EMFCSC INTERNATIONAL SCHOOL OF SUBNUCLEAR PHYSICS
54th Course: *The New Physics Frontiers in the LHC-2 Era*

Composite particles viewed as deformed oscillators, deformed Bose gas models and some applications

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Plan

1. Motivation for realization (modelling) of composite particles by deformed oscillators.
2. Operator realization of two-particle composite bosons by deformed oscillators (def. bosons).
3. Realization for "def. boson + fermion" composite fermi-particles (CFs). General solution for non-deformed constituents, and some particular ones.
4. Composite boson (quasiboson) as entangled bipartite system. The measures (characteristics) of entanglement between constituents of composite boson. Link with deformation parameter and energy dependence.
5. Application of deformed Bose gas models to effective account for the interaction/compositeness of particles: deformed virial expansion of EOS.
6. Momentum correlation function intercept of 2^{nd} order for $\tilde{\mu}, q$-Bose gas model.
7. Conclusions.

Composite (quasi)particles in diverse branches of physics

- *Mesons, baryons, nuclei* in nuclear or subnuclear physics;
- *Excitons, biexcitons, dropletons* in semiconductors and nanostructures (quantum dots etc.);
- *Trions* ("2 electrons + hole" or "2 holes + electron") in semiconductors;
- *Cooper pairs* in superconductors, in the study of electron transport;
- *Bipolarons* in crystals, organic semiconductors;
- *Composite fermions* ("electron + magnetic flux quanta" bound states [Jain]) appearing in fractional quantum Hall effect;
- *Biphotons* in quantum optics, quantum information;
- *Biphonons, triphonons, multiphonons* in crystals;
- *Atoms, molecules, ...* .

Deformed oscillators – nonlinear generalization of ordinary quantum oscillator defined by deformation structure function φ and

$$a^\dagger a = \varphi(N), \quad aa^\dagger = \varphi(N+1), \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a,$$

with $[a, a^\dagger] = 1 + \delta_\varphi(N)$ where $\delta_\varphi(N) \equiv \varphi(N+1) - \varphi(N) - 1$.

Fock space: $|n\rangle = \frac{1}{\sqrt{\varphi(n)!}}(a^\dagger)^n|0\rangle$, $\varphi(n)! \equiv \varphi(n) \cdot \ldots \cdot \varphi(1)$,

$$a^\dagger|n\rangle = \sqrt{\varphi(n+1)}|n+1\rangle, \quad a|n\rangle = \sqrt{\varphi(n)}|n-1\rangle, \quad N|n\rangle = n|n\rangle.$$

Motivational considerations

Composite particles

Creation & annihilation operators:

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_\alpha^{\mu\nu} a_\mu^\dagger b_\nu^\dagger,$$

$$A_\alpha = \sum_{\mu\nu} \overline{\Phi}_\alpha^{\mu\nu} b_\nu a_\mu,$$

$$N_\alpha = N_\alpha(a_\mu^\dagger, a_\mu, b_\nu^\dagger, b_\nu).$$

Commutator:

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta} - \Delta_{\alpha\beta}(\Phi_\gamma^{\mu\nu}).$$

Deformed particles

Generalized oscillator algebra:

$$\begin{cases} \mathcal{A}_\alpha^\dagger \mathcal{A}_\alpha = \phi(\mathcal{N}_\alpha), \\ [\mathcal{A}_\alpha, \mathcal{A}_\alpha^\dagger] = \phi(\mathcal{N}_\alpha + 1) - \phi(\mathcal{N}_\alpha), \\ [\mathcal{N}_\alpha, \mathcal{A}_\alpha^\dagger] = \mathcal{A}_\alpha^\dagger,\ [\mathcal{N}_\alpha, \mathcal{A}_\alpha] = -\mathcal{A}_\alpha. \end{cases}$$

Example. Arik-Coon deformation:

$$[\mathcal{A}_\alpha, \mathcal{A}_\alpha^\dagger] = 1 - (q-1)\mathcal{A}_\alpha^\dagger \mathcal{A}_\alpha,$$

$$\phi(\mathcal{N}_\alpha) = \frac{q^{\mathcal{N}_\alpha} - 1}{q - 1}.$$

The realization/modelling of composite bosons (e.g. mesons, excitons, cooperons etc.) by deformed oscillators allows to:

- considerably simplify the calculations;
- abstract away from the internal structure details;
- apply well developed *theory of deformed oscillators*.

So, the operators for the system of composite particles *map* onto the deformed oscillator operators. The internal structure information for such particles enters deformation parameter(s).

I. Operator realization of quasibosons by deformed bosons

Composite (or quasi-) boson creation/annihilation operators $A^\dagger_\alpha$, A_α (mode α) are realized on the states by certain def. oscillator operators

$$A^\dagger_\alpha = \sum_{\mu\nu} \Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu \ \rightarrow\ \mathcal{A}^\dagger_\alpha\,,\quad A_\alpha = \sum_{\mu\nu} \overline{\Phi^{\mu\nu}_\alpha} b_\nu a_\mu \ \rightarrow\ \mathcal{A}_\alpha\,, \quad (1)$$

where $\mathcal{A}^\dagger_\alpha$, $\mathcal{A}_\alpha$ are def. oscillator creation/annihilation operators:
$\mathcal{A}^\dagger_\alpha \mathcal{A}_\alpha = \varphi(\mathcal{N}_\alpha)$, $[\mathcal{A}_\alpha, \mathcal{A}^\dagger_\beta] = \delta_{\alpha\beta}\big(\varphi(\mathcal{N}_\alpha+1) - \varphi(\mathcal{N}_\alpha)\big) \equiv \delta_{\alpha\beta}(1 + \Delta_\varphi(\mathcal{N}_\alpha))$,
$\Phi^{\mu\nu}_\alpha$ is the composite's wavefunction, and constituent operators a_μ, b_ν satisfy fermionic (bosonic) commut. relations. – We look for such operators A_α, $A^\dagger_\alpha$, $N_\alpha \equiv \phi^{-1}(A^\dagger_\alpha A_\alpha)$, which behave *on a subspace of all quasiboson states* as the ones for a system of independent deformed oscillators with structure function $\phi(N)$:

$$[A_\alpha, A^\dagger_\beta] \equiv -\epsilon\Delta_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta,$$

$$[N_\alpha, A^\dagger_\alpha] = A^\dagger_\alpha, \quad [N_\alpha, A_\alpha] = -A_\alpha, \quad (2)$$

$$[A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta} - \epsilon\Delta_{\alpha\beta} = \phi(N_\alpha + 1) - \phi(N_\alpha),$$

with $\Delta_{\alpha\beta} \equiv \sum_{\mu\mu'} (\Phi_\beta \Phi^\dagger_\alpha)^{\mu'\mu} a^\dagger_{\mu'} a_\mu + \sum_{\nu\nu'} (\Phi^\dagger_\alpha \Phi_\beta)^{\nu\nu'} b^\dagger_{\nu'} b_\nu \leftrightarrow \Delta_\varphi(\mathcal{N}_\alpha)$.

