Conformal invariance without referring to metric

based on `arXiv:1406.5888`

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Outline

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Conformal invariance of field theoretical models

A field theoretical model is called conformally invariant in the usual sense if invariant to the spacetime metric rescalings (Weyl transformations):

$$g'_{ab} = \Omega^2 g_{ab}$$

A group action of the Weyl transformations must be given on all the fields in order to make sense of this.

The model may (or may not) be then seen to be invariant to this group action in terms of its field equations or the action functional.

We present an alternative formulation which does not refer to spacetime metric tensor field.
Instead, we formulate in terms of connexion.
Can be useful in constructions where metric is a derived quantity.

The approach treats metric (whenever present) as "just an other field".

Metric independent formulation of field theories

Details: `arXiv:1406.5888` or `arXiv:gr-qc/0403041` (Palatini type approach).

A classical field theory is a quartet $(M, V(M)\mathbf{dL}, S)$, where M is an oriented real 4-manifold, $V(M)$ some finite dim vector bundle (of fields) and

$$\begin{aligned}&\mathbf{dL} : \\ &\Gamma\left(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \wedge T^*(M) \otimes V(M) \otimes V^*(M)\right) \;\to\; \Gamma\left(\wedge^n T^*(M)\right), \\ &(v, Dv, F) \;\mapsto\; \mathbf{dL}(v, Dv, F)\end{aligned}$$

is the maximal form valued Lagrange form.

The action functional $S(K)$ over a compact region $K \subset M$ is defined via

$$\begin{aligned}&S(K) : \\ &\quad \Gamma(V(M)) \times D(V(M)) \;\to\; \mathbb{R}, \\ &\quad (v, \nabla) \;\mapsto\; S_{v,\nabla}(K) := \int_K \mathbf{dL}(v, \nabla v, F(\nabla))\end{aligned}$$

($D(V(M))$ are the covariant derivations over $V(M)$, $F(\nabla)$ is curvature of ∇.)

Solutions of the model are the fields (v, ∇) satisfying

$$D^\circ S_{v,\nabla}(K) = 0,$$

where $D^\circ S_{v,\nabla}(K) = 0$ is the Fréchet derivative of $S(K)$ at (v, ∇) along the closed subspace of fixed boundary value fields at ∂K.

These are, quite naturally, equivalent to Euler-Lagrange equations

$$\begin{aligned} D_1 \mathbf{dL}(v, \nabla v, F(\nabla)) - \tilde{\nabla}_a D_2^a \mathbf{dL}(v, \nabla v, F(\nabla)) &= 0, \\ D_2 \mathbf{dL}(v, \nabla v, F(\nabla))(\cdot)v - \tilde{\nabla}_a 2 D_3^{ab} \mathbf{dL}(v, \nabla v, F(\nabla))(\cdot) &= 0 \end{aligned}$$

with $\tilde{\nabla}$ being the torsion-free part of ∇.

(Note that we get 2 EL equations: one for the fields v, one for the covariant derivation ∇.)

Metric independent formulation of conformal invariance

Idea of measure lines.
[T.Matolcsi: Spacetime without reference frames (1993); Hungarian Acad. of Sci. Press.]

Special relativistic spacetime model: (M, η, L) triplet with M being 4-dim real affine space, L an 1-dim vector space (*measure line*), and η and $L \otimes L$-valued Lorentz metric:

$$\eta : \vee^2 \mathbb{M} \to L \otimes L,$$

$\mathbb{M}$ being the underlying vector space of M (“tangent space”).

This is simply formalization of dimensional analysis! L models the vec.space of lengths.
Quantities are not simply number valued, but tagged with physical dimensions.
(Take their values in tensor powers of measure line.)

Notation: $L^n := \otimes^n L$, $L^{-n} := \otimes^n L^*$ ($n \in \mathbb{N}$).

Idea of measure line bundles: general relativistic point of view of measure lines.
[A.László: `arXiv:1406.5888`]

Let $L(M)$ be an 1-dim fiber vector bundle over M (*measure line bundle*).
Field quantities should be not simply tensor valued but should carry physical dimensions in terms of powers of $L(M)$.

This is simply formalization of dimensional analysis!
But the physical dimensions are not a priori comparable in different points of spacetime, unless a connection on $L(M)$ given.

This is like making a gauge theory where the fields have internal degree of freedom in each spacetime point $p \in M$ residing in the one dimensional real vector space $L_p(M)$.

We call a model conformally invariant whenever the action functional is invariant to the choice of the covariant derivation over the measure line bundle $L(M)$.

