Conservation laws in the field theoretical formulation of gravity

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Plan

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Conservation laws.

- In any physical theory conserved quantities are of special interest.
- Energy and momentum are the most important of them, because their conservation is related to homogeneity of the space-time.
- Using these conservation laws one can obtain some useful information about physical system without solving its equations of motion.
The famous Noether’s theorem gives us the formula of the stress-energy tensor $T^{\mu\nu}$, for which

$$\partial_\nu T^{\mu\nu} = 0, \ \mu, \nu = 0, 1, 2, 3.$$  \hspace{1cm} (1)

Gauss’s theorem $\Rightarrow$ conserved integrals over the hypersurfaces $x^0 = \text{const}$:

$$P^\mu = \int d^3x \ T^{\mu0} = \text{const.}$$  \hspace{1cm} (2)

This is the energy-momentum vector, $P^0$ represents the energy and spatial components represent components of the momentum.
The problem of energy in general relativity.

In general relativity (for simplicity, without matter) one has

$$\partial_\nu \tau^{\mu\nu} = 0,$$  \hspace{1cm} (3)

where $\tau^{\mu\nu}$ is the stress-energy “tensor” of gravitation. The main trouble is that $\tau^{\mu\nu}$ is not a true tensor, so:

- Nonlocalizability of the energy density $\tau^{00}$.
- Nonuniqueness of $\tau^{\mu\nu}$.

There are many pseudotensors: Einstein, Møller, Landau-Lifshitz, Komar etc.
The splitting theory is one of the field theoretical formulations of the theory of gravity:


This theory considers a \((N - 4)\)-component real field \(z^A(y)\) in \(N\)-dimensional ambient Minkowski space, \(A = 1, \ldots, N - 4\).

- Each field configuration corresponds to the some splitting of ambient space into a system of 4-dimensional surfaces \(S : z^A(y) = \text{const}\).
- Surfaces don’t interact and don’t intersect.
- Any surface can be viewed as our spacetime.
Action of the splitting theory.

Action of the theory is the sum of the Einstein-Hilbert actions over all of the surfaces:

$$ S = \int dz \, S_G(z) = \int dz d^4x \sqrt{-g} \left( -\frac{1}{2\kappa} R \right), \quad (4) $$

Changing the variables \( \{ z^A, x^\mu \} \rightarrow \{ y^a \} \), we get

$$ S = -\frac{1}{2\kappa} \int dy \sqrt{|w|} \, R. \quad (5) $$

This action gives the so-called Regge-Teitelboim equations for each surface

$$ G^{cd} b^a_{\ cd} = 0, \ a, c, d = 0, \ldots, N - 1. \quad (6) $$

\( G^{cd} \) is the Einstein tensor, \( b^a_{\ cd} \) is the second fundamental form of the surface.
We have found the way to formulate gravity like a field theory in a flat spacetime.

This theory is explicitly diffeomorphism-invariant.

Such theory has no problems with the calculation of the stress-energy tensor, which will be a genuine tensor.
The canonical stress-energy tensor in the splitting theory.

The canonical (Noether) and the metrical (Hilbert) stress-energy tensors of the splitting theory were estimated. The canonical one is

\[ T^a_b = -\sqrt{|W|} \kappa G^{a b}. \]  

(7)

It vanishes for any solution of the Einstein equations:

\[ G^{a b} = 0 \Rightarrow T^{a b} = 0. \]  

(8)

So, it is not so interesting.
The metrical stress-energy tensor in the splitting theory.

The metrical one is

\[ T^{ab}_H = -\frac{\sqrt{|w|}}{\kappa} G^{ab} + \partial_g \psi^{gab}, \]  

where

\[ \psi^{gab} = \frac{\sqrt{|w|}}{\kappa} \left[ (\Pi^{ab} \Pi^{rs} - \Pi^{ar} \Pi^{bs}) b^{g,rs} + \right. \]

\[ \left. + (\Pi^{gr} \Pi^{bs} - \Pi^{ab} \Pi^{rs}) b^{a,rs} \right]. \]

\( \Pi^{ab} \) is the projector onto the surface. Both of these stress-energy tensors are true tensors.
Therefore

- The energy density $T^{00}$ is localizable.
- The energy $E = \int dy\, T^{00}$ is explicitly diffeomorphism-invariant.

But these quantities are related to an unobservable field $z^A(y)$ in ambient space.

- What are they in terms of our customary 4-dimensional language?
- For instance, what will this energy be for the Schwarzschild solution?
Conclusions

- Gravity can be formulated as a field theory in a flat Minkowski space.
- True stress-energy tensors are present in this theory.
- So, we have the localizable energy density and the diffeomorphism-invariant energy.
- We need to translate these results into usual 4-dimensional language and to compare them with the standard general relativity results like the ADM energy.
Thank you for your attention!