$\epsilon = +1/-1$ – for boson/fermion constituents respectively.

Remark that closed-form relation holds

$$[[A_\alpha, A^\dagger_\beta], A^\dagger_\gamma] = -\epsilon \sum_\delta C^\delta_{\alpha\beta\gamma}(\Phi)\, A^\dagger_\delta,$$

where $C^\delta_{\alpha\beta\gamma}$ depend on wavefunctions Φ.

Realization conditions (2) reduce to equations for Φ_α and $\varphi(n)$:

$$\Phi_\beta \Phi^\dagger_\alpha \Phi_\gamma + \Phi_\gamma \Phi^\dagger_\alpha \Phi_\beta = 0, \quad \alpha \neq \beta, \tag{3}$$

$$\Phi_\alpha \Phi^\dagger_\alpha \Phi_\alpha = \frac{f}{2}\Phi_\alpha, \quad \frac{f}{2} = 1 - \frac{1}{2}\phi(2), \tag{4}$$

$$\phi(n+1) = \sum_{k=0}^{n} (-1)^{n-k} C^k_{n+1} \phi(k), \quad n \geq 2. \tag{5}$$

Their solution yields:

- *deformation structure function* $\varphi_f(N) = (1 + \epsilon\frac{f}{2})N - \epsilon\frac{f}{2}N^2$

with

- *deformation parameter* $f = \frac{2}{m}$, $m \in \mathbb{N}$ (m is positive integer);
- *matrices*

$$\Phi_\alpha = U_1(d_a)\mathrm{diag}\{0..0, \sqrt{f/2}\, U_\alpha(m), 0..0\} U^\dagger_2(d_b), \tag{6}$$

where $U_1(d_a)$, $U_2(d_b)$, $U_\alpha(m)$ are arbitrary unitary matrices of dimensions $d_a \times d_a$, $d_b \times d_b$ and $m \times m$. $\epsilon = +1$ or -1 for fermionic resp. bosonic constituents.

Generalization to quasibosons, composed of q-fermions

Commutation relations for the constituent q-fermions:

$$a_\mu a^\dagger_{\mu'} + q^{\delta_{\mu\mu'}} a^\dagger_{\mu'} a_\mu = \delta_{\mu\mu'}, \qquad b_\nu b^\dagger_{\nu'} + q^{\delta_{\nu\nu'}} b^\dagger_{\nu'} b_\nu = \delta_{\nu\nu'}, \tag{7}$$

$$a_\mu a_{\mu'} + a_{\mu'} a_\mu = 0, \quad \mu \neq \mu', \quad b_\nu b_{\nu'} + b_{\nu'} b_\nu = 0, \quad \nu \neq \nu'. \tag{8}$$

The nilpotency is absent (as opposed to the previous case):

$$q < 1 \quad \Rightarrow \quad (a^\dagger_\mu)^k \neq 0, \quad (b^\dagger_\nu)^k \neq 0, \quad k \geq 2. \tag{9}$$

Solving the conditions analogous to (2) we obtain:

- Resulting expression for the structure function:

$$\phi(n) = ([n]_{-q})^2 = \left(\frac{1-(-q)^n}{1+q}\right)^2, \quad q < 1. \tag{10}$$

- Solution for matrices Φ_α at $q < 1$:

$$\Phi^{\mu\nu}_\alpha = \Phi^{\mu_0(\alpha)\nu_0(\alpha)}_\alpha \delta_{\mu\mu_0(\alpha)} \delta_{\nu\nu_0(\alpha)}, \qquad |\Phi^{\mu_0(\alpha)\nu_0(\alpha)}_\alpha| = 1. \tag{11}$$

For more details see:

[1] A. M. Gavrilik, I. I. Kachurik, Yu. A. Mishchenko, Ukr. J. Phys. **56**, 948 (2011).

[2] A. M. Gavrilik, I. I. Kachurik, Yu. A. Mishchenko, J. Phys. A: Math. Theor. **44**, 475303 (2011).

Realization of "def. boson+fermion" composite fermions

CFs' creation/annihilation operators are given by "ansatz" (1) with $a^\dagger_\mu, a_\mu$ resp. $b^\dagger_\nu, b_\nu$ – creation/annihilation operators for constituent bosons (deformed or not) resp. fermions such that

$$a^\dagger_\mu a_\mu = \chi(n^a_\mu), \quad [a_\mu, a^\dagger_{\mu'}] = \delta_{\mu\mu'}\big(\chi(n^a_\mu{+}1){-}\chi(n^a_\mu)\big); \quad [a^\dagger_\mu, a^\dagger_{\mu'}] = 0.$$

Fermionic nilpotency holds $(A^\dagger_\alpha)^2 = 0$. For *nondeformed constituent boson* ($\chi(n) \equiv n$) the anticommutator yields

$$\{A_\alpha, A^\dagger_\beta\} = \delta_{\alpha\beta} + \sum\nolimits_{\mu\mu'} (\Phi_\beta \Phi^\dagger_\alpha)^{\mu'\mu} a^\dagger_{\mu'} a_\mu - \sum\nolimits_{\nu\nu'} (\Phi^\dagger_\alpha \Phi_\beta)^{\nu\nu'} b^\dagger_{\nu'} b_\nu,$$

