A Massive Momentum-Subtraction Scheme

Ava Khamseh

RBC and UKQCD Collaborations
Peter Boyle and Luigi Del Debbio

June 17, 2016

Motivation for a Mass Dependent Scheme

Lattice	$48^3 \times 96$ Physical	$64^3 \times 128$ Physical	$48^3 \times 96$ Fine
$1/a$ (GeV)	1.73	2.36	2.8
Max M_{D_s} (GeV)	1.65533(19)	2.21677(13)	2.49572(24)

PDG $M_{D_S^{\pm}} = 1.86957(16) GeV$ MeV, $M_{D_S^0} = 1.86480(14)$ GeV

- massless quarks: $am \ll a\mu \ll 1$
- Reduction in lattice artefacts when performing continuum extrapolation in a massive scheme, by potentially removing mass dependent $\mathcal{O}(a^2)$ terms

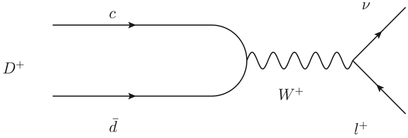
The Charm Project

Determine the decay constants f_D and f_{D_s} using

$$\langle 0 | A^{\mu}_{cq} | D_q(p) \rangle = f_{D_q} p^{\mu}_{D_q}$$

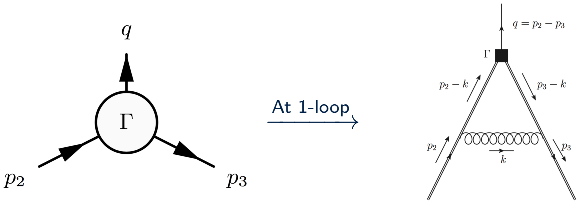
where $q = d, s$ and the axial current $A^{\mu}_{cq} = \bar{c}\gamma_{\mu}\gamma_5 q$.

To obtain the decay constant, we need to renormalize the bare axial current.



Kinematics

- Symmetric Minkowski momentum $p_2^2 = p_3^2 = q^2 = -\mu^2$ with $\mu^2 > 0$
- Vertex $G_\Gamma(p_3, p_2) = \langle O_\Gamma(q)\bar{\psi}(p_3)\psi(p_2)\rangle$, fermion bilinear $O_\Gamma = \bar{\psi}\Gamma\psi$
- Γ spans all the element of the basis of the Clifford algebra, $\Gamma = \mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$
- Propagator $S(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon}$
- Amputated vertex function $\Lambda_\Gamma(p_2, p_3) = S(p_3)^{-1} G_\Gamma(p_3, p_2) S(p_2)^{-1}$



Ward identities

- Bare WI: Noether currents associated to symmetry transformations
- Consider chiral transformations with a regulator that does not break the symmetry, e.g. dim-reg
- Noether currents:

$$\text{Vector } V_\mu = \bar{\psi}\gamma_\mu\psi$$
$$\text{Axial } A_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$$

- Vector WI: $q \cdot \Lambda_{\mathrm{V}} = iS(p_2)^{-1} - iS(p_3)^{-1}$
- Axial WI: $q \cdot \Lambda_{\mathrm{A}} = 2mi\Lambda_{\mathrm{P}} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$

Renormalization

$$\psi_R = Z_q^{1/2}\psi\,, \quad S_R(p) = Z_q S(p)\,, \quad m_R = Z_m m$$

$$\left[\bar{\psi}\Gamma\psi\right]_R = Z_\Gamma \bar{\psi}\Gamma\psi\,, \quad A_R^\mu = Z_A A^\mu\,, \quad V_R^\mu = Z_V V^\mu$$

Renormalization of Λ_Γ: $\Lambda_{\Gamma,R} = \frac{Z_\Gamma}{Z_q}\Lambda_\Gamma$

- In general, $Z = Z(g, a\mu, am)$
- Regulator a
- Renormalization scale μ
- Renormalization constants are determined by imposing renormalization conditions. e.g. RI/SMOM.

