Hopf Maps and Wigner's Little Groups

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Overview

1. Hopf Maps and division algebras

2. Hopf maps and Wigner's little groups

Hopf maps

Hopf maps(fibrations) are fibrations of spheres over spheres in very particular dimensions:

$$S^3/S^1 \simeq S^2, \quad S^7/S^3 \simeq S^4, \quad S^{15}/S^7 \simeq S^8.$$

For the first two cases, they are the first representatives of the corresponding family of fibrations:

$$S^{2n+1}/S^1 \simeq \mathbb{CP}^n, \quad S^{4n+3}/S^3 \simeq \mathbb{HP}^n.$$

Recall that

$$\mathbb{CP}^1 = S^2, \quad \mathbb{HP}^1 = S^4.$$

The explicit formulae

We start from a $2n$-dimensional Euclidian space without origin, where $n = 2, 4, 8$ are the dimensions of normed divisional algebras $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

$$\mathbb{R}^{2n} \setminus \{0\} = \mathbb{R}^+ \times S^{2n-1} = \mathbb{F}^2 \setminus \{0\}, \quad \mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$$

To get the sphere(total space T) we fix:

$$T = \{(\mathbf{u}_1, \mathbf{u}_2) : \mathbf{u}_1\bar{\mathbf{u}}_1 + \mathbf{u}_2\bar{\mathbf{u}}_2 = r\} \simeq S^{2n-1}.$$

The projection $P : T \mapsto B$ looks as follows:

$$p_0 + p_i\mathbf{e}_i = 2\bar{\mathbf{u}}_2\mathbf{u}_1, \quad p_{2n+1} = \mathbf{u}_1\bar{\mathbf{u}}_1 - \mathbf{u}_2\bar{\mathbf{u}}_2$$

which implies

$$\sum_{i=1}^{n+1} p_i^2 = (\mathbf{u}_1\bar{\mathbf{u}}_1 + \mathbf{u}_2\mathbf{u}_2)^2 = r^2 \qquad S^{2n-1} \mapsto S^n$$

The explicit formulae(fiber)

If $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, it is clear that the projection identifies all points which differ by a unit length element multiplication, i.e. the transformation

$$\mathbf{u}_\alpha \to \boldsymbol{\tau}\mathbf{u}_\alpha, \quad \boldsymbol{\tau}\bar{\boldsymbol{\tau}} = 1, \quad \alpha = 1,2 \tag{1}$$

does not affect the coordinates p of the base. On the other hand

$$\boldsymbol{\tau}\bar{\boldsymbol{\tau}} = 1 \text{ defines } S^{n-1}, \quad n = 2, 4.$$

If we choose $\mathbf{g} = \frac{\mathbf{u}_2}{|\mathbf{u}_2|}$, then the transformation (1) will look as follows:

$$\mathbf{g} \to \boldsymbol{\tau}\mathbf{g}.$$

For octonions, thus, the transformation looks as follows:

$$\mathbf{u}_1 \to \frac{(\boldsymbol{\tau}\mathbf{u}_2)(\bar{\mathbf{u}}_2\mathbf{u}_1)}{\mathbf{u}_2\bar{\mathbf{u}}_2}, \quad \mathbf{u}_2 \to \boldsymbol{\tau}\mathbf{u}_2, \quad \mathbf{u}_{1,2} \in \mathbb{O}$$

Notations

The generators of the Lorentz group $SO(1, d-1)$ are given by $d \times d$ matrices $\omega_{\mu\nu}$. The transformation of a vector p^μ is given by

$$\delta p_\mu = \omega_{\mu\nu} p^\nu, \quad d = 0, \dots, d-1$$

Spinors transform via spinor representation of the Lorentz algebra:

$$\delta\lambda = \omega_{\mu\nu}\Gamma^{\mu\nu}\lambda, \quad \Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu], \quad \{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}.$$

The generators $S^{\mu\nu}$, thus, satisfy the relations of $SO(1, d-1)$ algebra and

$$\dim \Gamma^{\mu\nu} = 2^{d/2}$$

The Massless Momentum

For any spinor λ we define a vector p^μ as follows:

$$p^\mu = \bar{\lambda}\Gamma^\mu\lambda, \quad \bar{\lambda} = \lambda^*\gamma^0 \tag{2}$$

It can be checked explicitly that

$$p_\mu p^\mu = 0$$

and for the chiral representation:

$$\left(\lambda^{*T}\lambda\right)^2 = p_i p_i, \quad i = 1, \ldots, d-1$$

(2) defines a map $S^{\dim\lambda - 1} \mapsto S^{d-2}$:

$$d = 4: \quad S^3 \mapsto S^2$$

$$d = 6: \quad S^7 \mapsto S^4$$

$$d = 10: \quad S^{15} \mapsto S^8$$

Question: *What are the transformations, which change λ but do not affect the p?*

Spinors and Stability Groups

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d	$W(\lambda)$	$W(p)$	$W(p)/W(\lambda)$
4	T^2	$SO(2) \ltimes T^2$	S^1
6	$SO(3) \ltimes T^4$	$SO(4) \ltimes T^4$	S^3
10	$SO(7) \ltimes T^8$	$SO(8) \ltimes T^8$	S^7

Decomposition

The $2k$-dimensional $SO(1,2k-1)$ gamma-matrices can be obtained from $SO(2k-1)$ euclidean ones as follows

$$\Gamma^0 = \begin{pmatrix} 0 & \imath\mathbf{1}_4 \\ -\imath\mathbf{1}_4 & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ -\gamma^i & 0 \end{pmatrix}, \quad \Gamma^{ij} = \begin{pmatrix} -\gamma^{ij} & 0 \\ 0 & -\gamma^{ij} \end{pmatrix}$$

$$\Gamma^{0j} = \begin{pmatrix} -\gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \quad \{\gamma^i, \gamma^j\} = \delta^{ij}$$

We always choose the spinor λ to be a Weyl one $\lambda = (Z, 0)$.

$\mathbf{g} \to \boldsymbol{\tau}\mathbf{g}$ transformation

- For d=4 we have $\gamma \equiv \imath\sigma$ and

$$\delta Z = \imath\varepsilon^{ijk}\omega_{jk}\left(Z\sigma_i\bar{Z}\right)Z$$

Putting $Z = (u_1, u_2)$ and $\boldsymbol{\tau} = \imath\varepsilon^{ijk}\omega_{jk}\left(Z\sigma_i\bar{Z}\right)$ we find

$$\mathbf{u}_\alpha \to \boldsymbol{\tau}\mathbf{u}_\alpha \tag{3}$$

- For $d = 6$

$$\delta Z = -(\bar{Z}\gamma^{ij}Z)\omega_{ij}Z + (ZC\gamma^{ij}Z)\omega_{ij}C\bar{Z}$$

To get (3) we should put

$$\boldsymbol{\tau} = (\bar{Z}\gamma^{ij}Z)\omega_{ij} + (ZC\gamma^{ij}Z)\omega_{ij}\mathbf{j}, \quad Z = (\mathbf{u}_1, \mathbf{u}_2)$$

- For $d = 10$

$$\delta Z = -\frac{1}{6}\omega_{ij}\left(ZC\gamma^{ijlm}Z\right)\gamma^{lm}Z$$