So, the concerned CFs' operators are *realized by fermion operators*. The validity of the realization on one-CF states yields

$$\begin{aligned}&(\Phi_\beta\Phi^\dagger_\alpha\Phi_\gamma)^{\mu\nu} - (\Phi_\gamma\Phi^\dagger_\alpha\Phi_\beta)^{\mu\nu} + \\ &\quad + \big(\chi(2)-2\big)\big[(\Phi_\beta\Phi^\dagger_\alpha)^{\mu\mu}\Phi^{\mu\nu}_\gamma - (\Phi_\gamma\Phi^\dagger_\alpha)^{\mu\mu}\Phi^{\mu\nu}_\beta\big] = 0. \qquad (12)\end{aligned}$$

For non-deformed constituent boson realization conditions (12), ... reduce to

$$\begin{cases} \mathrm{Tr}(\Phi_\beta\Phi^\dagger_\alpha) = \delta_{\alpha\beta}, \\ \Phi_\beta\Phi^\dagger_\alpha\Phi_\gamma - \Phi_\gamma\Phi^\dagger_\alpha\Phi_\beta = 0. \end{cases} \qquad (13)$$

a) Single CF mode α case. General solution:

$$\Phi_\alpha = U_\alpha \,\mathrm{diag}\{\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...\} V_\alpha^\dagger$$

with $\lambda_i^{(\alpha)} \geq 0$, $\sum_i (\lambda_i^{(\alpha)})^2 = 1$, and arbitrary unitary U_α, V_α.

General remark. The number of modes for *realized* composite fermions D_{CF} and the constituent fermions D_f satisfy: $D_{CF} \leq D_f$.

b) CFs in 2 modes $\&$ non-deformed constituents in 3 modes. The parametrization of two orthonormal vectors $(\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \lambda_3^{(\alpha)})$, $\alpha = \overline{1,2}$ follows from $SU(3)$ parametrization, e.g.

$$\lambda_1^{(1)} = \cos\theta_1 \cos\theta_2, \quad \lambda_2^{(2)} = \cos\theta_1 \sin\theta_2, \quad \lambda_3^{(3)} = \sin\theta_1;$$

$$\begin{aligned}
\text{and} \quad \lambda_1^{(2)} &= -\sin\theta_1 \cos\theta_2 \cos\theta_3 - \sin\theta_2 \sin\theta_3 e^{i\gamma}, \\
\lambda_2^{(2)} &= \cos\theta_2 \sin\theta_3 e^{i\gamma} - \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
\lambda_3^{(2)} &= \cos\theta_1 \cos\theta_3, \qquad 0 \leq \theta_1, \theta_2, \theta_3 \leq \pi/2, \quad 0 \leq \gamma \leq 2\pi.
\end{aligned} \tag{14}$$

Solution for any number of modes (non-def. CF constituents).

$$\Phi_\alpha = U \,\mathrm{diag}\{\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...\} V^\dagger$$

where U, V are fixed (for any α) unitary matrices, $\lambda^{(\alpha)} = (\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, ...)$ are complex orthonormal vectors. Compare with (6).

Interpretation of the involved parameters The concerned parameters θ_i, $i = \overline{1,3}$, and γ (the 3-mode case) should correspond to CF internal quantum numbers like spin, the ones determining CF binding energy, etc.

Possible applications: trions, baryons. The results of this work involving nontrivial deformation χ may be applied to the effective description of tripartite (three-component) composite particles e.g. when two constituents form a bound state modeled by χ-deformed boson. For such modeling of composite constituent boson, the quasibosons realization could be relevant. Proper combination of the two realizations, quasibosonic and CF ones, can provide an alternative effective description of tripartite complexes like trions (e.g. exciton-electron or "electron + electron + hole" composites) or baryons (viewed either as three quark bound state, or as quark-diquark composite particles) significantly simpler than their direct treatment. These issues deserve special detailed study.

II. Entanglement in composite boson vs deformation parameter [A. Gavrilik, Yu. Mishchenko, PLA **376**, 1596 (2012)]

The state of the quasiboson which can be realized by deformed oscillator is entangled (inter-component entanglement):

$$|\Psi_\alpha\rangle=\sum_{k=1}^{m}\frac{1}{\sqrt{m}}|v_k^\alpha\rangle\otimes|w_k^\alpha\rangle,\ \ |v_k^\alpha\rangle=U_1^{\mu k}|a_\mu\rangle,|w_k^\alpha\rangle=\tilde{U}_2^{k\nu}|b_\nu\rangle,$$
$$\lambda_k^\alpha=\lambda=\sqrt{f/2}=\sqrt{1/m}. \qquad (15)$$

Calculation of the entanglement characteristics yields:

- Schmidt rank $r=m$;
- Schmidt number (P – the purity of subsystems)
$$K=\left[\sum_k(\lambda_k^\alpha)^4\right]^{-1}=1/P=m; \qquad (16)$$
- Entanglement entropy $S_{\mathrm{entang}}=-\sum_k(\lambda_k^\alpha)^2\ln(\lambda_k^\alpha)^2=\ln(m)$;
- Concurrence $$C=\left[\frac{r}{r-1}\Big(1-\sum_k(\lambda_k^\alpha)^4\Big)\right]^{1/2}=1.$$

Remark. Strongly entangled composite boson (high m) approaches standard boson at small quantum numbers n:
$$\phi(n)\approx\phi_{boson}(n)\equiv n,\quad n\ll m,\quad m\gg 1.$$

Generalization to multi-quasiboson states

Example 1. Multi-quasiboson state, *one mode*

$$|n_\alpha\rangle = [\phi(n_\alpha)!]^{-1/2}(A_\alpha^\dagger)^{n_\alpha}|0\rangle \tag{17}$$

(ϕ-factorial $\phi(n)! \overset{def}{=} \phi(1)\cdot \ldots \cdot \phi(n)$).
Entanglement characteristics for (17):

$$K_{\epsilon=+1} = C_m{}^{n_\alpha}, \;\; K_{\epsilon=-1} = C^{n_\alpha}_{m+n_\alpha-1};$$
$$S_{\text{entang}}|_{\epsilon=+1} = \ln C_m{}^{n_\alpha}, \;\; S_{\text{entang}}|_{\epsilon=-1} = \ln C^{n_\alpha}_{m+n_\alpha-1}.$$

