdot \frac{n-r+1}{1+\mu(n-r+1)} e^{-\beta\hbar\omega n}}{(1-e^{-\beta\hbar\omega})^{r-1}\left(\sum_{n=0}^{\infty} \frac{n}{1+\mu n} e^{-\beta\hbar\omega n}\right)^r} - 1 =$$

$$= \lim_{\omega\to\infty} \frac{[r]_\mu! e^{-\beta\hbar\omega r} + \ldots}{\left(\frac{1}{1+\mu}\right)^r e^{-\beta\hbar\omega r} + \ldots} - 1 = \left(1+\mu\right)^r [r]_\mu! - 1,$$

$$[r]_\mu! \equiv [r]_\mu [r-1]_\mu \ldots [1]_\mu .$$

For $r=2$ and $r=3$ this result is in complete agreement with the corresponding asymptotical values of the μ-Bose gas intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$:

$$\lambda^{(2)}_{\mu,\,asympt} = \left(1+\mu\right)^2 [2]_\mu! - 1 = \frac{1}{1+2\mu},$$

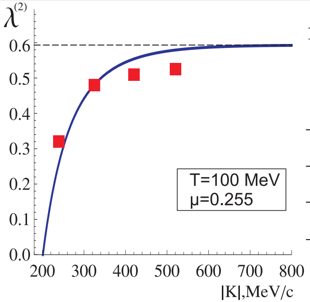
$$\lambda^{(3)}_{\mu,\,asympt} = \left(1+\mu\right)^3 [3]_\mu! - 1 = \frac{5+7\mu}{(1+2\mu)(1+3\mu)}.$$

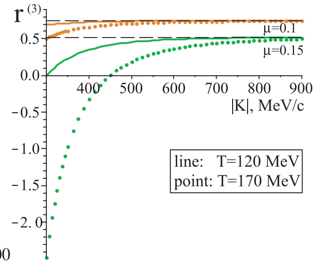
$$r^{(3)}(K) = \frac{1}{2}\frac{\lambda^{(3)}(K) - 3\lambda^{(2)}(K)}{(\lambda^{(2)}(K))^{3/2}}.$$

The importance of the function - all undesirable perverting effects are canceled (e.g. contributions from resonances).

The expression for the asymptotics of r-function

$$r^{(3)}_{as.}(\mu) = \frac{1-\mu}{1+3\mu}\sqrt{1+2\mu}.$$





B.I. Abelev *et al.* (STAR Collab.), Phys. Rev. C **80**, 024905 (2009).

Conclusions

- Deformed oscillators have unusual properties compared with ordinary quantum oscillator.
- On the set of μ-oscillators we realize the μ-Bose gas model.
- In the framework of μ-Bose gas model we obtained the intercepts of two- , three-particle momentum correlation function as well as the expression for the r-particle correlation function.
- The asymptotics of the intercepts are also obtained.

Thank you for your attention!

- Usual quantum oscillator: there is no level degeneracy in the energy spectrum ($d = 1$).
- In more general and complicated cases different types of energy level can exist (V.N. Zakhariev).

Spectrum of p, q-oscillator, $0 < p \leq 1, \quad 0 < q \leq 1.$

The degeneracy of energy levels exist:

$$\mathbf{E_n} = \mathbf{E_0},$$
$$\mathbf{E_n} = \mathbf{E_{n+1}},$$
$$\mathbf{E_n} = \mathbf{E_{n+2}},\ n \geq 2.$$

$$E.g.: \quad E_n^{(p,q)} - E_0^{(p,q)} = F_{n,0}(q,p) =$$

$$= \sum_{r=0}^{n} p^{n-r} q^r + \sum_{s=0}^{n-1} p^{n-1-s} q^s - 1 = 0,$$