Conformal invariance without referring to metric

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1. Conformal invariance of field theoretical models

A field theoretical model is called conformally invariant [1, 2, 3] in the usual sense if invariant to the spacetime metric rescalings:

$$g'_{ab} = \Omega^2 g_{ab}$$

A group action of the conformal metric rescalings must be given on all the fields in order to make sense of this. The model may (or may not) be then seen to be invariant to this group action in terms of its field equations or action functional. We present an alternative formulation which does not refer to metric. Instead, we formulate in terms of connexion. Can be useful in constructions where metric is a derived quantity.

2. Metric independent formulation of field theories

Details: [4, 5, 6, 7] (Palatini-like approach).

A classical field theory is a quartet $(M, V(M)\mathbf{dL}, S)$, where M is an oriented real 4-manifold, $V(M)$ some finite dim vector bundle (of fields) and

$$\begin{aligned}&\mathbf{dL}:\\ &\Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right) \to \Gamma\left(\wedge^n T^*(M)\right),\\ &(v, Dv, F) \mapsto \mathbf{dL}(v, Dv, F)\end{aligned}$$

is the maximal form valued Lagrange form.

The action functional $S(K)$ over a compact region $K \subset M$ is defined via

$$\begin{aligned}S(K):\quad &\Gamma(V(M))\times D(V(M)) \to \mathbb{R},\\ &(v,\nabla) \mapsto S_{v,\nabla}(K) := \textstyle\int_K \mathbf{dL}(v, \nabla v, F(\nabla))\end{aligned}$$

($D(V(M))$ are the covariant derivations over $V(M)$, $F(\nabla)$ is curvature of ∇.)

Solutions of the model are the fields (v,∇) satisfying

$$D^\circ S_{v,\nabla}(K) = 0,$$

where $D^\circ S_{v,\nabla}(K)$ is the Fréchet derivative of $S(K)$ at (v,∇) along the closed subspace of fixed boundary value fields at ∂K.

These are, quite naturally, equivalent to Euler-Lagrange equations

$$\begin{aligned}D_1\mathbf{dL}(v,\nabla v,F(\nabla)) - \tilde{\nabla}_a D_2^a\mathbf{dL}(v,\nabla v,F(\nabla)) &= 0,\\ D_2\mathbf{dL}(v,\nabla v,F(\nabla))(\cdot)v - \tilde{\nabla}_a 2D_3^{ab}\mathbf{dL}(v,\nabla v,F(\nabla))(\cdot) &= 0\end{aligned}$$

with $\tilde{\nabla}$ being the torsion-free part of ∇.

3. Metric indep. formulation of conformal invariance

Idea based on the notion of measure lines [8].

In that work, a special relativistic spacetime model is: an (M,η,L) triplet with M being 4-dim real affine space, L an 1-dim vector space (*measure line*), and η and $L\otimes L$-valued Lorentz metric:

$$\eta:\ \vee^2\mathbb{M} \to L\otimes L,$$

$\mathbb{M}$ being the underlying vector space of M ("tangent space").
This is simply formalization of dimensional analysis! L models the vec.space of lengths.
Quantities are not simply number valued, but tagged with physical dimensions.
(Take their values in tensor powers of measure line L. $\quad L^n := \otimes^n L$, $L^{-n} := \otimes^n L^*$.)

Idea of measure line bundles: general relativistic point of view of measure lines [5].

Let $L(M)$ be an 1-dim fiber vector bundle over M (*measure line bundle*). Field quantities should be not simply tensor valued but should carry physical dimensions in terms of powers of $L(M)$.

This is simply formalization of dimensional analysis! But the physical dimensions are not a priori comparable in different points of spacetime, unless a connexion on $L(M)$ given. We call a model conformally invariant whenever the action functional is invariant to the choice of the covariant derivation over the measure line bundle $L(M)$.

4. Example: conformally invariant vacuum GR

Slightly generalized version of Einstein-Hilbert Lagrangian [4, 5].

$$V(M) := L^{-1}(M)\ \times\ L^2(M)\otimes\vee^2 T^*(M) \quad \text{(vector bundle of fields)}$$

$$\begin{aligned}&\mathbf{dL}:\\ &\Gamma\left(V(M)\times T^*(M)\otimes V(M)\times T^*(M)\wedge T^*(M)\otimes V(M)\otimes V^*(M)\right) \to \Gamma\left(\wedge^4 T^*(M)\right),\\ &((\varphi, g_{ab}),\ (D\varphi_c, Dg_{def}),\ (r_{gh}, R_{ghi}{}^j)) \mapsto \mathbf{dv}(g)\varphi^2 g^{km}\delta^l{}_n R_{klm}{}^n.\end{aligned}$$

The symbol $\mathbf{dv}(g)$ denotes the volume form in $\Gamma\left(L^4(M)\otimes\wedge^4 T^*(M)\right)$ generated by the metric tensor field $g\in\Gamma\left(L^2(M)\otimes\vee^2 T^*(M)\right)$.