Example 2. n-quasiboson Fock states with 1 quasiboson per mode:

$$|\Psi\rangle = A^\dagger_{\gamma_1}\cdot \ldots \cdot A^\dagger_{\gamma_n}|0\rangle, \quad \gamma_i \neq \gamma_j, \; i\neq j, \; i,j = 1,\ldots,n.$$

Entanglement characteristics: $K_{\epsilon=+1} = K_{\epsilon=-1} = m^n;$

$$S_{\text{entang}}|_{\epsilon=+1} = S_{\text{entang}}|_{\epsilon=-1} = n\ln(m).$$

Example 3. For the coherent state (two bosonic constituents)

$$|\Psi_\alpha\rangle = \tilde{C}(\mathcal{A};m)\sum_{n=0}^{\infty}\frac{\mathcal{A}^n}{\phi(n)!}(A_\alpha^\dagger)^n|0\rangle, \quad A_\alpha|\Psi_\alpha\rangle = \mathcal{A}_\alpha|\Psi_\alpha\rangle$$

Schmidt number and Entanglement entropy were also calculated.

CFs: three modes $\mu, \nu = \overline{1,3}$ for constituents:

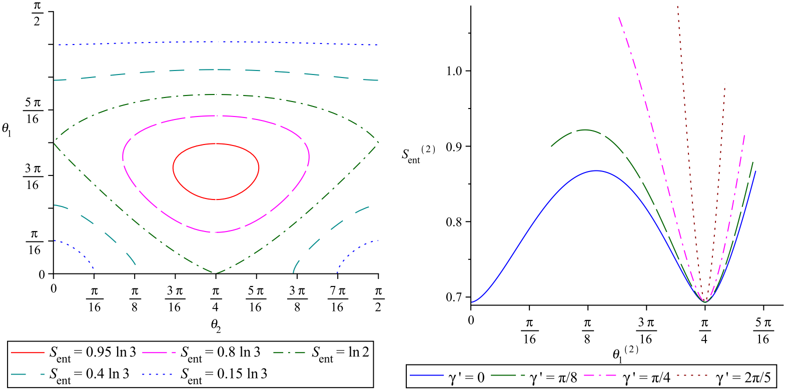


Figure 1: Left: Equi-entropic curves (for constant CF entanglement entropy $S_{\text{ent}}^{(\alpha)}$) versus parameters θ_1, θ_2, at a fixed mode $\alpha = 1$ or 2.
Right: Entanglement entropy $S_{\text{ent}}^{(2)}(\theta_1^{(2)}, \gamma')$, $\gamma' \equiv \arg(\lambda_1^{(1)} \lambda_2^{(1)} \overline{\phi_{11}} \phi_{22})$, for a CF in $\alpha = 2$ mode at fixed entanglement entropy $S_{\text{ent}}^{(1)} = \ln 3$ for $\alpha = 1$ mode of CF.

Energy dependence of entanglement entropy for the states of composite boson (quasiboson) systems

[A.M. Gavrilik, Yu.A. Mishchenko, J. Phys. A 46, 145301 (2013)]

Entanglement characteristics between the constituents of a quasiboson and their energy dependence are important in quantum information research: in quantum communication, entanglement production/enhancement, quantum dissociation processes, particle addition or subtraction in general and in the teleportation problem, etc.

We take the Hamiltonian of deformed oscillators system as

$$H = \frac{1}{2}\sum_{\alpha}\hbar\omega_{\alpha}\big(\varphi(N_{\alpha}) + \varphi(N_{\alpha}+1)\big). \qquad (18)$$

Single quasiboson case

$$S_{\mathrm{ent}}(E) = \ln\frac{\epsilon}{\frac{3}{2}-\frac{E}{\hbar\omega}} = \begin{cases} -\ln\Big(\frac{3}{2}-\frac{E}{\hbar\omega}\Big), \ \epsilon = 1, & \frac{1}{2} \le \frac{E}{\hbar\omega} \le \frac{3}{2}, \\ -\ln\Big(\frac{E}{\hbar\omega}-\frac{3}{2}\Big), \ \epsilon = -1, & \frac{3}{2} \le \frac{E}{\hbar\omega} \le \frac{5}{2}. \end{cases}$$

The corresponding plots are presented on Fig. 2 and Fig. 3.

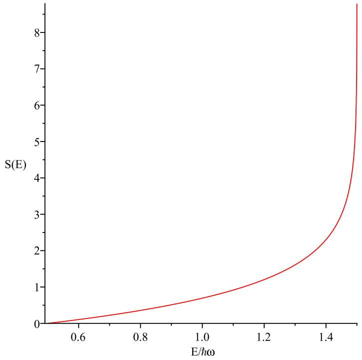


Figure 2: Dependence of the entanglement entropy S_{ent} on the energy E_α for a single composite boson in the case of fermionic components i.e. at $\epsilon = +1$.

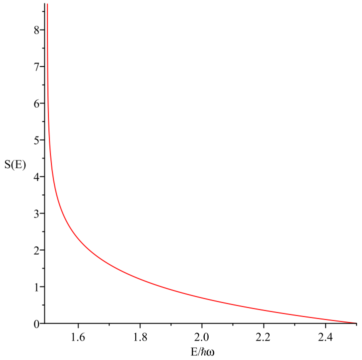


Figure 3: Dependence of the entanglement entropy S_{ent} on the energy E_α for a single composite boson in the case of bosonic components i.e. at $\epsilon = -1$.

