Some properties of BPS skyrmions

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1. Introduction: Skyrme models

The Skyrme model is a topological model of nuclear matter. One modification of it, the BPS Skyrme model has nice mathematical properties and zero binding energy. We investigate here the dynamics of radial pulsations of BPS Skyrmions.

1.1 Motivation

- Low-energy effective theories: σ-models

$$\mathcal{L}_2 = -\lambda_2 \,\mathrm{Tr}\, L_\mu L^\mu , \quad \text{where } L_\mu = U^\dagger \partial_\mu U .$$

where $U(x) \in SU(2)$, "pion field"

- Topological charge

$$B^\mu = \frac{1}{24\pi^2} \mathrm{Tr}\, \epsilon^{\mu\nu\rho\sigma} L_\nu L_\rho L_\sigma ,$$

yielding

$$Q = \int B^0 \mathrm{d}^3 x = -\frac{1}{24\pi^2} \int \mathrm{d}^3 x \epsilon_{ijk} L_i L_j L_k$$

- Scaling $U_\Lambda(x) = U(\Lambda x)$:

$$E[U_\Lambda] = \Lambda^{2-D} E_1$$

- Stabilise against scaling: Skyrme term [1]

$$\mathcal{L}_4 = -\lambda_4 \,\mathrm{Tr}([L_\mu, L_\nu]^2)$$

- Identify topological charge with baryon number

1.2 Skyrmions

Need a parametrisation of the group:

$$U(x) = \exp(i\phi\tau_\psi) ,$$

where ϕ is a real scalar, ψ a complex scalar,

$$\hat{\mathbf{n}} = \frac{1}{1+|\psi|^2}(\psi+\bar\psi, -i(\psi-\bar\psi), 1-|\psi|^2) ,$$

and $\tau_\psi = \hat{\mathbf{n}}\tau$, $\tau = (\tau^1, \tau^2, \tau^3)$ Pauli matrices
Charge density

$$B^0 = -\frac{\epsilon_{ijk}\partial_i\phi\partial_j\psi\partial_k\bar\psi}{\pi^2} \frac{i \sin^2\phi}{(1+|\psi|^2)^2} ,$$

pull-back of the volume form
Charge = winding number

Unit charge

$$\phi = \phi(r) , \quad \hat{\mathbf{n}} = \frac{\mathbf{r}}{r}$$

hedgehog Ansatz, spherically symmetric
Solve ODE for $f(r)$

Higher charge

- numerical solutions
- approximate:
- stereographic projection in physical space as well: $(r, \theta, \phi) \to (r, z, z^*)$
- $\phi = \phi(r)$, ψ rational function of z

Difficulties

- Structure (bunch, or crystal-like)
- Large binding energies (in contrast with the fluid drop model)
- Mathematically: topological energy bound cannot be satisfied

1.3 The BPS Skyrme model

Add other terms to the Skyrme Lagrangian [2,3]

$$\begin{aligned}\mathcal{L}_0 &= -\mu^2 V ,\\ \mathcal{L}_2 &= -\lambda_2 \,\mathrm{Tr}\, L_\mu L^\mu ,\\ \mathcal{L}_4 &= -\lambda_4 \,\mathrm{Tr}([L_\mu, L_\nu]^2) ,\\ \mathcal{L}_6 &= -\lambda_6^2 \pi^2 B^\mu B_\mu ,\end{aligned}$$

Different approaches

- properties of skyrmions in fn. of the coefficients [2]
- further fields
- Is there a special model?
  - small interaction energy
  - having all the nice properties of the Skyrme model

BPS Skyrme model [4,5]

- Keep only 6th order term
- Drop sigma model kinetic term
- Lagrangian

$$S = \int \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L} , \quad \mathcal{L}_{\mathrm{BPS}} = -\lambda^2\pi^2 B^\mu B_\mu - \mu^2 V ,$$

- Invariant to volume preserving diffeo both in physical and target space
- Has the same topological charge

Parametrisation [6]

$$U = \exp(i\phi\tau_\psi) = \cos\phi + i\tau_\psi \sin\phi .$$

yields for the energy density of static configurations

$$\mathcal{E} = \frac{\lambda^2 \sin^4\phi}{(1+|\psi|^2)^4} (\epsilon_{ijk}\partial_i\phi\partial_j\psi\partial_k\psi^*)^2 + \mu^2 V$$

BPS completion: find a complete square

$$\mathcal{E} = \left[\frac{\lambda \sin^2\phi}{(1+|\psi|^2)^2}(\epsilon_{ijk}\partial_i\phi\partial_j\psi\partial_k\psi^*) \pm \mu\sqrt{V}\right]^2 \mp 2\frac{\lambda\sin^2\phi}{(1+|\psi|^2)^2}(\epsilon_{ijk}\partial_i\phi\partial_j\psi\partial_k\psi^*)\mu\sqrt{V}$$

