Scaling in erosion of landscapes: Renormalization group analysis of a model with infinitely many couplings

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ISSP 54th Course:
The new physics frontiers in the LHC-2 Era
Erice
2016
Figure: The shores of the Troyitskii brook, Peterhof.
Erosion of landscapes and critical phenomena

Underlying phenomena include:

- Tectonic motion, surficial erosion from the flow of water or air over the landscape, etc.

Connection with critical phenomena [1, 2]

- Roughness exponent $\alpha$:

$$C(x) = \left\langle [h(x, t) - h(0, t)]^2 \right\rangle^{\frac{1}{2}} \sim x^\alpha,$$

where $h(x, t)$ is the height of the surface at position $x$ at the time $t$,
- $C(x)$ is the height-height static correlation function.

- Hence, universal aspects of landscape erosion can be described within the framework of relatively simple semiphenomenological models.
R. Pastor-Satorras and D. H. Rothman’s model (PSR)

Features of the model [1, 2]:
- describes erosion at the small length scales
- preferred direction—downhill—for the flux of eroded material
- the underlying soil is locally conserved

**Figure**: Anisotropic landscape for the case $d = 2$. 
Description of PSR model

Anisotropy

1. \( \mathbf{n} \) is a unit constant vector in the direction of the slope.
2. \( \mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\parallel \), where \( \mathbf{x}_\perp \cdot \mathbf{n} = 0 \).
3. \( \partial_\perp = \partial / \partial x_{\perp i} \) with \( i = 1 \ldots d - 1 \) (\( d \) is a space dimension).
4. \( \partial_\parallel = \mathbf{n} \cdot \partial \)

The stochastic differential equation for the height of the profile

\[
\partial_t h = \nu_\perp \partial_\perp^2 h + \nu_\parallel \partial_\parallel^2 h + \partial_\parallel^2 \lambda h^3 / 3 + f. \tag{2}
\]

- \( \nu_\parallel \) and \( \nu_\perp \) are topographic diffusion coefficients,
- \( f(x) \) is a Gaussian random noise with a zero mean and a prescribed pair correlation function (with some positive amplitude \( D \))

\[
\langle f(x)f(x') \rangle = D\delta(t - t')\delta^{(d)}(\mathbf{x} - \mathbf{x}'). \tag{3}
\]
The results of R. Pastor-Satorras and D. H. Rothman [1, 2]:

- The renormalization group used was the dynamic Wilsonian RG.
- The upper critical value of $d$ was established to be 4.
- An infrared attractive fixed point was found in the leading one-loop order.
- The calculated roughness exponents were in a good agreement with the experimental data obtained from sea floor measurements.

Source of controversy

- The upper critical value of $d$ is 2.
- Any truncated model is not suitable for renormalization analysis [3, 4].
Not truncated model of erosion

Description of the model

\[
\partial_t h = \nu_\perp \partial_\perp^2 h + \nu_\parallel \partial_\parallel^2 h + \partial_\parallel^2 V(h) + f. 
\] (4)

- \( V(h) \) is a series in powers of \( h \).
- Taylor expansion of \( V(h) \) was truncated on the leading \( h^3 \) term in [1, 2].

Field theoretic formulation of the model

The equation (4) \( \iff \) the field theoretic model with \( \Phi = \{ h, h' \} \) and action functional [5, 6]:

\[
S(\Phi) = h'h' + h' \left\{ -\partial_t h + \nu_0 \perp \partial_\perp^2 h + \nu_0 \parallel \partial_\parallel^2 h + \partial_\parallel^2 \sum_{n=2}^{\infty} \frac{\lambda_{n0} h^n}{n!} \right\} 
\] (5)

- \( D_0 \) and other factors of \( h'h' \) is scaled out and by adjusting the values of \( \lambda_{n0} \).
- Integrations over \( x = (t, x) \) are implied.
- The subscript "o" means that the parameters in (5) are not yet renormalized (bare).
Analysis of canonical dimensions

**Canonical dimensions** $d_F$ of $F$

- The dimension $[F]$: $[F] \sim [T]^{-d_F^\omega} [L_\perp]^{-d_F^\perp} [L_\parallel]^{-d_F^\parallel}$

- $L_\perp$ and $L_\parallel$ are independent length scales, $T$ is the time scale.