<u>**Hydrogen atom viewed as quasiboson.**</u> – Cannot be realized by quadratically deformed oscillators. The creation operator for hydrogen atom[1] with zero total momentum and quantum number n

$$A^{\dagger}_{\mathbf{0}n} = \frac{(2\pi\hbar)^{3/2}}{\sqrt{V}} \sum_{\mathbf{p}} \phi_{\mathbf{p}n} a^{(e)\dagger}_{\mathbf{p}} b^{(p)\dagger}_{-\mathbf{p}}, \qquad (19)$$

where $a^{(e)\dagger}_{\mathbf{p}}$ and $b^{(p)\dagger}_{-\mathbf{p}}$ are the creation operators for electron and proton respectively taken with opposite momenta. The momentum-space wavefunction $\phi_{\mathbf{p}n}$ is determined by the Schrodinger equation:

$$\phi_{\mathbf{p}n} = \int \frac{e^{\frac{i}{\hbar}\mathbf{p}\mathbf{r}}}{(2\pi\hbar)^{3/2}} \phi_n(\mathbf{r}) d^3\mathbf{r}; \quad -\frac{\hbar^2\nabla^2}{2m}\phi_n(\mathbf{r}) + U(\mathbf{r})\phi_n(\mathbf{r}) = E_n\phi_n(\mathbf{r}).$$

Expansion (19) can be viewed directly as the Schmidt decomposition for the state $A^{\dagger}_{\mathbf{0}n}|0\rangle$ with Schmidt coefficients $\lambda_{\mathbf{p}} = \frac{(2\pi\hbar)^{3/2}}{\sqrt{V}}\phi_{\mathbf{p}n}$. Then the entanglement entropy for the hydrogen atom is given by

$$S_{\text{ent}} = -\sum_{\mathbf{p}} |\lambda_{\mathbf{p}}|^2 \ln|\lambda_{\mathbf{p}}|^2 = -\sum_{\mathbf{p}} \frac{(2\pi\hbar)^3}{V}|\phi_{\mathbf{p}n}|^2 \ln\Big(\frac{(2\pi\hbar)^3}{V}|\phi_{\mathbf{p}n}|^2\Big).$$

[1] Note that similar ansatz is used for the excitonic creation operators

Let us consider the simplest case of quantum numbers $l=0$ & $m=0$.

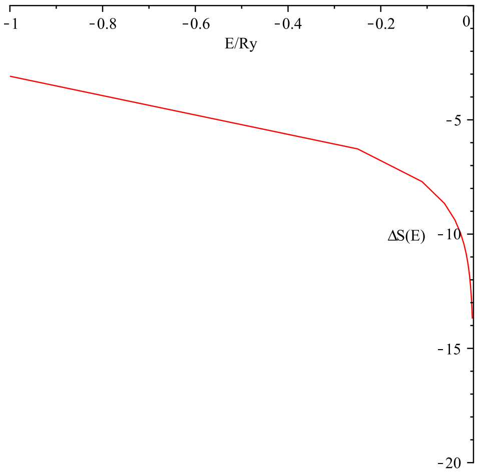


Figure 4: Dependence of the entanglement entropy $\Delta S = S_{\rm ent} - S_{\rm ent}^{(0)}$ on the energy E for Hydrogen atom.

III. Deformed Bose gas models. Preliminaries

Approach I. Standard relations *apply to deformed* physical quantities (dependent on deformation parameter(s) q):

$$F_i \to F_i^{(q)}, \quad R(F_i) = 0 \to R(F_i^{(q)}) = 0. \tag{20}$$

In [N. Swamy, 2009] q-Bose gas model involves the deformed particle number $N^{(q)}{=}z\mathcal{D}_z^{(q)} \ln Z^{(0)}$, obtained using deformed derivative $\mathcal{D}_z^{(q)}$. Then, other thermodyn. quantities can be derived. E.g., q-deformed virial expansion $\frac{Pv}{k_{\rm B}T} {=} \sum_{k=1}^{\infty} a_k(\epsilon)(\lambda^3/v)^{k-1}$, $\epsilon \equiv q-1$, along with a few virial coefficients $a_k(\epsilon)$ was found.

Approach II. Deform *defining relations* for a physical system (say commutation rel-ns): $\quad R(F_i) = 0 \to R_q(F_i) = 0.$

For instance, *deformed (nonlinear) oscillator* is defined by deformation structure function φ and the relations:

$$a^\dagger a = \varphi(N), \quad aa^\dagger = \varphi(N{+}1), \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a,$$

then $\quad [a, a^\dagger] = 1 + \delta_\varphi(N), \quad \delta_\varphi(N) \equiv \varphi(N{+}1){-}\varphi(N){-}1.$

Deformation of thermodynamics rel-s using φ-derivative

In φ-Bose gas model def. number of particles ($z = e^{\beta\mu}$ – fugacity)

$$N^{(\varphi)} = \varphi\Big(z\frac{d}{dz}\Big)\ \ln Z^{(0)} \equiv z\mathcal{D}_z^{(\varphi)} \ln Z^{(0)} = \frac{V}{\lambda^3}\sum_{n=1}^{\infty} \varphi(n)\frac{z^n}{n^{5/2}},$$

where $\ln Z^{(0)} = -\sum_i \ln(1 - ze^{-\beta\varepsilon_i})$ and the φ-derivative $\mathcal{D}_z^{(\varphi)}$ is used (analog of Jackson's q-derivative): $\quad z\frac{d}{dz} \to z\mathcal{D}_z^{(\varphi)} = \varphi\Big(z\frac{d}{dz}\Big)$.

Recovering def. partition function $Z^{(\varphi)}$ from $N^{(\varphi)} = \Big(z\frac{d}{dz}\Big)\ln Z^{(\varphi)}$ we obtain φ-deformed virial (λ^3/v)-expansion (let $\varphi(1) = 1$)

$$\frac{P^{(\varphi)}}{k_{\mathrm{B}}T} = \big(v^{(\varphi)}\big)^{-1}\Big\{\sum_{k=1}^{\infty} V_k^{(\varphi)}\Big(\frac{\lambda^3}{v^{(\varphi)}}\Big)^{k-1}\Big\} = \big(v^{(\varphi)}\big)^{-1}\Big\{1 - \frac{\varphi(2)}{2^{7/2}}\frac{\lambda^3}{v^{(\varphi)}} +$$
$$+\Big(\frac{\varphi(2)^2}{2^5} - \frac{2\varphi(3)}{3^{7/2}}\Big)\Big(\frac{\lambda^3}{v^{(\varphi)}}\Big)^2 + \Big(-\frac{3\varphi(4)}{4^{7/2}} + \frac{\varphi(2)\varphi(3)}{2^{5/2}3^{3/2}} - \frac{5\varphi(2)^3}{2^{17/2}}\Big)\Big(\frac{\lambda^3}{v^{(\varphi)}}\Big)^3 +$$
$$+\Big(-\frac{4\varphi(5)}{5^{7/2}} + \frac{\varphi(2)\varphi(4)}{2^{11/2}} - \frac{2\varphi(3)^3}{3^5} - \frac{\varphi(2)^2\varphi(3)}{2^3 3^{3/2}} + \frac{7\varphi(2)^4}{2^{10}}\Big)\Big(\frac{\lambda^3}{v^{(\varphi)}}\Big)^4 + \ldots\Big\},$$

where $v^{(\varphi)} = \frac{V}{N^{(\varphi)}}$ is specific volume, λ – thermal wavelength, $V_k^{(\varphi)}$, $k = 1, 2, \ldots$, are φ-deformed virial coefficients.

