Effects of random environment on a self-organized critical system: Renormalization group analysis of a continuous model

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Self-organised critical systems & turbulence (1)

Self-organised critical systems have several features:

- they are open nonequilibrium systems with dissipative transport;
- they are believed to be ubiquitous in the nature [1];
- they have no tuning parameter, thus, their behaviour differs from that of an equilibrium nearly-critical systems. Yet...

Self-organised critical systems under the influence of turbulence can be studied by the same methods!

Figure : The Abelian sandpile model was the first discovered example of a dynamical system displaying self-organised criticality.

The Method (2)

Start → Stochastic problem → Field theoretic formulation (De Dominicis-Janssen action functional [2]) → Renormalization (Dimensional analysis) → Feynman diagrams calculation → → Renormalization equations → Critical exponents → **Finish**

Description of the model (3)

The model of a self-organised critical system behavior is continuous equation for height transport with strong anisotropy [3, 4]:

$$\partial_t h = \nu_\perp \partial_\perp^2 h + \nu_\| \partial_\|^2 h - \partial_\| h^2/2 + f. \qquad (1)$$

- h is a height of the profile; $\nu_\perp, \nu_\| > 0$ are viscosity coefficients;
- $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\|$, $|\mathbf{n}| = 1$, $\mathbf{x}_\perp \mathbf{n} = 0$, $\mathbf{x} \in R^d$
- $f = f(x)$ is the Gaussian random noise with zero mean:

$$\langle f(x) f(x') \rangle = 2D_0 \delta_{tt'} \delta^{(d)}_{\mathbf{x}\mathbf{x}'};\ D_0 = g\nu_\perp^{3/2}\nu_\|^{3/2}$$

The turbulent motion of the environment is modeled by simple Gaussian statistics with zero mean, prescribed pair covariance with vanishing correlation time and strong anisotropy:

$$\langle v_i(t,\mathbf{x}) v_j(t',\mathbf{x}') \rangle = \delta_{tt'} D_{ij}(\mathbf{x}-\mathbf{x}'),$$

$$D_{ij}(\mathbf{r}) = B_0 \int_{k>m} \frac{d\mathbf{k}_\perp}{(2\pi)^{d-1}} \frac{1}{k_\perp^{d-1+\xi}} \exp(\mathrm{i}\mathbf{k}_\perp \cdot \mathbf{r}_\perp)$$

- $\mathbf{v} = \mathbf{n}v(\mathbf{x}_\perp, t)$; $\partial_i v_i = \partial_\| v(\mathbf{x}_\perp, t) = 0$; $B_0 > 0$ is an amplitude factor.

Field theoretic formulation of the model (4)

The stochastic problem (1) is equivalent to the field theoretic model with the action functional

$$\mathcal{S}(\{h, h', \boldsymbol{v}\}) = h' D_0 h' + h'\{-\partial_t h - v\partial_\| h + \nu_{\perp 0}\partial_\perp^2 h + \nu_{\|0}\partial_\|^2 h - \partial_\| h^2/2\} + \mathcal{S}_{\boldsymbol{v}}$$

$$\mathcal{S}_{\boldsymbol{v}} = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t,\mathbf{x}) D_{ij}^{-1}(\mathbf{x}-\mathbf{x}') v_j(t,\mathbf{x}'). \qquad (2)$$

The model has two interaction vertices: $h'\partial_\| h^2$ and $-h'(v\partial_\|)h$ (**Note**: h' is always under $\partial_\|$)

Diagrammatic representation (5)

- We will denote the model propagators $\langle hh \rangle_0$ as a straight line, $\langle hh' \rangle_0$ as a straight line with a small stroke and the velocity propagator as the wavy line.
- The coupling constants are g_0 and $w_0 = B_0/\nu_{\|0}$.

Two Galileyan symmetries (6)

- The symmetry of the original equation: $h \to h - u$, $u = const$;
- The symmetry of the problem augmented with the velocity field: $\mathbf{v} \to \mathbf{v} - \mathbf{n}u$, $u = const$.

The implementation:

- these and other considerations reduce the number of counter terms.
- canonical dimensions analysis coupled with the symmetries proves that our model is **multiplicatively renormalizable**.

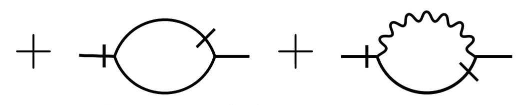
Renormalization (7)

- Renormalized action functional:

$$\mathcal{S}_R = Z_1 h' D h' + h'\{-Z_2\partial_t h - Z_3 v\partial_\| h + Z_4\nu_\perp\partial_\perp^2 h - Z_5\partial_\| h^2/2 + Z_6\nu_\|\partial_\|^2 h\} + \mathcal{S}_{\boldsymbol{v}}$$

- $Z_i = 1$, $i \neq 6$ (in all orders - thanks to the two symmetries).
- For Z_6 the calculation is done to the first order of the double expansion in ξ and $\varepsilon = 2 - d$ (one-loop approx.):

$$\langle h' h \rangle_{1-\mathrm{ir}} = \mathrm{i}\omega - \nu_\| p_\|^2 Z_6 + \ldots$$



$$Z_6 = 1 - \frac{g}{\epsilon}a - \frac{w}{\xi}b,\ (a, b > 0).$$

Three fixed points and scaling regimes (8)

- Dark space - The Gaussian fixed point.
- Vertical shading - The passively advected scalar field - the nonlinearity of the model is irrelevant: $\gamma^* = \xi$ (exact).
- Grey space - The advection is irrelevant: $\gamma^* = \frac{2}{3}\epsilon$ (exact).
- The boundaries between the regions are exact.

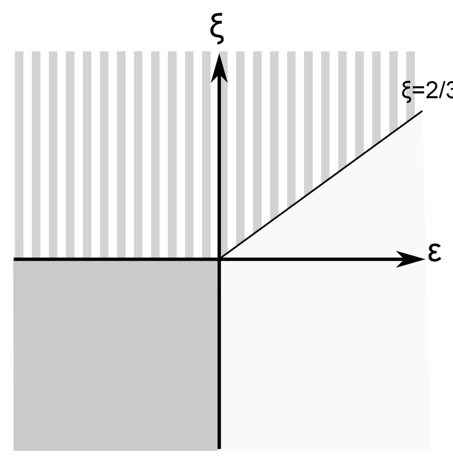


Figure : Regions of scaling regimes.

Conclusion (9)

- The most realistic values of $\xi = 4/3$ and $d = 4$ correspond to the universality class of passive scalar field.
- The critical exponents can be calculated for every regime and compared with experimental values.

Renormalization group analysis does allow us to study the influence of turbulence on self-organised critical systems.

References (10)

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