Highlights in Supergravity
(CCJ 47 years later)

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SUMMARY

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10. Rigid curved supersymmetry, SupergHiggs effect and the cosmological constant. No-scale supergravity.
In 1970, in occasion of the 8th course of the international School of Subnuclear Physics: “Elementary Processes at High Energies”, Sidney Coleman gave one lecture of his famous series on a “New Energy Momentum Tensor” following a paper by Callan, Coleman, Jackiw (CCJ) (A new improved energy-momentum tensor, published in Annals of Physics 59 (1970) 42-73). This paper solved the problem of defining an improved tensor $\Theta_{\mu\nu}$, different from the canonical one, such as it is conserved and symmetric

$$\partial_{\mu} \Theta_{\mu\nu} = 0, \quad \Theta_{\mu\nu} = \Theta_{\nu\mu}$$

as a consequence of Poincarè invariance in flat Minkowski space, but moreover it is also traceless

$$\Theta_{\mu}^{\mu} = 0$$

when the theory is scale and conformal invariant.
In this way the non vanishing of the trace is an operator which measures the departure from scale and conformal symmetry. This breaking can occur at the classical level due to dimensionfull terms in the scalar potential $V(\phi)$

$$\Theta_\mu = 4V - V_{,i} \phi^i$$

or can be due to quantum effects either perturbative (non-vanishing $\beta(g)$ function in perturbation theory) or non perturbative such as instanton effects in supersymmetric gauge theories.
It is the aim of these lectures to discuss the energy-momentum tensor (S.F., Samsonyan, Tournoy, van Proeyen) in supersymmetric field theories and its role in the breaking of “superconformal symmetry”. When considered in curved space, as a source of the gravitational fields, the supergravity coupling to stress tensor multiplet defines the (Super)-Einstein equations whose source is the “Supercurrent”, introduced by S.F. and Zumino (1975).
THE IMPROVED STRESS TENSOR, CONFORMAL ALGEBRA AND ITS NOETHER CURRENTS

The Noether currents of the “conformal algebra” can be written in a unified way in terms of the CCJ tensor

\[ J_\mu^\xi = \Theta_{\mu\nu}(x)\xi^\nu(x) \]

and the corresponding charges

\[ Q^\xi = \int d^3x J_0^\xi \]

which are conserved when \( \partial^\mu J_\mu^\xi = 0 \). This happens when \( \partial^\mu \Theta_{\mu\nu} = 0, \Theta_{\mu} = 0 \) provided the displacement \( \delta x^\mu = \xi^\mu(x) \) satisfies \( (d > 2) \)

\[ \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) = \frac{2}{d} \eta_{\mu\nu} \partial^\lambda \xi_\lambda(x) \]
For $d > 2$ and in particular $d = 4$ the solution of this equation is

$$\xi^\mu(x) = a^\mu + \omega^{\mu\nu}x_\nu + \lambda x^\mu + 2x^\mu x \cdot c - x^2 c^\mu$$

$a^\mu, \omega^{\mu\nu} = -\omega^{\nu\mu} \rightarrow$ Poincarè algebra (translation and Lorentz rotations)

$\lambda, c_\mu \rightarrow$ dilatation and special conformal (boosts) transformation

This algebra has 15 generators $P_\mu, M_{\mu\nu}, D, K_\mu$ satisfying the commutation relation of the Lie algebra $SO(4,2)$ (or $SU(2,2)$).

Note that for $a^\mu, \omega^{\mu\nu}$ only $\partial(\lambda \xi^\mu) = 0$ so the previous differential equation is empty. When $\Theta^\mu_{\mu} \neq 0$ the $D, K_\mu$ charges are not conserved.
\[ [M_{\mu\nu}, D] = 0, \quad [P_\mu, D] = iP_\mu, \quad [K_\mu, D] = -iK_\mu, \quad [K_\mu, K_\nu] = 0 \]
\[ [M_{\mu\nu}, K_\rho] = -i(g_\rho\mu K_\nu - g_\rho\nu K_\mu), \quad [P_\mu, K_\nu] = 2i(g_{\mu\nu} D - M_{\mu\nu}) \]

For finite transformations \( a_\mu, \Lambda_\mu^\nu, e^\lambda, c_\mu \) we have

