Are all supergravities the square of a gauge theory? OR
From QCD to gravitational waves
Erice June 2018

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based on

A. Anastasiou, L. Borsten, M. J. Duff, M. Hughes,
A. Marrani, S. Nagy and M. Zoccali]
Basic idea

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory (non-abelian generalisation of Maxwell). Gluons, W, Z and photons have spin 1.

- Gravitational force described by Einstein’s general relativity. Gravitons have spin 2.

- But maybe \((\text{spin } 2) = (\text{spin } 1)^2\). If so:
  1) Do global gravitational symmetries follow from flat-space Yang-Mills symmetries?
  2) Do local gravitational symmetries and Bianchi identities follow from flat-space Yang-Mills symmetries?
  3) What about twin supergravities with same bosonic lagrangian but different fermions?
  4) Are all supergravities Yang-Mills squared?
A recurring theme in attempts to understand the quantum theory of gravity and appears in several different forms:

Closed states from products of open states and KLT relations in string theory [Kawai, Lewellen, Tye:1985, Siegel:1988],

On-shell $D = 10$ Type IIA and IIB supergravity representations from on-shell $D = 10$ super Yang-Mills representations [Green, Schwarz and Witten:1987],

Vector theory of gravity [Svidzinsky 2009]


Local and global symmetries from Yang-Mills

- **LOCAL SYMMETRIES**: general covariance, local lorentz invariance, local supersymmetry, local p-form gauge invariance

- **GLOBAL SYMMETRIES**
  eg $G = E_7$ in $D = 4, \mathcal{N} = 8$ supergravity
  [A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]
  eg Spin-four $N = 7$ W-Supergravity, [arXiv:1805.10022] [S. Ferrara and D, Lust]

- **TWIN SUPERGRAVITIES FROM (YANG-MILLS)$^2$**
  [A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, A.Marrani, S. Nagy and M. Zoccali] [arXiv:1610.07192]
Local symmetries

- LOCAL SYMMETRIES
Most of the literature is concerned with products of momentum-space scattering amplitudes, but we are interested in products of off-shell left and right Yang-Mills field in coordinate-space

$$A_\mu(x)(L) \otimes A_\nu(x)(R)$$

so it is hard to find a conventional field theory definition of the product.

Where do the gauge indices go?

Does it obey the Leibnitz rule

$$\partial_\mu (f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

If not, why not?
Here we present a $G_L \times G_R$ product rule:

$$[A_{\mu}^i(L) \ast \Phi_{ii'} \ast A_{\nu}^{i'}(R)](x)$$

where $\Phi_{ii'}$ is the “spectator” bi-adjoint scalar field introduced by Hodges [Hodges:2011] and Cachazo et al [Cachazo:2013] and where $\ast$ denotes a convolution

$$[f \ast g](x) = \int d^4 y f(y)g(x-y).$$

Note $f \ast g = g \ast f, (f \ast g) \ast h = f \ast (g \ast h)$, and, importantly obeys

$$\partial_\mu (f \ast g) = (\partial_\mu f) \ast g = f \ast (\partial_\mu g)$$

and not Leibnitz

$$\partial_\mu (f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$
For concreteness we focus on

- $\mathcal{N} = 1$ supergravity in $D = 4$, obtained by tensoring the $(4 + 4)$ off-shell $\mathcal{N}_L = 1$ Yang-Mills multiplet $(A_\mu(L), \chi(L), D(L))$ with the $(3 + 0)$ off-shell $\mathcal{N}_R = 0$ multiplet $A_\mu(R)$.

- Interestingly enough, this yields the new-minimal formulation of $\mathcal{N} = 1$ supergravity [Sohnius, West: 1981] with its 12+12 multiplet $(h_{\mu\nu}, \psi_\mu, V_\mu, B_{\mu\nu})$

- The dictionary is,

$$Z_{\mu\nu} \equiv h_{\mu\nu} + B_{\mu\nu} = A_{\mu}^{\ i}(L) \star \Phi_{i i'} \star A_{\nu}^{\ i'}(R)$$

$$\psi_\nu = \chi^{\ i}(L) \star \Phi_{i i'} \star A_{\nu}^{\ i'}(R)$$

$$V_\nu = D^{\ i}(L) \star \Phi_{i i'} \star A_{\nu}^{\ i'}(R),$$
Yang-Mills symmetries

The left supermultiplet is described by a vector superfield $V^i(L)$ transforming as

$$\delta V^i(L) = \Lambda^i(L) + \bar{\Lambda}^i(L) + f^i_{jk} V^j(L) \theta^k(L)$$
$$+ \delta_{(a,\lambda,\epsilon)} V^i(L).$$

Similarly the right Yang-Mills field $A_{\nu}^i'(R)$ transforms as

$$\delta A_{\nu}^i'(R) = \partial_{\nu} \sigma^i'(R) + f^{i'}_{j'k'} A_{\nu}^{j'}(R) \theta^{k'}(R)$$
$$+ \delta_{(a,\lambda)} A_{\nu}^i'(R).$$

and the spectator as

$$\delta \Phi_{ii'} = -f^j_{ik} \Phi_{ji'} \theta^k(L) - f^{i'}_{j'k'} \Phi_{ij'} \theta^{k'}(R) + \delta_a \Phi_{ii'}.$$
Gravitational symmetries

\[
\begin{align*}
\delta Z_{\mu\nu} &= \partial_\nu \alpha_\mu (L) + \partial_\mu \alpha_\nu (R), \\
\delta \psi_\mu &= \partial_\mu \eta, \\
\delta V_\mu &= \partial_\mu \Lambda,
\end{align*}
\]

