Description of gravity in the model with independent nonsymmetric connection

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General Relativity - current description of gravity in modern physics

Consider modified gravity:

- presence of torsion: \( S_{\mu \nu}^\xi = \Gamma_{\mu \nu}^\xi - \Gamma_{\nu \mu}^\xi \neq 0 \);
- metric and connection are independent variables (Palatini Formalism):
  \[ \Gamma_{\rho \nu}^\alpha \neq \frac{1}{2} g^{\alpha \sigma} (\partial_\rho g_{\sigma \nu} + \partial_\nu g_{\rho \sigma} - \partial_\sigma g_{\nu \rho}) \].

the geometry of the space-time \( \rightarrow \) electromagnetic potential
Gravity field equations without matter

The Einstein-Hilbert action:

\[ S_1 = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \]  

(1)

where \( \kappa \) - Einstein’s constant, \( R \) - the scalar curvature, \( g \equiv \det g_{\mu\nu} \).

The minimal action principle gives:

\[ \begin{align*}
R^\nu{}^\mu + R^{\mu\nu} - R g^{\mu\nu} &= 0; \\
D_\rho g^\sigma{}^\nu &= \frac{1}{3} g^\sigma{}^\nu S^\alpha_{\rho\alpha} + \frac{1}{3} g^\sigma{}^\xi S^\alpha_{\xi\alpha} \delta^\nu_\rho + g^\xi{}^\sigma S^\nu_{\rho\xi}.
\end{align*} \]  

(2)
Gravity field equations without matter

Consider equations

\[ D_\rho g^{\sigma \nu} = \frac{1}{3} g^{\sigma \nu} S^\alpha_{\rho \alpha} + \frac{1}{3} g^{\sigma \xi} S^{\alpha}_{\xi \alpha} \delta^\nu_{\rho} + g^{\xi \sigma} S^\nu_{\rho \xi}. \]  

(3)

It is solved by

\[ \Gamma^\rho_{\mu \nu} = \hat{\Gamma}^\rho_{\mu \nu} + f_\mu \delta^\rho_{\nu}, \]  

(4)

where \( \hat{\Gamma}^\rho_{\mu \nu} = \frac{1}{2} g^{\alpha \sigma} (\partial_\rho g_{\sigma \nu} + \partial_\nu g_{\rho \sigma} - \partial_\sigma g_{\nu \rho}) \) - Christoffel symbols, \( f_\mu \) - arbitrary vector.

The invariance of the action under transformation:

\[ \Gamma^\rho_{\mu \nu} \rightarrow \Gamma^\rho_{\mu \nu} + \theta_\mu \delta^\rho_{\nu}. \]  

(5)
Addition of matter

Point-like particle

\[ S_2 = -m \int d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}. \] (6)

Interaction with the field of connection

\[ S_3 = -q \int d\tau \dot{x}^\mu \Gamma^\nu_{\mu\nu} \] (7)

Remark. Now the action \( S_1 + S_2 + S_3 \) is invariant under a narrow group of transformations:

\[ \Gamma^\rho_{\mu\nu} \rightarrow \Gamma^\rho_{\mu\nu} + \delta^\rho_\nu \partial_\mu \theta. \] (8)
Addition of kinetic term

Problem: incompatible equation of motion

\[ \dot{x} = 0. \]

Consequently, we need to add the kinetic term:

\[ S_4 = -\frac{1}{16\pi} \int d^4 x \sqrt{-g} \Omega_{\alpha\beta} \Omega_{\delta\gamma} g^{\alpha\delta} g^{\beta\gamma}, \]  

(9)

where \( \Omega_{\lambda\rho} \equiv R^\mu_{\mu\lambda\rho} = (\partial_\lambda \Gamma_\rho^\xi - \partial_\rho \Gamma_\lambda^\xi). \n
The total action

\[ S = S_1 + S_2 + S_3 + S_4 = -\frac{1}{2\kappa} \int d^4 x \sqrt{-g} R - m \int d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}} - q \int d\tau \dot{x}^\mu \Gamma_{\mu\nu}^\nu - \frac{1}{16\pi} \int d^4 x \sqrt{-g} \Omega_{\alpha\beta} \Omega_{\delta\gamma} g^{\alpha\delta} g^{\beta\gamma} \]  

(10)
Variation

Varying with respect to the metric $\delta g$:

$$\hat{R}^{\mu\nu} - \frac{1}{2} \hat{R} g^{\mu\nu} = \kappa T^{\mu\nu},$$

(11)

where $\hat{R}^{\mu\nu}$ and $\hat{R}$ - Riemannian geometry objects,

$$T^{\mu\nu} = \rho m u^{\mu} u^{\nu} - \frac{1}{4\pi} (\Omega^{\mu\alpha} \Omega^{\nu}_{\alpha} - \frac{1}{4} g^{\mu\nu} \Omega_{\alpha\beta} \Omega^{\alpha\beta})$$

(12)

stress-energy tensor,

$$u^{\mu} = \dot{x}^{\mu} \frac{1}{\sqrt{\dot{x}^{\alpha} \dot{x}^{\beta} g_{\alpha\beta}}}$$ - four-velocity, $\rho_m = m \int ds \delta(x - x(s)) \frac{1}{\sqrt{-g(x(s))}}$ - mass density.
Variation

Minimizing geodesic $x^\mu(\tau)$

$$m u^\mu \hat{D}_\mu u^\alpha = -q u_\xi \nabla^\xi \xi^\alpha.$$  \hspace{1cm} (13)

Varying with respect to the connection $\Gamma^\mu_{\nu\rho}$

\[
\begin{align*}
\Gamma^\rho_{\mu\nu} - \frac{1}{4} \nabla_\rho \delta^\rho_{\mu\nu} &= \hat{\Gamma}^\rho_{\mu\nu} - \frac{1}{4} \hat{\nabla}_\rho \delta^\rho_{\mu\nu}, \\
\hat{D}_\mu \nabla^\nu \mu &= 4\pi j^\mu,
\end{align*}
\]  \hspace{1cm} (14)

where $j^\mu = q \int ds u^\mu \delta(x - x(s)) \frac{1}{\sqrt{-g(x(s))}}$. 
Gravity field equations with matter

Finally we arrived at

\[
\begin{cases}
\hat{R}^{\mu\nu} - \frac{1}{2} \hat{R}^g_{\mu\nu} = \kappa T^{\mu\nu}; \\
mu^{\mu} \hat{D}_\mu u^\alpha = -qu_\xi \Omega^{\xi\alpha}; \\
\Gamma^{\rho}_{\mu\nu} - \frac{1}{4} \Gamma^{\alpha}_{\mu\alpha} \delta^\rho_\nu = \hat{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{4} \hat{\Gamma}^{\alpha}_{\mu\alpha} \delta^\rho_\nu; \\
\hat{D}_\mu \Omega^{\nu\mu} = 4\pi j^\mu,
\end{cases}
\]  

(15)

where \( \Omega_{\lambda\rho} \equiv R^\mu_{\mu\lambda\rho} = (\partial_\lambda \Gamma^{\xi}_{\rho\xi} - \partial_\rho \Gamma^{\xi}_{\lambda\xi}). \)

If \( q \) - charge of the particle, and \( \Gamma^{\xi}_{\rho\xi} \equiv A_\rho \) - the potential of the electromagnetic field, thus we constructed electrodynamics in a gravitational field.
In the model of gravity with independent nonsymmetric connection with presence of matter - point-like particle, we get a unified theory of gravity and electromagnetism.