Resurgence and Hydrodynamics in Gauss-Bonnet Holography

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Hydrodynamics in 3+1 Dimensions

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\[ \nabla_\mu T^{\mu\nu} = 0 \] (1)

where \( T^{\mu\nu} = T^{\mu\nu}(\epsilon, P, u^\mu) \) with \( \epsilon \) the energy density, \( P \) the Pressure, and \( u^\mu \) the fluid velocity.
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\[ T^{\mu \nu} = T^{\mu \nu}_{\text{ideal}} + c_1 \partial^\mu u^\nu + c_2 \partial^\nu u^\mu + c_3 \eta^{\mu \nu} \partial_\alpha u^\alpha + c_4 u^\mu u^\nu \partial_\alpha u^\alpha + \ldots \]  \hspace{1cm} (3)
Hydrodynamics in 3+1 Dimensions

When $\partial u$ is small we can order the series in derivatives of $u^\mu$

$$T^{\mu\nu} = T_{\text{ideal}}^{\mu\nu} + O(\sim \partial^\mu u^\nu) + O(\sim (\partial^\mu u^\nu)^2) + ...$$  \hspace{1cm} (4)

- This series is known as the Gradient Expansion.
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- This series is known as the Gradient Expansion.
- The co-efficients $c_i$ are known as transport co-efficients and uniquely specify our theory.
Bjorken Flow

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Figure: Head-on and Side profiles for a Lead-Lead collision. The overlapping region results in an energy density that evolves longitudinally according to hydrodynamics.
Bjorken Flow

- This energy density $T_{00} = \epsilon$ is a function of only the proper time, and the form is known to all orders:

$$\epsilon(\tau) = \tau^{-4/3}(\epsilon_0 + \epsilon_1\tau^{-2/3} + \epsilon_2\tau^{-4/3} + ...) \quad (5)$$
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- To gain some understanding of this evolving system analytically, we need a way to calculate the energy coefficients for a QCD-like theory at Strong Coupling.

- $N = 4$ SYM (a QCD-like theory) can be re-written at infinite coupling as a gravitational theory.
The Fluid-Gravity correspondence

We can perform classical gravity calculations to find strongly coupled QFT results.

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The Fluid-Gravity Correspondence

The geometry that is dual to Bjorken Flow Hydrodynamics in $N = 4$ SYM at infinite coupling is given by

$$ds^2 = -r^2 A(r, \tau) d\tau^2 + 2d\tau dr + (r\tau + 1)^2 e^{B(r, \tau)} dy^2 + r^2 e^{C(r, \tau)} dx^2_\perp$$

(6)

where $r$ is the radial distance towards the Black Hole, and $\tau$ is the proper time.
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where $r$ is the radial distance towards the Black Hole, and $\tau$ is the proper time. $A$, $B$ and $C$ are defined by:

$$A_{\text{pert}}(\tau, r) = \sum_{i=0}^{\infty} \tau^{-\frac{2}{3}i} A_i(s), \quad A_0 = 1 - s^4$$

$$B_{\text{pert}}(\tau, r) = \sum_{i=0}^{\infty} \tau^{-\frac{2}{3}i} B_i(s), \quad B_0 = 0$$

$$C_{\text{pert}}(\tau, r) = \sum_{i=0}^{\infty} \tau^{-\frac{2}{3}i} C_i(s), \quad C_0 = 0.$$  

with $s = r^{-1}\tau^{-1/3}$. (Kinoshita, Mukohyama & Nakamura [arXiv:0807.3797v2])
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(7)

where $r$ is the radial distance towards the Black Hole, and $\tau$ is the proper time. This looks a little like a space with a blackhole a horizon sinking into the radial direction.

$r = \infty$

$r = \tau^{-\frac{1}{3}}$

**Figure:** Schematic cartoon of the Geometry.
We can calculate \( \epsilon_i \) directly from the solution evaluated at the boundary \((s \rightarrow 0)\).

\[
\epsilon(\tau) = \tau^{-4/3}(\epsilon_0 + \epsilon_1 \tau^{-2/3} + \epsilon_2 \tau^{-4/3} + \ldots)
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But after some finite order, the co-efficients start to contribute more and more!
Resurgence

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But after some finite order, the co-efficients start to contribute more and more!

