Maxwell-Einstein-Scalar Theories

\[ \mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi) \partial_{\mu} \varphi^i \partial^{\mu} \varphi^j + \frac{1}{4} I_{\Lambda \Sigma}(\varphi) F_{\mu \nu}^\Lambda F_{\Sigma | \mu \nu} + \frac{1}{8 \sqrt{-G}} R_{\Lambda \Sigma}(\varphi) \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu}^\Lambda F_{\rho \sigma}^\Sigma. \]

**D=4 Maxwell-Einstein-scalar system** (with no potential)

[ may be the bosonic sector of D=4 (ungauged) sugra ]

Abelian 2-form field strengths

\[ H := (F^\Lambda, G_\Lambda)^T; \]

\[ *G_\Lambda|_{\mu \nu} := 2 \frac{\delta \mathcal{L}}{\delta F^\Lambda|_{\mu \nu}}. \]

static, spherically symmetric, asympt. flat, **extremal BH**

\[ ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} \left[ \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2} (d\theta^2 + \sin \theta d\psi^2) \right] \]

\[ \tau := -1/r \]

**Q :** \[ \int_{S^2_\infty} H = (p^\Lambda, q_\Lambda)^T; \]

**p^\Lambda :** \[ \frac{1}{4\pi} \int_{S^2_\infty} F^\Lambda, \ q_\Lambda = \frac{1}{4\pi} \int_{S^2_\infty} G_\Lambda. \]

**dyonic vector of e.m. fluxes**

**(BH charges)**
reduction $D=4 \rightarrow D=1$ : effective 1-dimensional (radial) Lagrangian

\[
S_{D=1} = \int [(U')^2 + g_{ij} \varphi^i \varphi^j + e^{2U} V_{BH}(\varphi(\tau), Q)] d\tau
\]

\[
V_{BH}(\varphi, Q) := -\frac{1}{2} Q^T \mathcal{M}(\varphi) Q
\]

BH effective potential

Ferrara, Gibbons, Kallosh

eoms

\[\left\{\begin{array}{l}
\frac{d^2 U}{d\tau^2} = e^{2U} V_{BH}; \\
\frac{d^2 \varphi^i}{d\tau^2} = g^{ij} e^{2U} \frac{\partial V_{BH}}{\partial \varphi^j}.
\end{array}\right\}
\]

Attractor Mechanism:

\[\partial_{\varphi} V_{BH} = 0 \iff \lim_{\tau \rightarrow -\infty} \varphi^a(\tau) = \varphi^a_H(Q)
\]

conformally flat geometry $AdS_2 \times S^2$ near the horizon

\[ds^2_{B-R} = \frac{r^2}{M^2_{B-R}} dt^2 - \frac{M^2_{B-R}}{r^2} (dr^2 + r^2 d\Omega)
\]

near the horizon, the scalar fields are stabilized purely in terms of charges

\[S = \frac{A_H}{4} = \pi V_{BH} \mid_{\partial_{\varphi} V_{BH}=0} = -\frac{\pi}{2} Q^T \mathcal{M}_H Q
\]

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

in $N=2$ ungauged sugra, hyper mults. decouple, and we thus disregard them: scalar fields belong to vector mults.
**Particular Class : Symmetric Scalar Manifolds**

A remarkable class of Einstein-Maxwell-scalar theories is endowed with scalar manifolds which are **symmetric cosets** $G/H$

[in presence of local SUSY : $N>2$ : general, $N=2$ : particular, $N=1$ : special cases ]

$H = $ isotropy group = *linearly* realized; scalar fields sit in an $H$-repr.  

$G = $ (global) **electric-magnetic duality** group, on-shell symmetry

**General Features in D=4**

The 2-form field strengths $(F,G)$ vector and the BH e.m. charges sit in a $G$-repr. $R$ which is **symplectic**:

\[
\exists! C_{[MN]} \equiv 1 \in R \times_a R; \quad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N C_{MN} = - \langle Q_2, Q_1 \rangle
\]

\[
C = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}
\]

**symplectic product**

in physics : **Gaillard-Zumino** embedding  
(generally maximal, but not symmetric)  
[ application of a Th. of Dynkin ]

\[
G \subset Sp(2n, \mathbb{R}) ; \quad R = 2n
\]
Let’s reconsider the starting **Maxwell-Einstein-scalar** Lagrangian density:

\[
\mathcal{L} = -\frac{R}{2} + \frac{1}{2}g_{ij}(\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi) F^\Lambda_{\mu\nu} F^\Sigma_{\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi) \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma}
\]

...and introduce the following real $2n \times 2n$ matrix:

\[
\mathcal{M} = \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}
\]

\[
\mathcal{M} = \mathcal{M}(R,I) = \mathcal{M}(\text{Re}(\mathcal{N}),\text{Im}(\mathcal{N})).
\]

\[
\mathcal{M}^T = \mathcal{M} \quad \mathcal{M}CM = C
\]

\[
\mathcal{M} = - (LL^T)^{-1} = -L^{-T}LL^{-1},
\]

$L$ = element of the $\text{Sp}(2n,R)$-bundle over the scalar manifold

( = *coset representative* for homogeneous spaces $G/H$)
By virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution** in *any* Maxwell-Einstein-scalar gravity theory in $D=4$:

\[ S(\varphi) \equiv CM(\varphi) \]

\[ S^2(\varphi) = CM(\varphi)CM(\varphi) = C^2 = -\mathbb{I}, \]

Ferrara, AM, Yeranyan; Borsten, Duff, Ferrara, AM

In turn, this allows to define an **anti-involution** on the dyonic charge vector $Q$, which has been named (scalar-dependent) **Freudenthal duality (F-duality)**

\[ \mathfrak{F}(Q) \equiv -S(\varphi)(Q). \]

\[ \mathfrak{F}^2 = -Id. \]