- $d_F = 2d_F^\omega + d_F^\perp + d_F^\parallel$

- The normalization conditions are:

  $d_{k_\perp} = -d_{x_\perp} = 1$, $d_{k_\parallel} = -d_{x_\parallel} = 0$, $d_{k_\perp}^\omega = d_{k_\parallel}^\omega = 0$, $d_\omega = -d_t = 1$.

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**Table:** Canonical dimensions of the fields and the parameters of the model (5).

<table>
<thead>
<tr>
<th>$F$</th>
<th>$h'$</th>
<th>$h$</th>
<th>$\nu_\perp$</th>
<th>$\nu_\parallel$</th>
<th>$\lambda_{n0}$</th>
<th>$g_{n0}$</th>
<th>$g_n$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_F^\omega$</td>
<td>1/2</td>
<td>$-1/2$</td>
<td>1</td>
<td>1</td>
<td>$(n + 1)/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$d_F^\parallel$</td>
<td>1/2</td>
<td>$1/2$</td>
<td>0</td>
<td>$-2$</td>
<td>$-(n + 3)/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$d_F^\perp$</td>
<td>$(d - 1)/2$</td>
<td>$(d - 1)/2$</td>
<td>$-2$</td>
<td>0</td>
<td>$(d - 1)(1 - n)/2$</td>
<td>$(2 - d)(n - 1)/2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$d_F$</td>
<td>$d/2 + 1$</td>
<td>$-(2 - d)/2$</td>
<td>0</td>
<td>0</td>
<td>$(2 - d)(n - 1)/2$</td>
<td>$(2 - d)(n - 1)/2$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Implications of canonical dimensions analysis

- All the coupling constants $g_{n0} = \lambda_{n0}^{-(n+3)/4} \nu_0^{-(n-1)/4}$ become simultaneously dimensionless at $d = 2 \Rightarrow d = 2$ is the upper critical dimension.
- $\varepsilon = 2 - d$ is expansion parameter.
- There is only one counter-term: $(\partial_\parallel^2 h') h^n$ (for any $n \geq 1$)

Hence, only not truncated model is multiplicatively renormalizable.

Renormalized action functional:

$$S_R(\Phi) = h' h' + h' \left\{ -\partial_t h + Z_\perp \nu_\perp \partial_\perp^2 h + Z_\parallel \nu_\parallel \partial_\parallel^2 h + \partial_\parallel^2 \sum_{n=2}^{\infty} \frac{Z_n \lambda_n h^n}{n!} \right\}, \quad (6)$$

$$\lambda_n = g_n \nu_\parallel^{(n+3)/4} \nu_\perp^{(n-1)/4} \mu^\varepsilon (n-1)/2. \quad (7)$$
One-loop counterterm

The model (5) involves infinitely many coupling constants...

One-loop counterterm can still be calculated - in an explicit closed form in terms of the function $V(h)$.

The trick

$\Gamma_R(\Phi)$ is the generating functional of the 1-irreducible Green’s functions of the model (5) in the number $p$ of loops:

$$\Gamma_R(\Phi) = \sum_{p=0}^{\infty} \Gamma^{(p)}(\Phi), \quad \Gamma^{(0)}(\Phi) = S_R(\Phi).$$  \hspace{1cm} (8)

The one-loop contribution:

$$\Gamma^{(1)}(\Phi) = -(1/2) \text{Tr} \ln(W/W_0),$$  \hspace{1cm} (9)

$$W(x, y) = -\delta^2 S_R(\Phi)/\delta \Phi(x) \delta \Phi(y).$$

$W$ and $W_0$ are 2 × 2-matrices in the pair $\Phi = \{h, h'\}$.

$W_0$ is the similar expression for the free parts of the action (5).
One-loop counterterm (cont’d)

Preliminary considerations

- Minimal subtraction scheme is used.
- We put \( Z = 1 \) in one-loop contribution \( \Gamma^{(1)}(\Phi) \).
- Leading-order terms are kept in \( g_n \) in the loopless contribution.
- \( g_n \approx g_n^{n-2} \)
- \( V(h) = \sum_{n=2}^{\infty} \frac{\lambda_n h^n(x)}{n!}, \quad V_R(h) = \sum_{n=2}^{\infty} \frac{Z_n \lambda_n h^n(x)}{n!} \)
- \( V', V'' \) is the derivatives of \( V \) with respect to the "variable" \( h(x) \).