**Poincarè** \[ x'_\mu = a_\mu + \Lambda_\mu^\nu x_\nu \quad (\Lambda_\mu^\nu \eta_{\nu\rho} \Lambda_\rho^\sigma = \eta_{\mu\sigma}) \]

**Dilatation** \[ x'_\mu = e^\lambda x_\mu \]

**Conf transf** \[ x'_\mu = \frac{x_\mu + c_\mu x^2}{1 + 2c \cdot x + c^2 x^2} \]

\[ (x'^2 = \frac{x^2}{1 + 2c \cdot x + c^2 x^2}, x^2 = 0 \Rightarrow x'^2 = 0) \]

in the infinitesimal \( \delta x^\mu = \xi^\mu(x) \).

For \( d > 4 \) \( SO(d, 2) \):
For \( d = 2 \) infinite dimensional vectors
\( M_{\mu\nu} \quad SO(d - 1, 1) \)
\( P_\mu, K_\mu \quad d \) vectors
\( D \) dilation (1 generator)
For certain systems the canonical and improved energy momentum tensors coincide. This is the case of electromagnetic field whose stress tensor is

\[ \Theta^{em}_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho}, \quad F_{\mu\nu}(A) = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \]

with the property \( \Theta^{em}_{\mu} = 0 \).

Moreover by using the cyclic identity \( \partial_{\rho}F_{\mu\nu} + \partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} = 0 \), one can show that

\[ \partial^{\mu}\Theta^{em}_{\mu\nu} = (\partial^{\mu}F_{\mu\rho})F_{\nu}^{\rho} = J_{\rho}F_{\nu}^{\rho}, \quad J_{\rho} = 0 \quad \text{for pure electromagnetism} \]

if \( J_{\rho} \neq 0 \), this term is cancelled by \( \partial^{\mu}\Theta^{M}_{\mu\nu} \) since \( \Theta_{\mu\nu} = \Theta^{em}_{\mu\nu} + \Theta^{M}_{\mu\nu} \)
The simplest theory where an improvement term occurs is the theory of a neutral scalar (or many) field $\varphi^i$ with a potential $V(\varphi^i)$. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^i \partial^\mu \varphi^i - V(\varphi^i)$$

and its “canonical” energy momentum tensor is

$$T_{\mu\nu} = \partial_\mu \varphi^i \partial_\nu \varphi^i - g_{\mu\nu} \mathcal{L}$$

$T_{\mu\nu}$ is conserved using $\varphi$ equations of motions

$$\partial^\mu T_{\mu\nu} = \partial_\nu \varphi \left( \square \varphi^i + \frac{\partial V}{\partial \varphi^i} \right) = 0$$

However it is not traceless since

$$T_{\mu}^\mu = -\partial^\lambda \varphi^i \partial_\lambda \varphi^i + 4V$$

and equations of motions can not be used.
Let's then define a new stress tensor $\Theta_{\mu\nu}$

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) \varphi^2$$

which is obviously conserved $\partial^\mu \Theta_{\mu\nu} = 0$ since both terms are conserved. However, the second term now contributes to its trace

$$\Theta^\mu_{\mu} = T^\mu_{\mu} + \frac{1}{2} \Box \varphi^2 = 4V - \frac{\partial V}{\partial \varphi^i} \varphi^i$$

using the equations of motions. And for a scale invariant theory $\Theta^\mu_{\mu} = 0$ since $V(\varphi)$ is homogeneous of degree 4.
The same argument follows for a $U(1)$ gauge theory of complex scalar fields $\varphi^i$. The Lagrangian is (the scalar part)

$$\mathcal{L}^M = (\partial_\mu - iA_\mu)\varphi^i(\partial^\mu + iA^\mu)\bar{\varphi}^i - V(\varphi, \bar{\varphi})$$

and its canonical energy momentum tensor

$$T^M_{\mu\nu} = (\partial_\mu - iA_\mu)\varphi^i(\partial^\nu + iA^\nu)\bar{\varphi}^i + (\mu \leftrightarrow \nu)$$

and the improvement term $\sim \frac{1}{3}(\partial_\mu \partial^\nu - g_{\mu\nu}\Box)\varphi^i\bar{\varphi}^i$. This time $\Theta^M_{\mu} \mu$ is as before

