Gluon Condensates from Hamiltonian Formalism

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Overview

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Gluon Condensates - why do we care?

The Gluon Condensate

$\langle G^2 \rangle_\varphi = \langle \varphi | G_{\mu\nu} G^{\mu\nu} | \varphi \rangle$

- Trace anomaly for gauge theories
- First-principle determination on lattice difficult $\rightarrow$ power divergences
- L. Del Debbio, R. Zwicky [arXiv:1306.4038] used RG equations to avoid the Hamiltonian:

$$g \frac{\partial}{\partial g} E_\varphi^2 = -\frac{1}{2} \langle \frac{1}{g^2} G^2 \rangle_\varphi$$

$$g \frac{\partial}{\partial g} \Lambda_{\mathrm{GT}} = -\frac{1}{2} \langle \frac{1}{g^2} G^2 \rangle_0$$

Feynman-Hellmann Theorem

From quantum mechanics:

$$\frac{\partial}{\partial\lambda}E = \langle\psi_E|\frac{\partial}{\partial\lambda}H(\lambda)|\psi_E\rangle \qquad \text{when } \langle\psi_E|\psi_E\rangle = 1$$

In QFT however:

$$\langle\psi(E',\vec{p}')|\psi(E,\vec{p})\rangle = 2E(\vec{p})(2\pi)^{D-1}\delta^{D-1}(\vec{p}-\vec{p}')$$

So when $\vec{p} \to \vec{p}'$ we get $(2\pi)^{D-1}\delta^{D-1}(\vec{p}-\vec{p}') \to \int d^{D-1}x = V$
A straight-forward application of F-H theorem to $\frac{1}{\sqrt{E(\vec{p})}}|\psi(E,\vec{p})\rangle$ yields:

F-H Theorem for the Hamiltonian density $\mathcal{H}$

$$\frac{\partial}{\partial\lambda}E_\phi^2 = \langle\phi|\frac{\partial}{\partial\lambda}\mathcal{H}(\lambda)|\phi\rangle$$

Gauge Theory Hamiltonian

- Need to work with canonical variables: $(A_i^a, \pi_i^a \equiv E_i^a)$
- A_0 plays the role of Lagrange multiplier and $\pi_0 \equiv 0$ (primary constraint)
    - $\frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \bar{q}(i\vec{\gamma} \cdot (\vec{\partial} + ig\vec{A}) - m)q$

$$\mathcal{H} = \mathcal{H}_g + \mathcal{H}_C + \mathcal{H}_{\mathcal{G}}$$

- Primary, secondary constraints
- Gauss constraint : $A_0(\vec{D}\cdot\vec{E} + \bar{q}\gamma_0 q)$
- Chromomagnetic field $B_i \equiv \frac{1}{2}\epsilon_{ijk}G^{jk} = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j + ig[A_j, A_k])$

- The constraints $\mathcal{H}_C$ and $\mathcal{H}_{\mathcal{G}}$ must vanish on physical states
- Use a simple (canonical) transformation : $\vec{A} \rightarrow \frac{1}{g}\vec{A}, \;\; \vec{E} \rightarrow g\vec{E}$

Taming the Beast

- The transformation leaves the functional measure and canonical commutation relations invariant with

$$\mathcal{H}_g = \frac{1}{2}(g^2\vec{E}^2 + \frac{1}{g^2}\vec{B}^2) - \overline{q}(i\vec{\gamma}\cdot\vec{D} + m)q \ ,$$

- Restore the Lorentz invariance:

$$g\frac{\partial}{\partial g}\mathcal{H}_g = g^2\vec{E}^2 - \frac{1}{g^2}\vec{B}^2 = -\frac{1}{2}\frac{1}{g^2}G_{\mu\nu}G^{\mu\nu} \ .$$

- Apply the F-H theorem:

$$g\frac{\partial}{\partial g}E^2_\varphi = -\frac{1}{2}\langle\frac{1}{g^2}G^2\rangle_\varphi$$

$$g\frac{\partial}{\partial g}\Lambda_{\rm GT} = -\frac{1}{2}\langle\frac{1}{g^2}G^2\rangle_0$$

- IMPORTANT- all the quantities/operators are renormalized!

Photon in Schwinger Model

- 2D massless QED
- Exactly solvable - J.Schwinger [Phys. Rev. 128, 2425]
- Spectrum contains massive photons:

$$M_\gamma^2 = \frac{e^2}{\pi} \; .$$

- Apply the formula:

$$e \frac{\partial}{\partial e} M_\gamma^2 = \frac{2e^2}{\pi} = -\frac{1}{2} \langle G^2 \rangle_\gamma \; .$$

- Interesting verification- the matrix element can be calculated directly to yield the same result

BPS Monopole in $\mathcal{N} = 2$ SYM

- 4D SU(2) gauge theory with four supercharges. Exact mass spectrum found by Seiberg and Witten [hep-th/9407087]. For 1 monopole state we have:

Known function of moduli parameter(coupling) on the S-W curve

$$M^2_{BPS} = \boxed{2|a_D|^2}$$

- The relevant Hamiltonian reads:

$$\mathcal{H}_{\rm BPS} = \frac{1}{g^2}\vec{D}\phi\cdot\vec{D}\phi + \frac{1}{2}\frac{1}{g^2}\vec{B}^2,$$

- Different from before - SUSY forces the coupling in front of matter
- Use the BPS condition: $\vec{D}\phi|{\rm BPS}\rangle = \frac{1}{\sqrt{2}}\vec{B}|{\rm BPS}\rangle$ with $\vec{E} = 0$:

$$g\frac{\partial}{\partial g}\mathcal{H}_{\rm BPS} = -2\frac{1}{g^2}\vec{B}^2 \overset{\vec{E}=0}{=} -\frac{1}{g^2}G^2 \quad \text{as before}$$

Potential Applications

- Evaluation via lattice, DSE or AdS/CFT $\rightarrow$ non-perturbative beta functions
- SUSY breaking
- QCD contribution to the cosmological constant - provided one can compute Λ_{GT} up to a g dependent constant

Summary

- Derived a useful relation that could serve as a definition of the gluon condensate
- Showed how to use it in practical calculations
- Interesting exercise in Hamiltonian approach

$$g\frac{\partial}{\partial g}E_{\varphi}^{2} = -\frac{1}{2}\langle\frac{1}{g^{2}}G^{2}\rangle_{\varphi}$$

$$g\frac{\partial}{\partial g}\Lambda_{\mathrm{GT}} = -\frac{1}{2}\langle\frac{1}{g^{2}}G^{2}\rangle_{0}$$

The Trace Anomaly

$$2M_\phi^2 \equiv \langle\phi|T_\mu^\mu|\phi\rangle = \frac{\beta}{2g}\langle F^2\rangle_\phi + (1+\gamma)m\bar{q}q$$

$$2E_\phi^2 = 2M_\phi^2 + 2p^2 \equiv \langle\phi|\mathcal{H}|\phi\rangle$$