Second virial coefficient for a gas with interaction

As known,

$$V_2-V_2^{(0)}=-8^{1/2}\sum_B e^{-\beta\varepsilon_B}-\frac{8^{1/2}}{\pi}\sum_l{}'(2l+1)\int_0^\infty e^{-\beta\frac{\hbar^2k^2}{m}}\frac{\partial\delta_l(k)}{\partial k}dk$$

where B runs over bound states, l is the angular momentum, and $\delta_l(k)$ partial wave phaseshift. In low-energy approximation we retain only the $l=0$ summand (s-*wave approximation*). Resp. phaseshift $\delta_0(k)$ is determined by Schrodinger eq. for a specified interaction.

Low-energy limit and s-wave approximation ($l=1$ effects are negligible). The following expansion – *effective range (or "shape-independent") approximation* holds:

$$k\,\mathrm{ctg}\,\delta_0=-\frac{1}{a}+\frac{1}{2}r_0k^2+\ldots,\quad r_0=2\int_0^\infty dr\left[\left(1-\frac{r}{a}\right)^2-\chi_0^2(r)\right],\quad (21)$$

where a is the scattering length, r_0 – effective range (radius), and $\chi_0(r)$ – the radial wavefunction of the lowest state multiplied by r.

$$\Rightarrow\qquad \partial\delta_0/\partial k=-a+(a-3r_0/2)a^2k^2+O(k^4).$$

For s-*wave approximation* we obtain

$$V_2-V_2^{(0)}=-8^{1/2}\sum_B e^{-\beta\varepsilon_B}+2\frac{a}{\lambda_T}-2\pi^2\left(1-\frac{3}{2}\frac{r_0}{a}\right)\left(\frac{a}{\lambda_T}\right)^3+\ldots.\quad (22)$$

• *Hard spheres interaction potential* [Pathria]:

$$U(r) = \begin{cases} +\infty, & r < D; \\ 0, & r > D. \end{cases} \;\Rightarrow\; V_2 - V_2^{(0)} = 2\frac{D}{\lambda_T} + \frac{10\pi^2}{3}\left(\frac{D}{\lambda_T}\right)^5 + \ldots \; (l = 0, 2).$$

We also considered: constant repulsive, square-well, anomalous scattering, scattering resonances, modified Pöschl-Teller and inverse power repulsive potentials.

With listed a and r_0 the second virial coefficient can be evaluated (22) for mentioned potentials of interparticle interaction.

The gas of non-interacting but composite bosons. Using the anzats $A^\dagger_\alpha = \sum_{\mu\nu} \Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu$ and the known formula

$$V_2 = \frac{1}{2!V}\left[(\mathrm{Tr}_1\, e^{-\beta H_1})^2 - \mathrm{Tr}_2\, e^{-\beta H_2}\right]. \tag{23}$$

if for all $(\mathbf{k}, n)$-modes $(A^\dagger_{\mathbf{k},n})^2|0\rangle \neq 0$ we obtain that in the absence of explicit interaction between *composite* bosons

$$V_2(T) - V_2^{(0)} = -\frac{1}{2^{5/2}}\Big(\sum\nolimits_n e^{-2\beta\varepsilon_n^{int}} - 1\Big). \tag{24}$$

Effective account for the interaction/compositeness of particles by q- resp. $\tilde{\mu}$-deformations

Interparticle interaction. In [N. Swamy, J.Stat.Mech (2009)] the q-deformation given by structure function $[N]_q \equiv \frac{1-q^N}{1-q}$ was interpreted as incorporating the effects of interparticle interaction. That lead to q-deformed virial expansion

$$\frac{Pv}{k_{\mathrm{B}}T} = \sum_{k=1}^{\infty} a_k(\epsilon)\Big(\frac{\lambda^3}{v}\Big)^{k-1}, \quad \epsilon = q-1, \tag{25}$$

with virial coefficients $a_k(\epsilon)$. They utilized Jackson derivative $\mathcal{D}_z^{(q)}$.

Compositeness of particles. In [Gavrilik *et al*, J.Phys.A (2011)] composite (two-fermion or two-boson) quasi-bosons with creation/annihilation operators (1) were realized by def. bosons with quadratic DSF $\varphi_{\tilde{\mu}}(N)$.

Unification of $\tilde{\mu}$- and q-deformations. The effective description of the *both* mentioned factors may be expected from a combination of DSFs $[N]_q$ and $\varphi_{\tilde{\mu}}(N)$. The simplest (*but non-unique*) variant is

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}([N]_q) = (1+\tilde{\mu})[N]_q - \tilde{\mu}[N]_q^2 \tag{26}$$

The respective $\tilde{\mu}, q$-deformed Bose gas model was recently proposed in [Gavrilik, Mishchenko, Ukr. J. Phys., 2013].

Deformed second virial coefficient vs. microscopically calculated one. Consequences for def. parameters

We use the obtained deformed virial expansion based on $\varphi_{\tilde{\mu},q}(N)$, see (26), to juxtapose with the resp. results in microscopic description. So, the first virial coefficients are

$$V_2^{(\tilde{\mu},q)} = -\frac{\varphi_{\tilde{\mu},q}(2)}{2^{7/2}}, \quad V_3^{(\tilde{\mu},q)} = \frac{\varphi_{\tilde{\mu},q}(2)^2}{2^5} - \frac{2\varphi_{\tilde{\mu},q}(3)}{3^{7/2}}. \tag{27}$$

In our approach, parameter q from $\varphi_{\tilde{\mu},q}(N)$ corresponds to effective account for interparticle interaction, and $\tilde{\mu}$ – for compositeness.