$$\Theta^M_{\mu} \mu = 4V - V_i\varphi^i - V_i\bar{\varphi}^i$$

but its divergence is not vanishing

$$\partial^\mu \Theta^M_{\mu\nu} = -\partial^\mu \Theta^{em}_{\mu\nu} = J^\mu F_{\mu\nu}(A)$$

with $J^\mu(\varphi, A) = i \left( \varphi^i D_\mu \bar{\varphi}^i - D_\mu \varphi^i \bar{\varphi}^i \right)$

$$D_\mu \varphi^i = (\partial_\mu - iA_\mu)\varphi^i$$

$$D_\mu \bar{\varphi}^i = (\partial_\mu + iA_\mu)\bar{\varphi}^i$$

In supergravity the $U(1)$ is the R-symmetry of the superconformal algebra (which is a gauge symmetry in the conformal part of the action.)
But a term $\Theta_{\mu\nu}^{em}$ is missing since in Einstein supergravity the $A_\mu$ gauge field is non-propagating and it is actually an auxiliary field (S.F., P. van Nieuwenhuizen; Stelle,West). It appears quadratically in the pure supergravity part of the action

$$\mathcal{L} = \sqrt{-g} \kappa^{-2} \left( \frac{1}{2} R + 3 A_\mu^2 + \ldots \right)$$

However, it contributes to the Einstein equations $\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = 0$ and indeed it contributes to the covariant divergence of the Einstein equations ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor)

$$\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = G_{\mu\nu} + 6 A_\mu A_\nu - 3 g_{\mu\nu} A_\rho A^\rho + \ldots = 0$$

so that the $A_\mu$ term contributes to both the trace and divergence of Einstein equation since

$$\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} g^{\mu\nu} = -R - 6 A_\mu^2 + \ldots = 0$$

$$\nabla^\mu \left( \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \right) = 6 \nabla^\mu A_\mu \cdot A_\nu + 6 A_\mu F_{\mu\nu}(A) + \ldots = 0$$

For superconformal matter it turns out that $R + 6 A_\mu^2 = 0$, $\nabla^\mu A_\mu = 0$ by using the matter field equations.
CCJ IN CURVED SPACE

To include the improvement term in curved space a modification of the minimal coupling to gravity is required. This coupling is called a “conformally coupled” scalar field

\[
(\sqrt{-g})^{-1} \mathcal{L} = \mathcal{L}_M - \frac{1}{12} \varphi^i \varphi^j R + \mathcal{L}_G
\]

\[
(\mathcal{L}_M = \frac{1}{2} \partial_\mu \varphi^i \partial_\nu \varphi^i g^{\mu\nu} - V(\varphi)
\]

\[
\mathcal{L}_G = \frac{1}{2} \kappa^{-2} R, \quad \kappa^{-1} = M_{Pl}
\]

which gives the Einstein equations:

\[
\frac{\kappa^{-2}}{2} G_{\mu\nu} = -\frac{1}{2} \Theta_{\mu\nu} + \frac{1}{12} G_{\mu\nu} \varphi^i \varphi^j = -\frac{1}{2} \Theta^c_{\mu\nu}
\]

with \( \Theta^c_{\mu\nu} = T_{\mu\nu} - \frac{1}{6} \left( \nabla_\mu \partial_\nu - g_{\mu\nu} \nabla^2 \right) \varphi^i \varphi^j - \frac{1}{6} G_{\mu\nu} \varphi^i \varphi^j \)

where \( \nabla^\mu \Theta^c_{\mu\nu} = 0 \) as a consequence of the modified matter field equations.
Note that the term $-\frac{1}{3}G_{\mu\nu}\partial^{\mu}\phi^{i}\phi^{i}$ in $\nabla^{\mu}\Theta_{\mu\nu}^{c}$ is cancelled by a term coming from $\nabla^{\mu}T_{\mu\nu}$ (which cancels the $\frac{1}{6}R\partial_{\nu}\phi^{i}\phi^{i}$) and a term coming from $-\frac{1}{6}(\nabla^{2}\partial_{\nu} - \partial_{\nu}\nabla^{2})\phi^{i2}$ (which cancels the $-\frac{1}{3}R_{\mu\nu}\partial^{\mu}\phi^{i}\phi^{i}$ term). If the matter system is conformal, then

$$\Theta_{\mu}^{\mu} = 0 \quad \text{which implies} \quad R = 0$$
CCJ AND SUPERGRavity