The matrix \( W \)

\[
W = \begin{pmatrix} -\partial^2 h' \cdot V'' & L^T \\ L & -2 \end{pmatrix}
\]

(10) is a symbolically representation of \( W \) (under condition that \( Z = 1 \)).

- \( L \equiv \partial_t - \nu_{||} \partial^2_{||} - \nu_{\perp} \partial^2_{\perp} - \partial^2_{||} V' \),
- \( L^T \equiv -\partial_t - \nu_{||} \partial^2_{||} - \nu_{\perp} \partial^2_{\perp} - V' \partial^2_{||} \) is the transposed operation.
One-loop counterterm (cont’d)

Calculation

- The divergent part of $\Gamma_R(\Phi)$ has the form $\int dx \partial^2_\| h'(x) R(h(x))$.
- $R(h)$ is similar to $V(h)$.
- $Tr \ln$ in (9) has to be calculated only to the first order in its $hh$-element $-\partial^2_\| h' \cdot V''$.
- $\delta(Tr \ln K) = Tr(K^{-1} \delta K)$ [7], then

$$\int dx \partial^2_\| h'(x) R(h(x)) \simeq -Tr [D_{hh} V'' \partial^2_\| h'] =$$

$$= - \int dx \ D^{(hh)}(x, x) V''(h(x)) \partial^2_\| h'(x), \quad (11)$$

- $D_{hh} = (W^{-1})_{hh}$ at $h' = 0$,
- $D^{(hh)}$ is the ordinary propagator $\langle hh \rangle$ of the model (6) with $Z = 1$ and $\nu_\| \partial^2 + \nu_\perp \partial^2 + \partial^2 V'$ substituted for $\nu_\| \partial^2 + \nu_\perp \partial^2$. 

(x)
Results of the calculation

Finishing touches

- All the external momenta can be set to zero: after $\partial_\parallel^2$ is moved to the external factor $h'$, only a logarithmically divergent expression remains in the counterterm.

- Then $\partial_\parallel^2 h'(x)$ and $h(x)$ can be assumed to be constant in (11) ⇒

$$D_{hh}(x, x) = \frac{S_d}{(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon} \frac{1}{\sqrt{\nu_\perp(\nu_\parallel + V')}} + \ldots.$$ (12)

- The ellipsis stands for the UV-finite part.

- $S_d = 2\pi^d/\Gamma(d/2)$ is the area of the unit sphere in $d$ dimensions.

Result: Divergent part of $\Gamma^{(1)}(\Phi)$

$$\Gamma^{(1)}(\Phi) \approx \frac{S_d}{2(2\pi)^d} \frac{\mu^{-\varepsilon}}{\varepsilon} \int dx \frac{V''(h(x))}{\sqrt{\nu_\perp(\nu_\parallel + V'(h(x)))}} \partial^2 h'(x).$$ (13)
Introducing $r_n$

Let us introduce the representation

$$\frac{V''(h(x))}{\sqrt{\nu_\perp (\nu_\parallel + V'(h(x)))}} = \sum_{n=0}^{\infty} \frac{\mu \varepsilon(n+1/2) \nu_\perp(n-1)/4 \nu_\parallel(n+3)/4 r_n h_n^n}{\nu_\parallel n!}, \quad (14)$$

$r_n$ are completely dimensionless coefficients – polynomials in the charges $g_n$. Then:

**Constants**

$$Z_\perp = 1, \quad Z_\parallel = 1 - \frac{r_1 S_d}{2(2\pi)^d \varepsilon} + \ldots,$$

$$Z_n = 1 - \frac{r_n S_d}{g_n 2(2\pi)^d \varepsilon} + \ldots$$

**Functions**

$$\gamma_\parallel = a\mathcal{D}_g r_1/2, \quad a \equiv \frac{S_d}{2(2\pi)^d};$$

$$\beta_n = -\varepsilon \frac{n - 1}{2} g_n + \frac{n + 3}{4} g_n \gamma_\parallel - \frac{a}{2} (\mathcal{D}_g - n + 1) r_n.$$
First terms of the Taylor expansion