By recalling

\[ V_{BH}(\varphi, Q) \equiv -\frac{1}{2}Q^T M(\varphi)Q, \]

**F-duality** is the **symplectic gradient** of the **effective BH potential**:

\[ \mathfrak{F} : Q \rightarrow \mathfrak{F}(Q) \equiv C\frac{\partial V_{BH}}{\partial Q}. \]
All this enjoys a nice physical interpretation when evaluated at the BH horizon:

**Attractor Mechanism**

\[ \partial_\varphi V_{BH} = 0 \Leftrightarrow \lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(Q) \]

**Bekenstein-Hawking entropy**

\[ S = \frac{A_H}{4} = \pi V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2} Q^T M_H Q \]

By evaluating the matrix \( M \) at the horizon:

\[ \lim_{\tau \to -\infty} M(\varphi(\tau)) = M_H(Q) \]

one can define the **horizon F-duality** as:

\[ \lim_{\tau \to -\infty} \mathfrak{F}(Q) =: \mathfrak{F}_H(Q) = -\mathcal{C} M_H Q = \frac{1}{\pi} \mathcal{C} \frac{\partial S_{BH}}{\partial Q} =: \tilde{Q}, \]

\[ \mathfrak{F}^2_H(Q) = \mathfrak{F}_H(\tilde{Q}) = -Q \]

It is a **non-linear (scalar-independent) anti-involutive map** on \( Q \) (hom of degree 1)

Bek.-Haw. entropy is **invariant** under its own **non-linear symplectic gradient** (*i.e.*, F-duality):

\[ S(Q) = S(\mathfrak{F}_H(Q)) = S\left(\frac{1}{\pi} \mathcal{C} \frac{\partial S}{\partial Q}\right) = S(\tilde{Q}) \]

This can be extended to include **at least all quantum corrections** with **homogeneity 2 or 0** in the BH charges \( Q \)

Ferrara, AM, Yeranyan (and late Raymond Stora)
Lie groups of type $E_7$ : $(G,R)$

- the (ir)repr. $R$ is symplectic:
  \[ \exists ! C_{[MN]} \equiv 1 \in R \times_a R; \quad \langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N C_{MN} = - \langle Q_2, Q_1 \rangle; \]
  symplectic product

- the (ir)repr. admits a completely symmetric invariant rank-4 tensor
  \[ \exists K_{MNPQ} = K_{(MNPQ)} \equiv 1 \in (R \times R \times R \times R)_s \quad \text{(K-tensor)} \]
  $G$-invariant quartic polynomial
  \[ I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon |I_4|, \quad S_{BH} = \pi \sqrt{|I_4|} \]

- defining a triple map in $R$ as
  \[ T : R \times R \times R \to R \quad \langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q \]
  it holds
  \[ \langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q \]

this third property makes a group of type $E_7$ amenable to a description as automorphism group of a Freudenthal triple system (FTS)
Evidence: all electric-magnetic duality groups of $D=4$ ME(S)GT’s with symmetric scalar manifolds (and at least 8 supersymmetries) are of type $E_7$.

$(E_7, 912 –$ embedding tensor $N=8/N=2$ exc, $D=4$) satisfies the first two Brown’s axioms, but not the third one!

```
I_4(p, q) = (I_2(p, q))^2
```

```
S_{BH} = \pi \sqrt{|I_4(p, q)|} = \pi |I_2(p, q)|.
```
In $D=4$ sugras with the previous electric-magnetic duality group of type $E_7$, the $G$-invariant $K$-tensor determining the extremal BH Bekenstein-Hawking entropy

\[ S_{BH} = \pi \sqrt{|I_4|} \]

\[ I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon |I_4|, \]

can generally be expressed as adjoint-trace of the product of $G$-generators (dim $R = 2n$, and dim $Adj = d$)

\[ K_{MNPQ} = - \frac{n (2n + 1)}{6d} \left[ t^{\alpha}_{MN} t_{\alpha|PQ} - \frac{d}{n(2n + 1)} C_{M(P} C_{Q)N} \right] \]

The horizon $F$-duality can be expressed in terms of the $K$-tensor

\[ \mathcal{F}_H(Q)_M = \tilde{Q}_M = \frac{\partial \sqrt{|I_4(Q)|}}{\partial Q^M} = \epsilon \frac{2}{\sqrt{|I_4(Q)|}} K_{MNPQ} Q^N Q^P Q^Q \]

Borsten, Dahanayake, Duff, Rubens

and the invariance of the BH entropy under horizon $F$-duality can be recast as

\[ I_4(Q) = I_4(\mathcal{C} \tilde{Q}) = I_4 \left( \mathcal{C} \frac{\partial \sqrt{|I_4(Q)|}}{\partial Q} \right) \]
Hints for the Future…

❖ **F-Duality** applied to **homogeneous non-symmetric** PSK manifolds
  [deWit, Van Proeyen; Alekseevsky, Cortes, …]
  and investigation of the role to **T-Algebras** [Vinberg, Cecotti]

❖ extension to **“small”** U-orbits: how to define **“small”** F-duality?
  [for intrinsically quantum black holes]

❖ extension to **multi-centered** (extremal) BH solutions:
  Yeranyan; Ferrara, AM, Shcherbakov, Yeranyan

❖ into the **quantum regime** of gravity [e.m. duality over **discrete** fields]:
  **F-duality** for **integer, quantized charges**?
  Borsten, Duff *et al.*

❖ towards a **classification of groups of type E_7**
  (non-semisimple? reducible (reprs.)?)
  [ non-reductive FTS’s? Relation to **intermediate** composition algebras (sextonions, …)? ]
Thank You!