A further modification of $\Theta_{\mu \nu}^c$ occurs if the matter fields $\varphi^i$ are coupled to a $U(1)$ gauge field, as it occurs in supergravity theory due to the $U_R(1)$ symmetry of the $N = 1$ superconformal algebra. In this case the first term in $\Theta_{\mu \nu}^c$, i.e. the canonical stress tensor $T_{\mu \nu}$ contains the gauge fields and the equations of motions of the scalar fields since an extra term $\nabla^\mu \Theta_{\mu \nu}^c = J^\mu(\varphi, A)F_{\mu \nu}(A)$, which thank to the equations of motions $A_\mu = J_\mu(\varphi, A)$ produce a term which is exactly cancelled by the two terms obtained by varying the supergravity action

$$\nabla^\mu \left( \frac{2}{\sqrt{-g}} \frac{\delta L_{SG}}{\delta g^{\mu \nu}} \right) = 6A^\mu F_{\mu \nu}(A)$$

In other words if we add to $\Theta_{\mu \nu}^c$ a term proportional to $A_\mu A_\nu - \frac{1}{2}g_{\mu \nu}A_\rho^2$ then $\nabla^\mu \hat{\Theta}_{\mu \nu}^c = 0$ which is consistent with the generalized Einstein equations

$$\frac{1}{2\kappa^2}G_{\mu \nu} = -\frac{1}{2}\hat{\Theta}_{\mu \nu}^c$$

$$\hat{\Theta}_{\mu \nu}^c = T_{\mu \nu} - \frac{1}{6} \left( \nabla_\mu \partial_\nu - g_{\mu \nu} \nabla_\rho \nabla_\rho \right) \varphi^i \bar{\varphi}^i - \frac{1}{6} G_{\mu \nu} \varphi^i \bar{\varphi}^i + A_\mu A_\nu - g_{\mu \nu} A_\rho^2$$
Conformal symmetry in flat space implies Weyl symmetry in curved space, for the vierbein $e_{a\mu}(x)$ the e.m. field $F_{\mu\nu}$ and the scalar field $\phi$ we have

\[
\begin{align*}
e'_{a\mu} &= e^{\lambda(x)} e_{a\mu} \\
F'_{\mu\nu} &= F_{\mu\nu} \\
\phi'^i(x) &= e^{-\lambda(x)} \phi^i(x) \\
R' &= e^{-2\lambda(x)} R + \partial\lambda \text{ terms}
\end{align*}
\]

So for zero scalar masses the e.m. system and the scalars (coupled to gauge fields) are Weyl invariant. The Weyl symmetry of a scalar system allows actually to consider some of them "not" dynamical, since if $\phi(x) \neq 0$ by choosing the Weyl parameter such that $\phi(x) = e^{\lambda(x)}$, we can set $\phi'(x) = 1$ (even when by introducing a scale $\kappa^{-1} = M_P$). In absence of the Einstein term $L_G$, the Lagrangian, using Weyl symmetry becomes

\[
L(\phi' = \kappa^{-1}) = -1/12\kappa^{-2} R \quad (\text{since } L_M(\phi') = 0, \partial\phi' = 0).
\]
This gives the Einstein term with a wrong sign. So, we must introduce a matter field with wrong sign of the metric: \( \phi_0 \) with
\[
\mathcal{L}_{\phi_0} = -\frac{1}{2} \partial_\mu \phi_0 \partial_\nu \phi_0 g^{\mu\nu} + \frac{1}{12} \phi_0^2 R - \lambda \phi^4.
\]
In analogy with the Einstein action obtained by a gauge fixed Weyl action we can obtain supergravity by a gauge-fixed (chiral) multiplet action which is superconformal invariant in flat space and super Weyl invariant in curved superspace
\[
X^0 = (\kappa^{-1}, \psi^0 = 0, F^0 = u).
\]
The superconformal algebra in flat space is a simple superalgebra (in mathematical terms) \( SU(2, 2/1) \) with (Lie Algebra) part \( SU(2, 2) \times U_R(1) \) when \( SU(2, 2) \sim SO(4, 2) \) and the R-symmetry is part of the algebra. The (anticommuting) supersymmetry generators \( Q_\alpha, S_\alpha \) by anticommuting them, generate the even-part of the superalgebra
\[
\begin{align*}
\{Q, \bar{Q}\} &\to P_\mu, & \{S, \bar{S}\} &\to K_\mu \\
\{Q, \bar{S}\} = \{\bar{Q}, S\} &\to 0, & \{Q, S\} &\to M_{\mu\nu}, D, \Pi \\
\{Q, D\} &\to \frac{1}{2} Q, & \{S, D\} &\to -\frac{1}{2} S' \\
\{Q, \Pi\} &\to Q, & \{S, \Pi\} &\to -S
\end{align*}
\]
The Noether currents of the supercharges \((Q, S)\) can be written, in a unified way, by introducing a \(x\)-dependent (anticommuting) spinor parameter \(\epsilon(x) = \epsilon_0 + \not{x}\epsilon_1\) and write the spinor Noether currents for \(Q\) and \(S\) supersymmetry as follows \(\bar{\epsilon}^\alpha(x)J_{\alpha\mu}\) whose conservation implies