Spherical symmetry, $r \to \left(\frac{\lambda n}{\sqrt{2}\pi\mu}\right)^{1/3} r$ scale away all constants, n:

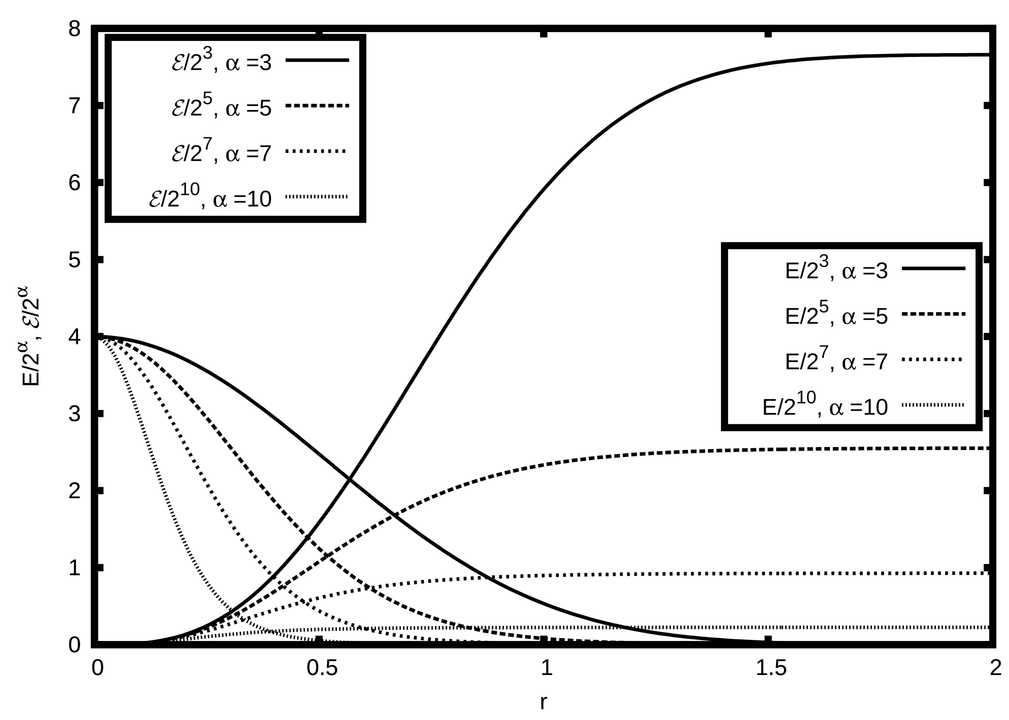
$$\mathcal{E} = \frac{\mu^2}{2}\left[\frac{\sin^4\phi}{r^4}(\phi')^2 + 2V\right] ,$$

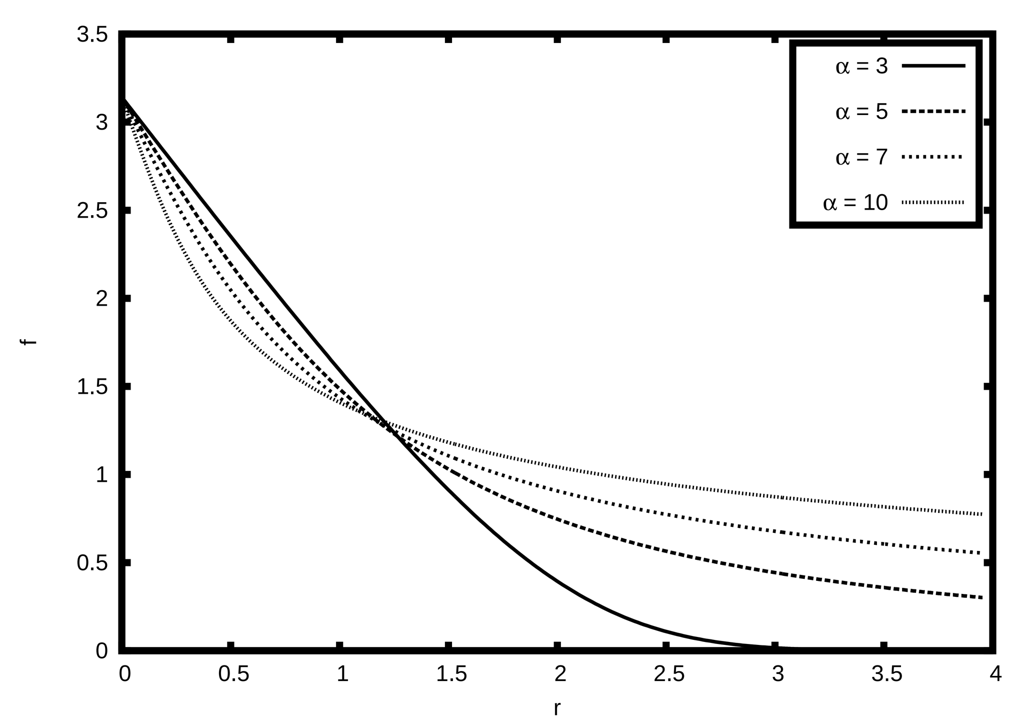
and look for complete squares (Bogomol'nyi completion):

