Lecture 2
Project 3-Point: An AdS/CFT Puzzle Resolved

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Summary of Lecture 1

The $\mathcal{N}=1$ truncation of $\mathcal{N}=8$ SG has three scalar fields

$z^\alpha = A^\alpha + iB^\alpha$.

The A^α are dual to three $\Delta = 1$ operators

$\mathcal{O}_\alpha(x)$ in truncated $\mathcal{N}=8$ ABJM theory.

GOAL: To calculate $\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{L^2}{4\pi^4 G_4}\frac{1}{|x-y||y-z||z-x|}$ from "hidden" $A^1A^2A^3$ vertex in on-shell bulk action. This result will "lift" to parent $\mathcal{N}=8$ theory.

Evidence for hidden vertex from Bogomolny argument:

$$\begin{aligned} S_3 &= -\frac{1}{4\pi G_4}\int d^3x\, e^{3r_0/L} e^{K/2}|W_{SG}| \\ W_{SG} &= (1+z^1z^2z^3)/L \end{aligned}$$

We now want a more direct argument.

c. **Find CT's by extension of local SUSY to the boundary.**

i. In usual proofs of invariance in SG, one is happy to achieve invariance up to total derivatives, i.e. $\delta S = \int d^4x \partial_\mu [\sqrt{-g} \bar{\epsilon}(x) X^\mu]$

ii. Now, however, we collect these bdy terms and write

$$\int d^4x \partial_\mu [\sqrt{-g}\, \bar{\epsilon}(x) X^\mu] = \int_{r=r_0} d^3x \sqrt{-h}\, \bar{\epsilon}\, X^r \equiv \delta S_{bdy}.$$

iii. Find set of CTs: $S_{CT} = \int d^3x \sqrt{-h}\, \mathcal{L}_{CT}$, such that $\delta_{SUSY} S_{CT} = -\delta S_{bdy}$.

iv. Similarly, require consistency of Euler-Lagrange variational principle at the bdy. Find bdy. conditions on the bulk fields.

d. First work out bdy. terms and CT's in global limit of any $\mathcal{N} = 1$ SG model. This is a limit in which the back reaction of the matter fields is consistently suppressed, so the gravitino can be dropped. Result is an action that has global SUSY on AdS_4.
Similar to construction of Festuccia and Seiberg, 1105.0689

i. In this global limit, the SUSY parameters are AdS Killing spinors. Killing spinors satisfy

$$(D_\mu + \frac{1}{2L}\gamma_\mu)\epsilon(r,x) = 0 \implies (\gamma^\mu D_\mu + \frac{2}{L})\epsilon(r.x) = 0$$

They can be found explicitly for the AdS_4 metric $ds^2 = dr^2 + e^{2r/L}\eta_{ij}dx^i dx^j$. Their leading components grow at the bdy. as $\epsilon(r,x) \sim e^{r/2L}$.

ii. This limiting procedure works for any Kähler metric and any supot. of the form $W_{SG} = (1 + W(z^\alpha))$ with cubic $W(z^\alpha)$. This guarantees that the SG model has an AdS stationary point with cos. const. $\Lambda = 3/L^2$, the SUSY value.

iii. Further simplifications: info on CT's that we need is captured by case of one chiral multiplet $z, \ \chi$ with a flat Kähler potential $K = z\bar{z}$ and cubic $W = z^3/3$ or $z^1 z^2 z^3$.

iv. Result is a simple (off-shell) action.

$$
\begin{aligned}
S &= S_{kin} + S_F + S_{\bar{F}} \\
S_{kin} &= \int d^4x\sqrt{-g}\Big[-\partial_\mu z \partial^\mu \bar{z} - \frac{1}{2}\bar{\chi}\gamma^\mu D_\mu \chi \\
&\quad + (F + z/L)(\bar{F} + \bar{z}/L) + 2z\bar{z}/L^2 \Big] \\
S_F &= \int d^4x\sqrt{-g}[FW' - \frac{1}{2}W''\bar{\chi}P_L\chi + 3W/L] \\
S_{\bar{F}} &= (S_F)^* .
\end{aligned}
$$