\[
\epsilon_0 \rightarrow \partial^\mu J_{\alpha\mu} = 0 \quad \epsilon_1 \rightarrow \partial^\mu (\not{x}J_{\mu}) = 0 \rightarrow \gamma^\mu J_{\mu\alpha} = 0
\]

\(Q_\alpha, S_\alpha\) generators \(\rightarrow \bar{\epsilon}^\alpha(x)J_{\alpha\mu} \rightarrow \int d^3x \bar{\epsilon}^\alpha(x)J_{\alpha0}(x)\). So, two superconformal transformations, of parameters \(\epsilon(x), \eta(x)\) generate a space-time conformal transformation of parameter \(\xi^\mu(x) = \bar{\epsilon}(x)\gamma^\mu\eta(x)\), which indeed satisfies

\[
\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2}\eta_{\mu\nu}\partial^\lambda \xi_\lambda\]

as a consequence of \(\epsilon(x) = \epsilon_0 + \not{x}\epsilon_1, \eta(x) = \eta_0 + \not{x}\eta_1\) (Wess, Zumino).
THE SUPERCURRENT AND SUPER CONSERVATION LAWS

\[ L_{improved}^M = \sqrt{-g} \left( \frac{1}{2} \partial_\mu \phi^i \partial_\nu \phi^i - \frac{1}{12} R \phi^i \phi^i \right) - \sqrt{-g} V(\phi) \]

with Einstein equations

\[ \frac{1}{2\kappa^2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{1}{2\kappa^2} G_{\mu\nu} = -\frac{1}{2} T_{\mu\nu}^{improved} = -\frac{1}{2} \Theta^c_{\mu\nu} \]

\[ \left( T_{\mu\nu} = \frac{1}{2\sqrt{-g}} \frac{\delta L^M}{\delta g^{\mu\nu}} \right) \] in supergravity these equations are modified because of the \( R \)-symmetry in the superconformal algebra.
Supercurrent for a chiral multiplet (S.F., B. Zumino)

\[ J_{\alpha\dot{\alpha}}^{s} = i S^{i} \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\partial}_{\mu} S^{\dot{i}} + \frac{1}{2} D_{\alpha} S \bar{D}_{\dot{\alpha}} S^{\dot{i}} \]

Supercurrent for a vector multiplet

\[ J_{\alpha\dot{\alpha}}^{V} = W_{\alpha} \bar{W}_{\dot{\alpha}} \quad (W_{\alpha} = \bar{D}^{2} D_{\alpha} V) \]

\[ \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}^{V} = 0 \] as a consequence of \[ \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} = 0 \] (Super Maxwell eqs)

\[ \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}^{s} \sim D_{\alpha} Y \] with \[ Y = W - \frac{1}{3} W_{i} S^{i} = \Delta W \]

so that \[ Y = 0 \] for cubic \[ Y \]. The \[ Y \] multiplet satisfying the (partial) conservation equation is made by \( (Y, \Psi_{Y}, F_{Y}) \) with

\[ Y = W - \frac{1}{3} W_{i} S^{i}, \quad \Psi_{Y} = (\gamma^{\mu} J_{\mu})_{\alpha}, \quad F_{Y} = \theta_{\lambda}^{\chi} + i \partial^{\mu} J_{\mu}^{5} \]

so that \[ Y = 0 \] for the superconformal case.
In supergravity the multiplet which contains the stress tensor, the supercurrent and $R$-symmetry