The field $\varphi\in\Gamma\left(L^{-1}(M)\right)$ plays the role of inverse Planck length. Just it is not a constant, but a section of a line bundle. The vector space of length values are not initially the same in particular points of spacetime. In order to compare length values in different points, one would need a covariant derivation on $L(M)$. The model is conf.inv in the sense that action does not depend on covariant derivation of $L(M)$.

The derived Euler-Lagrange field equations are:

$$\begin{aligned}\tilde{\nabla}_a\left(\varphi^2 g_{bc}\right) &= 0 \qquad (\tilde{\nabla}_a \text{being the torsion-free part of } \nabla_a),\\ E(\tilde{\nabla},\varphi^2 g)_{ab} &= \mathcal{T}(\nabla,\varphi^2 g)_{ab}.\end{aligned}$$

Here, $E(\tilde{\nabla},\varphi^2 g)_{ab}$ is the Einstein tensor of $\tilde{\nabla}_a$ and the rescaled metric $\varphi^2 g_{ab}$, and $\mathcal{T}(\nabla,\varphi^2 g)_{ab} :=$

$$\frac{1}{4}\left(2\tilde{\nabla}_{(a}T(\nabla)^g_{b)g} + T(\nabla)^h_{ga}T(\nabla)^g_{bh} - \frac{1}{2}(\varphi^2 g_{ab})(\varphi^{-2}g^{ef})\left(2\tilde{\nabla}_e T(\nabla)^g_{fg} + T(\nabla)^h_{ge}T(\nabla)^g_{fh}\right)\right)$$

which is the contribution of the torsion of ∇ (i.e. of $T(\nabla)^c_{ab}$) to energy-momentum tensor. (Actually, contribution $T(\nabla)^c_{ab}$ can be zeroed out initially, if variation is performed on the closed subspace of torsion-free covariant derivations. If not zeroed out, obeys slightly different field equations than in Einstein-Cartan-Sciama-Kibble model [9] because of our Palatini-like formulation.)

This is nothing but an ordinary vacuum Einstein equation for the dimensionless metric $\varphi^2 g_{ab}$ measured in units of square Planck length φ^{-2}. (With possible torsion contribution.)

Can be also re-expressed in terms of the dimensional metric g_{ab}:

$$\begin{aligned}\tilde{D}_a(g_{bc}) &= 0 \qquad (\tilde{D}_a \text{ is the torsion-free part of } D_a),\\ E(\tilde{D},g)_{ab} &= \mathcal{T}(\nabla,\varphi^2 g)_{ab}\\ &\quad +2\varphi^{-1}\tilde{D}_{(a}\tilde{D}_{b)}\varphi - 2g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e\tilde{D}_f(\varphi)\\ &\quad -4\varphi^{-1}\tilde{D}_a(\varphi)\,\varphi^{-1}\tilde{D}_b(\varphi) + g_{ab}\,g^{ef}\varphi^{-1}\tilde{D}_e(\varphi)\,\varphi^{-1}\tilde{D}_f(\varphi),\\ g^{ab}\tilde{D}_a\tilde{D}_b\varphi - \tfrac{1}{6}\mathcal{R}(\tilde{D},g)\varphi &= \tfrac{1}{6}g^{ab}\mathcal{T}(\nabla,\varphi^2 g)_{ab}\varphi.\end{aligned}$$

Here, $\mathcal{R}(D,g)$ is the Ricci scalar of D_a and g_{ab}.
(Again, contribution of torsion can be zeroed out, eventually.)

This is nothing but the conformally invariant coupled Einstein-Klein-Gordon equations. Which is of course conformally invariant also in the usual sense. Thus we have a definition of conformal invariance not referring to metric but to connexion.

Can be useful to generate conformally invariant Lagrangians in frameworks where spacetime metric is a derived quantity, not fundamental field. E.g. spinorial formulation of GR.

5. Summary

It is possible to formulate conformal invariance of a model in terms of connexion instead of metric. Quantities must be tagged with location dependent physical dimensions (with *measure line bundle*, i.e. line bundle of lengths). The model is conformally invariant in terms of connexion whenever the action functional is invariant to choice of connexion over the measure line bundle.

6. References

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