RI/SMOM Conditions

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p} \right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1} \right]\big|_{p^2=-\mu^2} - \frac{1}{2}\, \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\, \gamma_5 \right]\big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R})\ \not{q} \right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\, \gamma_5\ \not{q} \right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R} \gamma_5 \right]\big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R} \right]\big|_{\mathrm{sym}} = 1$$

Tree level values

RC are consistent with trivial renormalizations at tree level e.g.

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p} \right] \Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12p^2} Z_q^{-1} \mathrm{Tr}\left[iS(p)^{-1} \not{p} \right] \Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12p^2} Z_q^{-1} \mathrm{Tr}\left[(\not{p} - m) \not{p} \right] \Big|_{p^2=-\mu^2} = 1$$

at tree level $Z_q = 1$, same for all the others.

This is a property we wish to preserve in the massive scheme

$Z_V = 1$ in SMOM

Bare Vector WI: $q \cdot \Lambda_{\mathrm{V}} = iS(p_2)^{-1} - iS(p_3)^{-1}$

Rewriting in terms of renormalized quantities using,

$$S_R(p) = Z_q S(p) \quad \text{and} \quad \Lambda_{\mathrm{V},R} = \frac{Z_{\mathrm{V}}}{Z_q}\Lambda_{\mathrm{V}} \quad \Rightarrow$$

$$\frac{Z_q}{Z_{\mathrm{V}}}\ q \cdot \Lambda_{\mathrm{V},R} = i\ Z_q\ S_R(p_2)^{-1} - i\ Z_q\ S_R(p_3)^{-1}$$

multiplying by $\not{q}$ and taking trace, using

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R})\ \not{q}\right]\Big|_{\mathrm{sym}} = 1$$

gives $\frac{Z_q}{Z_{\mathrm{V}}} = Z_q \quad \Rightarrow Z_{\mathrm{V}} = 1$

Heavy-Heavy RI/mSMOM Conditions

$$\lim_{m_R \to \bar{m}} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p} \right] \bigg|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1} \right] \Big|_{p^2=-\mu^2} - \frac{1}{2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} \right) \gamma_5 \right] \Big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{V},R} \right) \not{q} \right] \Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} - 2m_R i \Lambda_{\mathrm{P},R} \right) \gamma_5 \not{q} \right] \Big|_{\mathrm{sym}} = 1$$

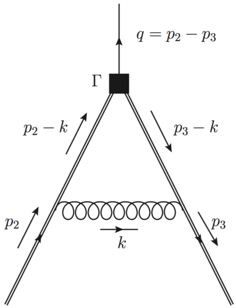
$$\lim_{m_R \to \bar{m}} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R} \gamma_5 \right] \Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R} \right] - \frac{1}{6q^2} \mathrm{Tr}\left[2im_R \Lambda_{P,R} \gamma_5 \not{q} \right] \bigg|_{\mathrm{sym}} = 1$$

Check at 1-loop in perturbation theory using dim reg

Dimensional Regularization, $D = 4 - 2\epsilon$

$$\Lambda_\Gamma^{(1)} = -ig^2 C_2(F) \int_k \frac{\gamma_\mu[\not{p}_2 - \not{k} + m]\Gamma[\not{p}_3 - \not{k} + m]\gamma^\mu}{k^2[(p_2 - k)^2 - m^2][(p_3 - k)^2 - m^2]}$$



Results

$$Z_q = 1 + \frac{\alpha}{4\pi} C_2(F) \left(\frac{1}{\epsilon} - \gamma_E + 1 - \frac{m^2}{\mu^2} - \frac{m^4}{\mu^4} \ln\left(\frac{m^2}{m^2+\mu^2} \right) - \ln\left(\frac{m^2+\mu^2}{\tilde{\mu}^2} \right) \right)$$

$$\Lambda_{\mathrm{V}}^{(1)\sigma}(p_2, p_3) = \frac{\alpha}{4\pi} C_2(F) \Big[A_{\mathrm{V}} \frac{1}{\mu^2} \; (i\epsilon^{\sigma\rho\alpha\beta}\gamma_\rho\gamma^5 p_{3\alpha} p_{2\beta}) + B_{\mathrm{V}}\gamma^\sigma + C_{\mathrm{V}} \frac{1}{\mu^2} (p_2^\sigma \; \not{p}_2 + p_3^\sigma \; \not{p}_3)$$
$$+ D_{\mathrm{V}} \frac{1}{\mu^2} (p_2^\sigma \; \not{p}_3 + p_3^\sigma \; \not{p}_2) + E_{\mathrm{V}} \frac{1}{\mu} (p_2^\sigma + p_3^\sigma) \Big]$$