The 3 terms $S_{kin}, \ S_F, \ S_{\bar{F}}$ are *separately* invariant under:

$$
\delta z = \bar{\epsilon}P_L\chi \qquad \delta P_L\chi = P_L(\gamma^\mu\partial_\mu z + F)\epsilon \qquad \delta F = \bar{\epsilon}(\gamma^\mu D_\mu - 1/L)P_L\chi .
$$

S_F is very simple and so is its SUSY variation. It vanishes in flat spacetime, and the remaining AdS terms give

$$\delta S_F = \int d^4x\sqrt{-g}[\nabla_\mu(\bar{\epsilon}\gamma^\mu W' P_L\chi) - \bar{\epsilon}(\overleftarrow{D}_\mu \gamma^\mu - 2/L)W' P_L\chi].$$

Last term vanishes by adjoint of Killing spinor eqtn.
First term is the bdy term we are looking for! It is cancelled by CT

$$S_{cubic} = -\int d^3x\sqrt{-g}[W(z) + \bar{W}(\bar{z})].$$

With addition of bdy term from δS_{kin} and with change to previous normalization, one reproduces the CT S_3 from Bogomolny argument.

Total CT for $\mathcal{N}=1$ truncation of $\mathcal{N}=8$ SG

$$S_{CT} = -\frac{1}{4\pi G_4 L}\int d^3x\, e^{3r_0/L} e^{K/2}[1+\frac{1}{2}(z^1z^2z^3+c.c.)]$$

IMPORTANT CLAIM– Results in the $\mathcal{N}=1$ truncation extend to $\mathcal{N}=8$ SG, which is a MUCH more complicated theory. Please see our paper for an $\mathcal{N}=8$ Bogomolny argument and explicit analysis in the $\mathcal{N}=8$ SG theory (not the global limit).

How to calculate

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{L^2}{4\pi^4 G_4}\frac{1}{|x-y||y-z||z-x|}$$

from S_{CT} above.

Ingredients:
i. Use (Euclidean) AdS metric $ds^2 = \frac{L^2}{w_0^2}[dw_0^2 + dw_i dw_i]$
with $w_i = x^i$ and $w_0 = Le^{-r/L}$.

ii. Using $z^\alpha = (A^\alpha + iB^\alpha)/\sqrt{2}$, transform A^3 term in S_{CT} to new coordinates:

$$S_3 = -\frac{1}{8\pi G_4 L}\frac{L^3}{\sqrt{2}}\int \frac{d^3w}{w_0^3} A^1(w_0,w)A^2(w_0,w)A^3(w_0,w).$$

iii. In alt. quant., A^α is field dual to $\Delta = 1,\ \mathcal{O}_\alpha$ operator and has bulk-bdy propagator

$$K_1(w_0, w-x) = -\frac{1}{2\pi^2}\frac{w_0}{w_0^2 + (w-x)^2}$$

$$\implies A^\alpha(w_0,w) = \int d^3x\, K_1(w_0, w-x)\mathcal{A}^\alpha(x).$$

where $\mathcal{A}^\alpha(x)$ is the boundary value of the bulk field $A^\alpha(w_0,w)$.

iv. The 3-point correlator we need is $\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle$

It commonly occurs in AdS/CFT that the canonically normalized bulk field differs from the actual source of the dual CFT operator by a constant called c. By studying the details of the fit to the free energy in 1302.7310, one finds $c = 2\sqrt{2}$.