$$J^5_\mu(x), \Theta_{\mu\nu}(x), J_{\mu\alpha}(x) \rightarrow J^5_\mu + \theta^\alpha J_{\mu\alpha} + \bar{\theta}^{\dot{\alpha}} J^{\dot{\alpha}}_{\mu} + \theta^\nu \bar{\theta}^\alpha \Theta_{\mu\nu} + \ldots$$

should couple to the supergravity fields. The matter supercurrent satisfies a (super)-conservation equation

$$\overline{D}^{\dot{\alpha}} \sigma^\mu_{\alpha\dot{\alpha}} J_\mu(x, \theta, \bar{\theta}) = D_\alpha Y \quad (\overline{D}_\dot{\alpha} Y = 0)$$

when $Y \neq 0$ superconformal symmetry is broken.
SUPERSPACE DESCRIPTION OF SUPERGRAVITY

\begin{align*}
&e_{a\mu} \quad \text{(vielbein field)}, \quad \psi_{\mu\alpha} \quad \text{(gravitino)} \\
&g_{\mu\nu} = e_{a\mu} e_{b\nu} \eta^{ab} \quad \text{(metric field)} \\
&A_{\mu}, \ u : \text{auxiliary fields} \\
&\text{Curvature superfields:}\ Townsend, \ van \ Nieuwenhuizen; \ S.F., \ Zumino; \ S.F., \ Villasante
\end{align*}

In superconformal gravity the gauge fields \( e_{a\mu}, \psi_{\mu\alpha}, A_{\mu} \) gauge the superconformal algebra (Weyl in superspace). \( u \to \) complex scalar residual of the superconformal compensator \( X^0 = (\kappa^{-1}, 0, u) \) (to restore superWeyl symmetry)
Geometry of superspace \((x_\mu, \theta_\alpha), \theta_\alpha \theta_\beta = -\theta_\beta \theta_\alpha\)

Three basic multiplets: (S.F., Zumino; Wess, Zumino)

- **Chiral** \(\mathcal{R}\) contains scalar curvature \(R\)
- **Einstein** \(E_\mu\) contains Einstein tensor \(G_{\mu\nu}\)
- **(Chiral)Weyl** \(W_{\alpha\beta\gamma} = \sigma^\mu_\alpha W_{\mu\nu\gamma}\) contains Weyl tensor \(C_{\alpha\beta\gamma\delta}\)

(Traceless part of the Riemann tensor)

\[
\mathcal{R} = \bar{u} + \theta^\alpha \gamma^\mu R_\mu + \theta^2 \left( -\frac{1}{6} R - A^2_\mu - 2u\bar{u} - iD^\mu A_\mu \right)
\]
\[
E_\mu = \sigma^\alpha_\mu \bar{E}_{\alpha\dot{\alpha}} = A_\mu + \theta^\alpha Z_{\mu\alpha} + \theta^\alpha \sigma^\nu_\alpha B_{\mu\nu} \bar{\theta}^\dot{\alpha} + \ldots
\]
\[
B_{\mu\nu} = 3R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R - 6A_\mu A_\nu + 3g_{\mu\nu} A^2_\rho + 3g_{\mu\nu} u\bar{u}
\]
\[
W_{\alpha\beta\gamma} = \ldots \theta^\delta (C_{\alpha\beta\gamma\delta} + F_{\alpha\beta\epsilon\gamma}) + \theta^2 \text{(fermion)}
\]
SUPERCONFORMAL MATTER AND SUPERCURRENTS IN CURVED SUPERSPACE

The Einstein equations are coming from a supermultiplet of equations
\[ E_{\alpha\dot{\alpha}} = -\kappa^2 J_{\alpha\dot{\alpha}} \]
whose first component is \( \frac{\delta L}{\delta A_{\alpha\dot{\alpha}}} = 0 \) and their trace and conservation of the Einstein tensor come from the fundamental superspace identity
\[ \overline{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = D_\alpha \mathcal{R} \]
in superconformal formulation
\[ \overline{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = (X^0)^3 D_\alpha \frac{\mathcal{R}}{X^0} \]
they contain the identity \( \nabla^\mu G_{\mu\nu} = 0 \)