By considering the expansion (14) we can calculate:

\[ r_1 = g_3 - \frac{1}{2} g_2^2, \]
\[ r_2 = g_4 - \frac{3}{2} g_2 g_3 + \frac{3}{4} g_2^3, \]
\[ r_3 = g_5 - 2 g_2 g_4 - \frac{3}{2} g_3^2 + \frac{9}{2} g_2^2 g_3 - \frac{15}{8} g_2^4, \]
\[ r_4 = g_6 - \frac{5}{2} g_2 g_5 + \frac{15}{2} g_2 g_4 - 5 g_3 g_4 + \frac{45}{4} g_2 g_3^2 - \frac{75}{4} g_2^2 g_3 + \frac{105}{16} g_2^5, \ldots \]

And then calculate renormalization functions:

\[ \gamma_\parallel = \frac{a}{2} (2 g_3 - g_2^2), \]
\[ \beta_2 = -\frac{\varepsilon}{2} g_2 + a (-g_4 + \frac{11}{4} g_2 g_3 - \frac{1}{8} g_2^3), \]
\[ \beta_3 = -\varepsilon g_3 + a (-g_5 + 2 g_2 g_4 + 3 g_3^2 - \frac{21}{4} g_2 g_3 + \frac{15}{8} g_2^4), \ldots \]
Regions of stability

- A point $g_*$ is infrared (IR) stable if the real parts of all the eigen-numbers of the matrix $\omega_{nm} = \partial \beta_n / \partial g_m |_{g_*}$ are strictly positive.
- All $\omega_{nn}$ be positive is the necessary condition for that.

\[
\omega_{22} = -\frac{\varepsilon}{2} + a \left[ \frac{11}{4} g_{3*} - \frac{3}{8} g_{2*}^2 \right], \quad \omega_{33} = -\varepsilon + a \left[ 6 g_{3*} - \frac{21}{4} g_{2*}^2 \right],
\]

\[
\omega_{nn} = -\varepsilon \frac{n - 1}{2} + a \frac{(n + 1)^2 + 2}{4} g_{3*} - \frac{a}{8} (n(3n + 4) + 3) g_{2*}^2, \quad n \geq 4
\]

- In a region $g_{3*} \geq 7 g_{2*}^2 / 8 + \varepsilon / 6$ all these quantities are positive.

- This two-dimensional surface of fixed points might contain a region of IR stability.

Critical exponents $\Delta$

- $2\Delta_h = d - 1 + \Delta_\parallel - \Delta_\omega$ (the exact result),
- $\Delta_\parallel = 1 + a(2g_{3*} - g_{2*}^2) / 4$, $\Delta_h = a(2g_{3*} - g_{2*}^2) / 8$ (one-loop approximation).
Comparison with truncated models

Model of [1, 2]

- The model of [1, 2] is not suitable for renormalization analysis.
- The naive approach of putting the corresponding coupling constants in \( V(h) \) to zero shows that there is no agreement with the results of [1, 2].

Model with odd \( V(h) \)

- From the symmetry considerations, and from the explicit expression for the counterterm (13), it is clear that this case is renormalizable in itself.
- To "solve" this model one can simply set all the odd couplings \( g_{2n+1} \) and the corresponding \( \beta \)-functions equal to zero.
- A two-dimensional surface of fixed points will be reduced to a curve.
Not truncated model of erosion can be reformulated as a renormalizable field theoretic model with an infinite set of independent renormalization constants (thus, infinite set of coupling constants).

The one-loop counterterm can still be derived.

There is a two-dimensional surface of fixed points which is likely to contain IR attractive region(s).

Experimental results [2] indicate two wide ranges of roughening exponent value which might be explained by the existence of two different IR attractive regions.

If the surface of fixed points contains IR attractive regions, than the model exhibits scaling behaviour.

The corresponding scaling exponents are nonuniversal because of their dependence on the coordinates of specific fixed point on the surface (curve) – $g_2^*$ and $g_3^*$.

The exponents satisfy certain exact relations.


Thank you for your attention!

For more details see: arXiv:1602.00432