$$\mathcal{E} = \frac{\mu^2}{2}\left\{\left[\frac{\sin^2\phi}{r^2}\phi' \pm \sqrt{2V}\right]^2 \mp 2\frac{\sqrt{2V}\sin^2\phi\phi'}{r^2}\right\} .$$

Another nice property: an infinite number of conserved quantities

- Consider spherically symmetric solutions
- Charge n only rescales solution
- BPS $\phi' = \mp\frac{\sqrt{2V}r^2}{\sin^2\phi}$
- Potential $V = \left\{\frac{1}{2}(2 - \mathrm{Tr}\, U)\right\}^\alpha$ ($\alpha = 1$: Skyrme)
- Large r asymptotics: compacton for $\alpha < 3$, exponential localization for $\alpha = 3$, power law for $\alpha > 3$





Compacton cases (e.g., Skyrme $\alpha = 1$): coefficient of highest derivatives vanishes outside the skyrmion! [4,5,6]

2. Dynamics of the BPS Skyrme model

2.1 BPS Skyrme model dynamics

Fitting parameters to properties of the proton: quasi-classical quantization of internal oscillations and rotations
Radial oscillations: $\phi = \phi(r, t)$:

$$S = 2\pi\mu^2 \int \mathrm{d}t \int \mathrm{d}r\, r^2 \left[\frac{\sin^2\phi}{r^4}\left(\dot\phi^2 - \phi'^2\right) - 2V\right] ,$$

equation of motion:

$$\frac{\sin^4\phi}{r^4}\left(\ddot\phi - \phi'' + \frac{2}{r}\phi'\right) + \frac{2\sin^3\phi\cos\phi}{r^4}\left(\dot\phi^2 - \phi'^2\right) + \frac{\partial V}{\partial\phi} = 0.$$

Dynamics

- $\alpha < 3$: cannot linearise about $\phi = 0$ (outside compactons): the coefficient of the highest derivatives vanishes there
- for $\alpha \geq 3$, well behaved

2.2 Soliton perturbations [7]

Approximate with radial pulsation:

$$\phi(r, t) = \phi_0\left(\frac{r}{\rho(t)}\right) ,$$

where ϕ_0 is the static solution.

$$L_\rho = K\frac{\dot\rho^2}{\rho^3} - U_1\frac{1}{\rho} + U_2\rho^3 ,$$

with

$$\begin{aligned}K &= 4\pi \int \mathrm{d}r\, \sin^4\phi_0\, \phi_0'^2 ,\\ U_1 &= 4\pi \int \mathrm{d}r \frac{\sin^4\phi_0}{r^2}\phi_0'^2 ,\\ U_2 &= 8\pi \int \mathrm{d}r\, r^2 V(\phi_0) .\end{aligned}$$

For $\rho = 1 + \delta$: harmonic oscillator $\omega_{\mathrm{P}}^2 = \frac{6U_2}{K}$; first order: $U_1 = U_2$.

2.3 Numerical solution

Method of lines solution of the radial equation

α	E	K	U_1	U_2	ω_{P}
4	69.789	19.066	34.894	34.894	3.314
5	81.711	17.632	40.850	40.850	3.728
6	97.731	16.603	48.851	48.851	4.202
7	118.881	15.863	59.417	59.417	4.741
8	146.611	15.340	73.275	73.275	5.354
9	182.892	14.987	91.411	91.411	6.049
10	230.371	14.772	115.148	115.148	6.839

a	ω_{P}	ω	T
4	3.314	3.34	5.17
5	3.728	3.90	1.46
6	4.202	4.71	0.83
7	4.741	5.64	0.58
8	5.354	6.69	0.44
9	6.049	7.90	0.34
10	6.839	9.28	0.28

Result of the comparison

- A fairly good approximation
- Errors grow with increasing α
- Radial decay of the solution becomes slower with increasing α

2.4 Further interesting properties

It is possible to construct solutions with fractional charge (connected with an infinitely thin band to infinity). This points to the necessity of introducing terms breaking volume preserving diffeomorphism invariance.

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