Example: conformally invariant vacuum GR

Slightly generalized version of Einstein-Hilbert Lagrangian.

$$V(M) := L^{-1}(M) \times L^2(M) \otimes \vee^2 T^*(M) \quad \text{(vector bundle of fields)}$$

$\mathbf{dL}$:

$$\Gamma\left(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \wedge T^*(M) \otimes V(M) \otimes V^*(M)\right) \to \Gamma\left(\wedge^4 T^*(M)\right),$$
$$((\varphi, g_{ab}),\ (D\varphi_c, Dg_{de\,f}),\ (r_{gh}, R_{ghi}{}^{j})) \mapsto \mathbf{dv}(g)\varphi^2 g^{km} \delta^l{}_n R_{klm}{}^n.$$

The symbol $\mathbf{dv}(g)$ denotes the volume form in $\Gamma\left(L^4(M) \otimes \wedge^4 T^*(M)\right)$ generated by the metric tensor field $g \in \Gamma\left(L^2(M) \otimes \vee^2 T^*(M)\right)$.

The field $\varphi \in \Gamma\left(L^{-1}(M)\right)$ plays the role of inverse Planck length.
Just it is not a constant, but a section of a line bundle.

The vector space of length values are not initially the same in particular points of spacetime.
In order to compare length values in different points, one would need a covderiv on $L(M)$.

The model is conf.inv in the sense that action does not depend on covderiv of $L(M)$.

The derived EL field equations are:

$$\begin{aligned} \tilde{\nabla}_a \left(\varphi^2 g_{bc}\right) &= 0 \\ E(\tilde{\nabla}, \varphi^2 g)_{ab} &= \mathcal{T}(\nabla, \varphi^2 g)_{ab}. \end{aligned}$$

($\tilde{\nabla}_a$being the torsion-free part of ∇_a),

Here, $E(\tilde{\nabla}, \varphi^2 g)_{ab}$ is the Einstein tensor of $\tilde{\nabla}_a$ and the rescaled metric $\varphi^2 g_{ab}$.
$\mathcal{T}(\nabla, \varphi^2 g)_{ab} :=$

$$\frac{1}{4}\left(2\tilde{\nabla}_{(a} T(\nabla)^g_{\;b)g} + T(\nabla)^h_{\;ga} T(\nabla)^g_{\;bh} - \frac{1}{2}(\varphi^2 g_{ab})(\varphi^{-2} g^{ef}) \left(2\tilde{\nabla}_e T(\nabla)^g_{\;fg} + T(\nabla)^h_{\;ge} T(\nabla)^g_{\;fh}\right)\right)$$

which is the contribution of the torsion of ∇ (i.e. of $T(\nabla)^c_{\;ab}$) to energy-momentum tensor.

(Actually, contribution $T(\nabla)^c_{\;ab}$ can be zeroed out initially, if variation is performed on the closed subspace of torsion-free covariant derivations.)

This is nothing but an ordinary vacuum Einstein equation for the dimensionless metric $\varphi^2 g_{ab}$ measured in units of square Planck length φ^{-2}. (With possible torsion contribution.)

Can be also re-expressed in terms of the dimensional metric g_{ab}:

$$\begin{aligned} \tilde{D}_a(g_{bc}) &= 0 \qquad\qquad (\tilde{D}_a \text{ is the torsion-free part of } D_a), \\ E(\tilde{D},g)_{ab} &= \mathcal{T}(\nabla,\varphi^2 g)_{ab} \\ &\quad +2\varphi^{-1}\tilde{D}_{(a}\tilde{D}_{b)}\varphi - 2g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e\tilde{D}_f(\varphi) \\ &\quad -4\varphi^{-1}\tilde{D}_a(\varphi)\,\varphi^{-1}\tilde{D}_b(\varphi) + g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e(\varphi)\,\varphi^{-1}\tilde{D}_f(\varphi), \\ g^{ab}\tilde{D}_a\tilde{D}_b\varphi - \frac{1}{6}\mathcal{R}(\tilde{D},g)\varphi &= \frac{1}{6}g^{ab}\mathcal{T}(\nabla,\varphi^2 g)_{ab}\varphi. \end{aligned}$$

Here, $\mathcal{R}(D,g)$ is the Ricci scalar of D_a and g_{ab}.
(Again, contribution of torsion can be zeroed out, eventually.)

This is nothing but the conformally invariant coupled Einstein-Klein-Gordon equations.
Which is conformally invariant also in the usual sense, i.e. in terms of Weyl transformations.

Thus we have a definition of conformal invariance not referring to metric but to connexion.

Can be useful to generate conformally invariant Lagrangians in frameworks where spacetime metric is a derived quantity, not fundamental field. E.g. spinorial formulation of GR.

Summary

- A metric independent formulation of conformal invariance was proposed for field theoretical models.
- Based on the idea of *measure line bundle*.
- The model is called conformally invariant whenever the action functional is invariant to the choice of the connexion of the measure line bundle.
- Consistent with usual metric dependent definition.
- Can be especially useful for models with emergent metric (e.g. spinorial formulation of GR).
- Approach shows that conformal invariance has a further deeper meaning in terms of dimensional analysis.
- Also it has a meaning in terms of gauge theory analogy.