Effective account for interaction up to $(\lambda^3/v)^2$. Microscopic treatment yields, see (22),

$$V_2 - V_2^{(0)} = -8^{1/2}\sum_B e^{-\beta\varepsilon_B} + 2\frac{a}{\lambda_T} - 2\pi^2\Big(1 - \frac{3}{2}\frac{r_0}{a}\Big)\Big(\frac{a}{\lambda_T}\Big)^3 + \ldots. \tag{28}$$

According to our interpretation $V_2^{(\tilde{\mu},q)} - V_2^{(0)}|_{\tilde{\mu}=0} = \frac{2-\varphi_{\tilde{\mu},q}(2)}{2^{7/2}}|_{\tilde{\mu}=0} = \frac{1-q}{2^{7/2}}$. Equating to (28) yields

$$q = q(a, r_0, T) = 1 - 2^{9/2}\frac{a}{\lambda_T} + 2^{9/2}\pi^2\Big(1 - \frac{3}{2}\frac{r_0}{a}\Big)\Big(\frac{a}{\lambda_T}\Big)^3 + \ldots + 2^5\sum_B e^{-\beta\varepsilon_B},$$

where $\lambda_T = h/\sqrt{2\pi m k_B T}$ is thermal wavelength.

Effective account for the compositeness up to $(\lambda^3/v)^2$-terms. In the absence of explicit interaction between quasibosons, cf. (24),

$$V_2(T) = -\frac{1}{2^{5/2}} \sum\nolimits_n e^{-2\beta\varepsilon_n^{int}}$$

On the other hand, according to our interpretation $V_2^{(\tilde{\mu},q)} - V_2^{(0)}|_{q=1} = \frac{2-\varphi_{\tilde{\mu},q}(2)}{2^{7/2}}|_{q=1} = \frac{\tilde{\mu}}{2^{5/2}}$. After equating,

$$\Rightarrow \ \tilde{\mu} = \tilde{\mu}(\varepsilon_n^{int}, \Phi_\alpha^{\mu\nu}, T) = 1 - \sum\nolimits_n e^{-2\beta\varepsilon_n^{int}}. \qquad (29)$$

Structure function $\varphi_{\tilde{\mu},q}(N)$ with $q = q(a, r_0, T)$, $\tilde{\mu} = \tilde{\mu}(\varepsilon_n^{int}, \Phi_\alpha^{\mu\nu}, T)$ is chosen for effective account (in certain approximation) for the factors of interaction and of composite structure of particles.

$*$ The temperature dependence of def. parameter $q(\ldots, \underline{T})$ appears unexpected since in our interpretation def. parameter characterizes the nonideality of deformed Bose gas.

III. Correlation function intercept for $\tilde{\mu}, q$-Bose gas model

[Nucl. Phys. B **891**, 466 (2015)]

STAR (RHIC) experiment on $\pi\pi$-correlations shows that for intercept $\lambda^{(2)}(\mathbf{k}) \equiv \frac{n^{(2)}_{\mathbf{k},\mathbf{k}}}{(n_{\mathbf{k}})^2} - 1$:
1) $\lambda^{(2)}(k) \neq \mathrm{const}$ (depends on momentum);
2) all exper. values are < 1 (for usual Bose gas $\lambda^{(2)} = \mathrm{const} = 1$).

The $\tilde{\mu}, q$-deformed Bose gas model is based on a system of $\tilde{\mu}, q$-deformed oscillators with structure function $\varphi_{\tilde{\mu},q}(N)$, see (26) (just the commut. relations are deformed). Intercept of 2*nd* order as momentum $\mathbf{k}$ function is defined in the model as:

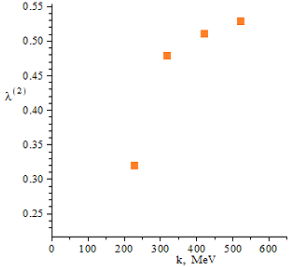


Figure 5: STAR (RHIC)-data

$$\lambda^{(2)}_{\varphi}(\mathbf{k}) = \frac{\langle (a^\dagger_{\mathbf{k}})^2 (a_{\mathbf{k}})^2 \rangle}{\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \rangle^2} - 1 = \frac{\langle \varphi(N_{\mathbf{k}}) \varphi(N_{\mathbf{k}} - 1) \rangle}{\langle \varphi(N_{\mathbf{k}}) \rangle^2} - 1, \qquad (30)$$

where $\langle \ldots \rangle$ is statistical average: $\langle F \rangle = \frac{\mathrm{Tr}\, F \exp(-\beta H)}{\mathrm{Tr} \exp(-\beta H)}$.
Hamiltonian is taken linear: $H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} N_{\mathbf{k}}$.

Deformed analogs of one- and two-particle distributions in $\tilde{\mu}, q$-deformed Bose gas model

Here we calculate the deformed one- and two-particle distributions for $\tilde{\mu}, q$-def. Bose gas model defined by DSF

$$\varphi_{\tilde{\mu},q}(N) = \varphi_{\tilde{\mu}}([N]_q) = (1+\tilde{\mu})[N]_q - \tilde{\mu}([N]_q)^2 \equiv [N]_{\tilde{\mu},q},$$

$[N]_q \equiv \frac{1-q^N}{1-q}$, with the interpretation of q and $\tilde{\mu}$ as responsible resp. for the interaction [N. Swamy, 2009] and compositeness [Gavrilik *et al*, 2011]. The definition of the distributions formally coincides with the non-deformed ones,

$$n_{\mathbf{k}} = \langle a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \rangle, \quad n^{(2)}_{\mathbf{k},\mathbf{k}'} = \langle a^{\dagger}_{\mathbf{k}} a^{\dagger}_{\mathbf{k}'} a_{\mathbf{k}'} a_{\mathbf{k}} \rangle, \tag{31}$$

but involves creation and annihilation operators, $a^{\dagger}_{\mathbf{k}}$ resp. $a_{\mathbf{k}}$ for the $\varphi_{\tilde{\mu},q}$-def. bosons which obey the following set of commutation relations given by DSF $\varphi_{\tilde{\mu},q}$:

$$[N_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} a^{\dagger}_{\mathbf{k}}, \quad [N_{\mathbf{k}}, a_{\mathbf{k}'}] = -\delta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}},$$
$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}}] = \varphi_{\tilde{\mu},q}(N_{\mathbf{k}}+1) - \varphi_{\tilde{\mu},q}(N_{\mathbf{k}}), \quad a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} = \varphi_{\tilde{\mu},q}(N_{\mathbf{k}}).$$