$$A_{\mathrm{V}} = \frac{4}{3} \left[\left(\frac{1}{2} - \frac{m^2}{\mu^2} \right) C_0 + \left(1 + \frac{m^2}{\mu^2} \right) \log\left(\frac{m^2}{m^2+\mu^2} \right) - \sqrt{1 + 4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}} - 1}{\sqrt{1+4\frac{m^2}{\mu^2}} + 1} \right) \right]$$

$$B_{\mathrm{V}} = \frac{1}{\epsilon} - \gamma_E + \frac{1}{3} \Big[-C_0 \left(1 - 4\frac{m^2}{\mu^2} - 2\frac{m^4}{\mu^4} \right) + 2\left(3 - \frac{m^2}{\mu^2} \right) \frac{m^2}{\mu^2} \log\left(\frac{m^2}{m^2+\mu^2} \right) + \left(1 - 4\frac{m^2}{\mu^2} \right) \log\left(\frac{m^2}{\tilde{\mu}^2} \right)$$
$$-4\left(1 - \frac{m^2}{\mu^2} \right) \log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2} \right) - \left(1 - 2\frac{m^2}{\mu^2} \right) \sqrt{1+4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}} - 1}{\sqrt{1+4\frac{m^2}{\mu^2}} + 1} \right) \Big]$$

Results

$$C_{\rm V} = -\frac{2}{3}\left[\left(1-\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)+\left(1-2\frac{m^2}{\mu^2}\right)\sqrt{1+4\frac{m^2}{\mu^2}}\log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}}-1}{\sqrt{1+4\frac{m^2}{\mu^2}}+1}\right)\right.$$
$$\left.+\left(2-\frac{m^2}{\mu^2}\right)-2C_0\frac{m^2}{\mu^2}\left(1+\frac{m^2}{\mu^2}\right)-\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2}{\tilde{\mu}^2}\right)+\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2}\right)\right]$$

$$D_{\rm V} = \frac{2}{3}\left[(1+C_0)\left(1-2\frac{m^2}{\mu^2}\right)-2\left(1+\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)\right]$$

satisfies bare WI, **and** ... $Z_V = 1$!
Similarly for Z_A and all other identities.

In particular no μ dependence for the renormalization constant of Noether currents.

Summary

- Developed a massive scheme to potentially remove lattice artefacts
- Non-perturbative derivation and checked at 1-loop perturbation theory
- Both for heavy-heavy and heavy-light vertex functions
- Numerical implementation
- Will be tested on renormalizing matrix elements used to obtain decay constants

Backup Slides

Heavy-Light RI/mSMOM Conditions

$$\lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{V},R} - (M_R - m_R)\Lambda_{\mathcal{S},R}\right)\not{q}\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(i\zeta^{-1}S_{H,R}(p_2)^{-1} - i\zeta S_{l,R}(p_3)^{-1}\right)\not{q}\right]$$

$$\lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R} - (M_R + m_R)i\Lambda_{\mathcal{P},R}\right)\gamma_5\not{q}\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \frac{1}{12q^2}\mathrm{Tr}\left[\left(-i\gamma^5\zeta^{-1}S_{H,R}(p_2)^{-1} - i\zeta S_{l,R}(p_3)^{-1}\gamma^5\right)\gamma_5\not{q}\right]$$

$$\lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \frac{1}{12i}\mathrm{Tr}\left[\Lambda_{\mathcal{P},R}\gamma_5\right]\Big|_{\mathrm{sym}} = \lim_{\substack{m_R\to 0\\ M_R\to\bar{m}}} \left\{\frac{1}{12(M_R+m_R)}\left\{\mathrm{Tr}\left[-i\zeta^{-1}S_{H,R}(p)^{-1}\right]\Big|_{p^2=-\mu^2} - \frac{1}{2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R}\right)\gamma_5\right]\Big|_{\mathrm{sym}}\right\} + \frac{1}{12(M_R+m_R)}\left\{\mathrm{Tr}\left[-i\zeta S_{l,R}(p)^{-1}\right]\Big|_{p^2=-\mu^2} - \frac{1}{2}\mathrm{Tr}\left[\left(q\cdot\Lambda_{\mathcal{A},R}\right)\gamma_5\right]\Big|_{\mathrm{sym}}\right\}\right\}.$$