**Figure:** Energy density co-efficients $\epsilon_n^{1/n}$ as a function of order $n$. 

Note that $(n!)^{1/n} \sim n$ for large $n$. [arXiv:1302.0697v2]
Resurgence

Using the definition of the Gamma Function:

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(\tau^{-\frac{2}{3}})^n = \int_0^\infty du \left( \frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \frac{u^n}{n!}
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as an integral of a series with finite radius of convergence

\[
\epsilon(\tau) = \int_0^\infty du \left( \frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \left( \frac{\epsilon_0}{2!} u^2 + \frac{\epsilon_1}{3!} u^3 + \frac{\epsilon_2}{4!} u^4 + \ldots \right)
\]
\[
\epsilon_B(u) \text{ known as the Borel sum}
\]  \hspace{1cm} (11)
Resurgence

This convergent series is called the Borel Sum

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If we study \( \epsilon_B(u) \) in the complex plane we can see that the series diverges at discrete points.

**Figure:** Poles in \( \epsilon_B(u) \) which lead to the non-convergence of \( \epsilon(\tau) \)
Resurgence

But in principle we could have defined $\epsilon_B(u)$ through an integral along the any line in the complex plane:

$$\epsilon(\tau) = \int_0^\infty du \left( \frac{e^{-u\tau^2/3}}{\tau^{2/3}} \right) \epsilon_B(u)$$

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Figure: Poles in $\epsilon_B(u)$ which lead to the non-convergence of $\epsilon(\tau)$
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$$\epsilon(\tau) = \int_0^\infty du \left( \frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \epsilon_B(u) \overset{!}{=} \int_{C} du \left( \frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \epsilon_B(u) \quad (14)$$

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This motivates a contribution to our perturbative ansatz of the form:

$$\epsilon(\tau) \sim \int_{C'} du \left( \frac{e^{-u\tau^{2/3}}}{\tau^{2/3}} \right) \frac{1}{u - \omega} \sim \tau^{-2/3} e^{-\omega\tau^{2/3}}. \quad (16)$$

(The result of evaluating the contribution from the first pole $\omega$ in integral above).
Resurgence

We study a gravity solution with given by:

\[ A(\tau, r) = A_{\text{pert}}(\tau, r) + \tau^{-2/3} e^{-\omega \tau^{2/3}} \psi_A(s) \]
\[ B(\tau, r) = B_{\text{pert}}(\tau, r) + \tau^{-2/3} e^{-\omega \tau^{2/3}} \psi_B(s) \]
\[ C(\tau, r) = C_{\text{pert}}(\tau, r) + \tau^{-2/3} e^{-\omega \tau^{2/3}} \psi_C(s) \]

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with \( s = r^{-1} \tau^{-1/3} \). This implies \( \psi_A(s) = \psi_C(s) = 0 \) and an eigenvalue problem with solutions for a discrete set of \( \omega_i \):

\[ (s(1 - s^4) \partial_s^2 + (-3 + s^4) \partial_s) \psi_B(s) = i\omega_i (3 - 2s) \psi_B(s) \]

(17)

These \( \omega_i \) correspond exactly to poles in the Borel Sum!
Resurgence: non-Perturbative Modes

Figure: Poles in $\epsilon_B(u)$ (Gray) plot with $\omega_i$ (Red)
Strong (but finite) Coupling

In this analysis we found the perturbative series for infinitely coupled $\mathcal{N} = 4$ SYM with classical gravity:

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Strong (but finite) Coupling

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$$S = \int d^5 x \sqrt{-g} \left( R + 12 \right)$$  \hspace{1cm} (18)

We want to find them for finitely coupled $\mathcal{N} = 4$ SYM with higher derivative (Gauss-Bonnet) gravity:

$$S = \int d^5 x \sqrt{-g} \left( R + 12 + \frac{\lambda}{2} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right)$$  \hspace{1cm} (19)
Figure: Poles of the $\epsilon_B(u)$ for $\lambda = 0$
**Resurgence**

**Figure:** Poles of the $\epsilon_B(u)$ for $\lambda = -0.1$
Resurgence

Figure: Poles of the $\epsilon_B(u)$ for $\lambda = -0.2$
Figure: Poles of the $\epsilon_B(u)$ for $\lambda = -0.5$
Resurgence

**Figure:** Poles of the $\epsilon_B(u)$ for $\lambda = -1$
Conclusion

The hydrodynamic expansion for infinitely coupled Bjorken Flow diverges but can be used to gain non-perturbative information.

We've found that the same non-perturbative modes exist at finite coupling and can be described in the same way.

Since we can predict the locations of these non-perturbative modes, our next step will be to write down the full non-perturbative solution at finite coupling.
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