One- and two-particle deformed distributions

Detailed analysis of the applicability of chosen struct. function yields two possibilities: *discrete* or *continuous* $(\tilde{\mu}, q)$. In the *discrete case*:

$$\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle = \frac{1}{z-q} + \frac{(\varphi_{\tilde{\mu},q}(2) - [2]_q)\left(1 - \frac{[N_{\max}+1]_{q^2}}{[N_{\max}+1]_z}\right)}{(z-q)(z-q^2)} = \frac{1}{z-q} + \frac{\delta\varphi_{\tilde{\mu},q}(2)(1-R)}{(z-q)(z-q^2)},$$

where $\delta\varphi_{\tilde{\mu},q}(n) \equiv \varphi_{\tilde{\mu},q}(n) - [n]_q, \quad R \equiv R_{\tilde{\mu},q}(z) = \frac{[N_{\max}+1]_{q^2}}{[N_{\max}+1]_z},$ (32)

$z = e^x$, $x = \beta\hbar\omega_{\mathbf{k}}$, $\beta = (k_B T)^{-1}$. For $\langle (a^\dagger_{\mathbf{k}})^2 a^2_{\mathbf{k}}\rangle$ we find:

$$\langle (a^\dagger_{\mathbf{k}})^2 a^2_{\mathbf{k}}\rangle = \frac{\varphi_{\tilde{\mu},q}(2)}{(z-q)(z-q^2)} \left\{ 1 + (1-R)\frac{\delta\varphi_{\tilde{\mu},q}(3)\left(z + q^2 - \frac{q^2[4]_q}{\varphi_{\tilde{\mu},q}(2)}\right)}{(z-q^3)(z-q^4)} - \right.$$
$$\left. -q^2[2]_q R \frac{\left(\frac{[2]_q[3]_q}{\varphi_{\tilde{\mu},q}(2)} - 2q - 1\right)(z-q^2) + q^2(q-1)^2}{(z-q^3)(z-q^4)} \right\}. \qquad (33)$$

In the *continuous case*, take the limit $N_{\max} \to \infty$ to obtain

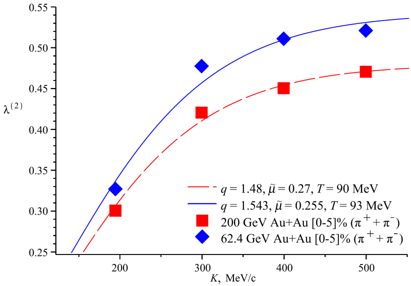
$$\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle = \frac{z + \varphi_{\tilde{\mu},q}(2) - [3]_q}{(z-q)(z-q^2)},$$

$$\langle (a^\dagger_{\mathbf{k}})^2 a^2_{\mathbf{k}}\rangle = \frac{\varphi_{\tilde{\mu},q}(2)}{(z-q)(z-q^2)} \left\{ 1 + \frac{(\varphi_{\tilde{\mu},q}(3) - [3]_q)\left(z - q^2\left(\frac{[4]_q}{\varphi_{\tilde{\mu},q}(2)} - 1\right)\right)}{(z-q^3)(z-q^4)} \right\}.$$

Intercept of two-particle correlation function and comparison with experiment

Plugging (32)-(33) in (30) yields the resulting *expression for the intercept* $\lambda^{(2)}(\mathbf{k})$.

The dependence $\lambda^{(2)}(K)$ on the momentum $K = |\mathbf{k}|$ for some values of $\tilde{\mu}$, q and temperature T versus exper. data for π-meson intercepts from RHIC/STAR is shown in Fig. 6, where we take $\hbar\omega = \sqrt{m^2 + K^2}$, and for m the π-meson mass (139.5 *MeV*).



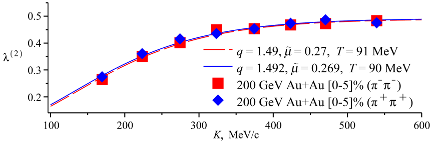


Figure 6: Intercept $\lambda^{(2)}(K)$ vs. momentum K, for different values of q, $\tilde{\mu}$ and T chosen to fit experimental data. Exper. dots [STAR] are shown by boxes.

Conclusions

- For quasibosons built of 2 fermions, 2 bosons or 2 q-fermions their operator realization by deformed oscillators (deformed bosons) with quadr. struct. function is found. The "fermion + def. boson" composite fermi-particles are also treated.

- The deformation parameter is established to be unambiguously determined by the entanglement characteristics for the realized composite bosons. Thus, inter-component entanglement reveals the physical meaning of the deformation parameter.

- The relation of the 2nd vir. coefficient of the $\tilde{\mu}, q$-def. Bose gas model to the parameters of interaction and compositeness is found. Def. parameter q is linked to the *scattering length and effective radius* of interaction, and $\tilde{\mu}$ relates to *internal energy levels* of quasibosons.

- Deformed 2-particle distribution and correlation function intercept in resp. $\tilde{\mu}, q$-deformed Bose gas model are explicitly obtained. The comparison of the obtained 2-particle correlation intercept with 2-pion intercepts from RHIC/LHC experiments is made. It shows a qualitative agreement.

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3. Gavrilik A. M. and Mishchenko Yu. A. *Energy dependence of the entanglement entropy of composite boson (quasiboson) systems*. J. Phys. A: Math. Theor., 2013, **46**, 145301.
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5. Gavrilik A. M. and Mishchenko Yu. A. *Correlation function intercepts for $\tilde{\mu}, q$-deformed Bose gas model implying effective accounting for interaction and compositeness of particles.* Nucl. Phys. B **891**, 466 (2015).

Thank you for your attention!