which implies
\[ \mathcal{R} + \frac{Y}{(X^0)^2} = 0 \]
\[ \left( \mathcal{R} + \kappa^2 Y = 0 \text{ after gauge fixing } X^0 = \kappa^{-1} \right) \]
\[ \mathcal{R} = Y = 0 \text{ for superconformal matter} \]
In this setup the supergravity Lagrangian is given by the following expression

\[ \mathcal{L}_{SG}(e, \psi, A, u) = \frac{1}{\kappa^2} \left( \frac{1}{2} R - \frac{1}{2} \bar{\psi}^\mu R_\mu - 3u\bar{u} + 3A^2_\mu \right) \]

so that the bosonic contribution to the (super)Einstein tensor is

\[ \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = \frac{1}{2} G_{\mu\nu} + \frac{3}{2} g_{\mu\nu} u\bar{u} + 3A_\mu A_\nu - \frac{3}{2} g_{\mu\nu} A^2_\rho \]

This means that the matter part will have improvement terms to balance the \( G_{\mu\nu} \) term in \( \nabla^\mu G_{\mu\nu} = 0 \quad \nabla^\mu (G_{\mu\nu} + ...) = 0 \).
In pure supergravity $A_\mu = u = 0$, but in matter coupled systems the improved stress tensor will get supergravity corrections

$$\hat{\Theta}^c_{\mu \nu} = \Theta^c_{\mu \nu} + \left[ 6A_\mu A_\nu - 3g_{\mu \nu} A^2_\rho + 3g_{\mu \nu} u \bar{u} \right] \kappa^{-2}$$

S.F., Samsonyan, Van Proeyen, Tournoy

such that

$$\frac{1}{\kappa^2} G_{\mu \nu} + \hat{\Theta}^c_{\mu \nu} = 0$$

will be consistent with the matter conservation laws

$$\nabla^\mu \hat{\Theta}^c_{\mu \nu} = 0$$

with the additional property $\hat{\Theta}^c_{\mu} = 0$ in the superconformal case.
RIGID SUPERSYMMETRY BREAKING

Given a generic “superfield” $\phi(x, \theta)$ with a given $n_{MAX}$, its component expression is (symbolically)

$$\phi(x, \theta) = \phi_0(x) + \theta \phi_1(x) + \ldots + \theta^{n-1} \phi_{n-1}(x) + \theta^n \phi_n(x) + \theta^{n+1} \phi_{n+1}(x) + \ldots \theta^{last} \phi_{last}(x)$$

The supersymmetry transformations are

$$\delta_\epsilon \phi_n(x) = \epsilon \partial_x \phi_{n-1}(x) + \epsilon \phi_{n+1}(x)$$

So if $\langle \phi_{n+1} \rangle \neq 0$ then supersymmetry is broken.

$$\langle \delta_\epsilon \phi_n(x) \rangle = \langle \epsilon \{Q, \phi_{n+1}(x)\} \rangle \neq 0 \quad \Rightarrow \quad Q|0\rangle \neq 0$$

The only field which can have a susy preserving vev is $\phi_0(x)$. 
By applying this argument to the supercurrent multiplet $\langle \Theta_{\rho\mu} \rangle \neq 0$ implies that supersymmetry is broken. In a Lorentz and translational invariant vacuum $\langle \Theta_{\rho\mu} \rangle = \eta_{\rho\mu} \langle V \rangle \geq 0$ ($V$ scalar potential) ($V_0 > 0$ broken susy).

A supersymmetry preserving vacuum has $V_0 = 0$.

This argument is not valid in curved space as we know that $AdS$ (Anti de Sitter) is (can be) a supersymmetry preserving vacuum.
Rigid vacua in curved geometries have recently been studied (starting with Festuccia, Seiberg; Dumitrescu, Festuccia, Seiberg)