Putting all the pieces together, including factor c^3, one gets

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{L^3}{8\pi G_4}\frac{2}{\pi^6}\int \frac{d^3w}{w_0^3} \times \left[\frac{w_0}{w_0^2+(w-x)^2}\frac{w_0}{w_0^2+(w-y)^2}\frac{w_0}{w_0^2+(w-z)^2}\right].$$

This is the correlator evaluated at the cutoff w_0 near the AdS boundary. There are some cases, notably 2-point functions, in which this cutoff is essential, but in this case, the limit $w_0 \to 0$ is smooth, and we get

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{L^2}{4\pi^7 G_4}\, I(x,y,z)$$

where $I(x,y,z)$ is the integral

$$I(x,y,z) = \int \frac{d^3w}{(w-x)^2(w-y)^2(w-z)^2} = \frac{\pi^3}{|x-y||y-z||z-x|}$$

Integral easily done using Feynman parameters and/or conformal inversion. The final result

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{L^2}{4\pi^4 G_4}\frac{1}{|x-y||y-z||z-x|}$$

Perfect match to field theory result in the $\mathcal{N}=1$ truncation and can be extended to the full $\mathcal{N}=8$ theory!

Using Conformal Invariance to Evaluate the Integral

$$I(x, y, z) = \int \frac{d^3w}{(w-x)^2(w-y)^2(w-z)^2}$$

We will use the fundamental INVERSION symmetry: defined as the change of coordinates in 3-dim. Euclidean (or Minkowski) space:

$x_i = \frac{x'_i}{|x'|^2}$ so that $|x|^2 = \frac{1}{|x'|^2}$.

A scalar operator $\mathcal{O}_\Delta(x)$ of dimension Δ is mapped into a new operator $\mathcal{O}'_\Delta(x')$ by

$$\mathcal{O}_\Delta(x) = |x'|^{2\Delta}\, \mathcal{O}'_\Delta(x')$$

Thus (for three operators with $\Delta = 1$)

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = (|x'|^2|y'|^2|y'|^2)\, \langle \mathcal{O}'_1(x')\mathcal{O}'_2(y')\mathcal{O}'_3(z')\rangle$$

Apply inversion to $I(x,y,z) = \int \frac{d^3w}{(w-x)^2(w-y)^2(w-z)^2}$:

Invert $w_i,\ x_i,\ y_i,\ z_i$, using Jacobian $d^3w = \frac{d^3w'}{|w'|^6}$ and $\frac{1}{(w-x)^2} = \frac{|w'|^2|x'|^2}{(w'-x')^2}$. We get

$$\begin{aligned} I(x,y,z) &= \int \frac{d^3w'}{|w'|^6} \frac{|w'|^2|x'|^2}{(w'-x')^2} \frac{|w'|^2|y'|^2}{(w'-y')^2} \frac{|w'|^2|z'|^2}{(w'-z')^2} \\ &= (|x'|^2|y'|^2|y'|^2)\, I(x',y',z') \end{aligned}$$

We learned:

1. The 3-pt. correlator transforms properly under inversion!
2. But, no progress in evaluating the integral!

Use translation invariance: $u = x - z, \quad v = y - z$

$$I(x,y,z) = I(u,v,0) = \int d^3w \frac{1}{w^2} \frac{1}{(w-u)^2} \frac{1}{(w-v)^2}.$$

Inversion again: $w_i = \frac{w'_i}{|w'|}, \quad u_i = \frac{u'_i}{|u'|}$, etc.

$$I(u,v,0) = |u'|^2|v'|^2 \int d^3w' \frac{1}{(w'-u')^2(w-v')^2}$$

Now only two denominators, very easy with Feynman parameters:

$$\begin{aligned} I(u,v,0) &= \pi^3 \frac{|u'|^2|v'|^2}{|u'-v'|} = \pi^3 \frac{1}{|x-z|^2|y-z|^2} \frac{|x-z||y-z|}{|x-y|} \\ &= \pi^3 \frac{1}{|x-y||y-z||z-x|}. \end{aligned}$$

Repristinate (eng.) = Repristinare (it.)