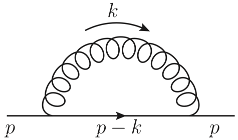
where M and m refer to heavy and light quark masses respectively and

$$\zeta = \frac{\sqrt{Z_l}}{\sqrt{Z_H}}$$

Finiteness of the ζ ratio

$$\zeta = \frac{\sqrt{Z_l}}{\sqrt{Z_H}}$$

- BPHZ theorem: Remove all the divergences of a graph, G, using **local subtractions** only $\Longrightarrow$
  coeffs. multiplying the divergent part are local.
- Possible structure of the coeffs: $1, \underbrace{p^2/m^2}_{\text{IR div}}, \underbrace{m^2/p^2}_{\text{non-local}}, \underbrace{\ln\left(\frac{m^2}{p^2}\right)}_{\text{non-local}}$



Lattice Regularization

Lattice WI for chiral symmetry

$$\nabla^*_\mu \langle A_\mu(x)\psi(y)\bar{\psi}(z)\rangle = 2m\langle P(x)\psi(y)\bar{\psi}(z)\rangle + \text{contact terms}$$
$$+\langle X(x)\psi(y)\bar{\psi}(z)\rangle$$

- X explicit chiral symmetry breaking by lattice regulator
- Reproduces usual continuum result when regulator is removed
  $\Rightarrow\ X^a(x) = aO_5(x)$
- Renormalize operators, $O_5(x)$ mixed with lower-dim operators
- Testa: power divergencies do not contribute to the anomalous dimensions
  $\Rightarrow\ A_{R,\mu} = Z_A\,(g, am)\,A_\mu$

$$O_{5R}(x) = Z_5\left[O_5(x) + \underbrace{\frac{\bar{m}}{a}P(x) + \frac{Z_A - 1}{a}\nabla^*_\mu A_\mu(x)}_{\frac{\tilde{Z}}{a}\tilde{O}(x)}\right]$$

Example: $Z_A = 1$ for Heavy-Heavy Vertex

Bare axial WI:

$$q \cdot \Lambda_{\mathrm{A}} = 2mi\Lambda_{\mathrm{P}} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$$

Rewriting in terms of renormalized quantities

$$\frac{1}{Z_{\mathrm{A}}} q \cdot \Lambda_{\mathrm{A},R} - \frac{1}{Z_m Z_{\mathrm{P}}} 2m_R i\Lambda_{\mathrm{P},R} = -\left\{\gamma_5 iS_R(p_2)^{-1} + iS_R(p_3)^{-1}\gamma_5\right\}$$

Example: $Z_A = 1$ for Heavy-Heavy Vertex

① Trace with $\gamma^5 \not{q}$

$$(Z_\mathrm{A} - 1) = \left(1 - \frac{Z_A}{Z_m Z_\mathrm{P}}\right) C_{mP}\,,$$

$$C_{m\mathrm{P}} = \lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[2im_R \Lambda_{\mathrm{P},R} \gamma_5 \not{q}\right]\big|_\mathrm{sym}$$

② Trace with γ^5

$$(Z_\mathrm{A} - 1) C_{qA} = -2 Z_\mathrm{A} \left(1 - \frac{1}{Z_m Z_\mathrm{P}}\right)\,,$$

$$C_{q\mathrm{A}} = \lim_{m_R \to \bar{m}} \frac{1}{12m_R} \mathrm{Tr}\left[q \cdot \Lambda_{\mathrm{A},R} \gamma_5\right]\big|_\mathrm{sym}$$

Together give $Z_A = 1$ and $Z_m Z_p = 1$.