For vacua preserving maximal (4 charges) supersymmetry in curved 4d space an efficient method to find them is to look for the solution of the curved superspace identity

\[ \overline{D}^{\dot{\alpha}} E_{\alpha \dot{\alpha}} = D_\alpha \mathcal{R} \]

and ask for vacua for which the higher \( \theta \) components of the geometric superfields \( \mathcal{R}, E_{\alpha \dot{\alpha}}, W_{\alpha \beta \gamma} \) are vanishing, but the lowest components are not.
Other than Minkowski space one finds two vacua with four supersymmetries:

\[
\text{AdS}_4 = \frac{SO(3,2)}{SO(3,1)} \rightarrow \mathcal{R} = (\langle \tilde{u} \rangle \neq 0, 0, 0), \quad E_{\alpha\dot{\alpha}} = 0
\]

the Ricci tensor is \( R_{\mu\nu} = -3g_{\mu\nu}|u|^2 \) and it is obtained by \( B_\mu = 0 \)

\[
S^3 \times L \rightarrow \sigma_\mu^{\alpha\dot{\alpha}} E_{\alpha\dot{\alpha}} = (\langle A_\mu \rangle, 0, 0, ..., 0), \quad \langle A_\mu \rangle = (A_0, \vec{0})
\]

Both solutions satisfy \( \bar{D}^{\dot{\alpha}} E_{\alpha\dot{\alpha}} = D_\alpha \mathcal{R} = 0 \).

The Einstein curvature of these two spaces is retrieved by the vanishing of the upper components of \( \mathcal{R} \) and \( E_{\alpha\dot{\alpha}} \) respectively given by

\[
\frac{1}{6} R + A_\mu^2 + 2u\tilde{u} = \frac{1}{6} B_\mu^\mu \quad \text{and} \quad B_{\mu\nu}
\]

For \( S^3 \times L, \ u = 0 \) and the Ricci tensor \( R_{\mu\nu} \) is computed by the equation \( B_{\mu\nu} = 0 \). For the non trace part we have

\[
R_{00} = R_{0i} = 0, \quad R_{ij} = 2\delta_{ij} A_0^2
\]
These spaces also satisfy

\[ B_{\mu\nu} = 0 \quad \text{vanishing of upper components of } E_{\alpha\dot{\alpha}} \]

\[ uA_\mu = 0, \quad \partial_\mu A_\nu = 0, \quad \partial_\mu u = 0 \]

\[ C_{\alpha\beta\gamma\delta} = 0 \quad \text{(conformally flat spaces) from the vanishing of the } \theta \text{ (upper) component of the Weyl multiplet} \]

In a similar fashion one can prove that the other maximally symmetric space \( dS = \frac{SO(4,1)}{SO(3,1)} \) is not supersymmetric. Indeed for this space the upper components of the chiral scalar curvature multiplet \( \mathcal{R} = ...\theta^2 \left( -\frac{1}{6} R - 2u\bar{u} \right) \) is not vanishing.
If one combines the ansatz for the $AdS$ and $dS$ curvatures which give a cosmological constant

$$V(\mu, \lambda) = \frac{1}{3} \kappa^{-4} \left( \mu^2 - 9\lambda^2 \right)$$

one gets for the first and last component of the scalar curvature multiplet $\mathcal{R}$ respectively

$$\kappa \mathcal{R} \Big|_{\text{first}} \simeq \lambda, \quad \kappa \mathcal{R} \Big|_{\text{last}} \simeq -\frac{2}{9} \kappa^{-1} \mu^2$$

which show that the $\mu$ term breaks supersymmetry. Depending whether $\mu^2 \leq 9\lambda^2$, $\mu^2 > 9\lambda^2$ these configurations break susy in $AdS$, Minkowski and $dS$. Susy is unbroken whenever $\mu = 0$. 
No-scale supergravity (Cremmer, S.F., Kounnas, Nanopoulos)

In the conformal setting no-scale supergravity arises as a particular deformation $\Delta W$ of the conformal(cubic) superpotential

$$\sigma W_S = 3\Delta W = 3W - SW_S$$

$$W = \frac{1}{2} (\sigma + S)^3$$

Using the general formula

$$V(S, \bar{S}) = \frac{1}{3} \left( (\Phi^{-1}_M)^{ij} W_i \bar{W}_j - |3\Delta W|^2 \right)$$

the potential becomes

$$V(S, \bar{S}) = \frac{1}{3} W_S \bar{W}_S (1 - |\sigma|^2)$$

For $\sigma = 1$, $V = 0$ with $W_S \neq 0$ and the no-scale structure is obtained.
Among other things these results show that supergravity, in its full “off-shell formulation”, is the appropriate tool to study the interplay between particle physics and cosmology and the nature of supersymmetry breaking during the inflationary era as well after the exit of inflation at energies explored at LHC, looking for supersymmetry as the most plausible physics beyond the Standard Model.