where

\[
\begin{align*}
\alpha_\mu (L) &= A^{i'}_\mu (L) \ast \Phi_{ii'} \ast \sigma^{i''} (R), \\
\alpha_\nu (R) &= \sigma^i (L) \ast \Phi_{ii'} \ast A^{i'}_\nu (R), \\
\eta &= \chi^i (L) \ast \Phi_{ii'} \ast \sigma^{i''} (R), \\
\Lambda &= D^i (L) \ast \Phi_{ii'} \ast \sigma^{i'} (R),
\end{align*}
\]

illustrating how the local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills.
New minimal supergravity also admits an off-shell Lorentz multiplet \((\Omega_{\mu ab}^-, \psi_{ab}, -2V_{ab}^+)\) transforming as

\[
\delta V^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a,\lambda,\varepsilon)} V^{ab}.
\]  

(1)

This may also be derived by tensoring the left Yang-Mills superfield \(V^i(L)\) with the right Yang-Mills field strength \(F^{abi'}(R)\) using the dictionary

\[
V^{ab} = V^i(L) \star \Phi_{ii'} \star F^{abi'}(R),
\]

\[
\Lambda^{ab} = \Lambda^i(L) \star \Phi_{ii'} \star F^{abi'}(R).
\]
Bianchi identities

- The corresponding Riemann and Torsion tensors are given by

\[ R^+_{\mu\nu\rho\sigma} = -F_{\mu\nu}^i(L) \Phi_{ii'} \ast F_{\rho\sigma}^{i'}(R) = R^-_{\rho\sigma\mu\nu}. \]

\[ T^+_{\mu\nu\rho} = -F_{[\mu\nu}^i(L) \Phi_{ii'} \ast A_{\rho]}^{i'}(R) = -A_{[\rho}^i(L) \Phi_{ii'} \ast F_{\mu\nu]}^{i'}(R) = -T^-_{\mu\nu\rho}. \]

- One can show that (to linearised order) both the gravitational Bianchi identities

\[ DT = R \wedge e \quad (2) \]

\[ DR = 0 \quad (3) \]

follow from those of Yang-Mills

\[ D_{[\mu}(L)F_{\nu]\rho]^{i}(L) = 0 = D_{[\mu}(R)F_{\nu]\rho]^{i'}(R) \]
To do

Clearly two important improvements would be

- To generalise our results to the full non-linear transformation rules (difficult!)
- To address the issue of dynamics as well as symmetries. For this we need BRST
BRST: The gauge theories

Consider two copies of linearised Yang-Mills theory with BRST
Lagrangian

\[ \mathcal{L}_A = \text{tr} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial^\mu A_\mu)^2 - \bar{c} \Box c \right) \]

The gauge group indices will be suppressed in the following. While not
necessary, we work within the one-parameter, \( \xi \), family of general linear
Lorentz covariant gauges. Variation of the above leads to the equations
of motion

\[ \Box A_\mu - \frac{\xi + 1}{\xi} \partial_\mu \partial A = j_\mu(A), \quad \Box c^\alpha = j^\alpha(c) \]

where, for simplicity, we have added the sources directly into the
equations and introduced the \( OSp(2) \) ghost-antighost doublet \( c^\alpha \), where
\( c^1 = c, c^2 = \bar{c} \). The theory is invariant under the BRST transformations

\[ QA_\mu = \partial_\mu c, \quad Qc = 0, \quad Q\bar{c} = \frac{1}{\xi} \partial^\mu A_\mu. \]
First, we give the BRST Lagrangians for the graviton $h_{\mu\nu}$ and dilaton $\varphi$ and the two-form, $B_{\mu\nu}$. In Einstein frame, it reads

$$\mathcal{L}_{h,\varphi,B} = -\frac{1}{4} h_{\mu\nu} E_{\mu\nu} + \frac{1}{2\xi(h)} \left( \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_{\mu} h \right)^2$$

$$- \frac{1}{4} (\partial \varphi)^2 - \bar{c}^\mu \Box c_\mu$$

$$- \frac{1}{24} H^\mu_{\nu\rho} H_{\mu\nu\rho}$$

$$+ \frac{1}{2\xi(B)} \left( \partial_\mu B^{\mu\nu} + \partial^\nu \eta \right)^2 - \bar{d}_\nu \Box d^\nu$$

$$+ \frac{\xi(d) - m(d)}{\xi(d)} \bar{d}_\mu \partial_\mu \partial^\nu d_\nu + m(d) \bar{d} \Box d$$

where $E_{\mu\nu}$ is the linearised Einstein tensor and we average over de Donder gauge fixings, controlled by $\xi(h)$. Note the first-level ghosts for the two-form gauge invariance, $(d_\mu, \bar{d}_\mu)$ at ghost number $(1, -1)$, as well as the second-level bosonic ghosts, $(d, \bar{d}, \eta)$ at ghost number $(2, -2, 0)$ respectively.
The equations of motion are given by

\[ \Box h_{\mu \nu} - 2 \xi'_(h) \partial^\rho \partial_{(\mu} h_{\nu)} \rho + \xi'_(h) \partial \mu \partial \nu h = j_{\mu \nu}(h) \]

\[ \Box B_{\mu \nu} - \xi'_(B) \partial^\rho \partial_{[\mu} B_{\nu] \rho} = j_{\mu \nu}(B) \Box \varphi = j(\varphi) \]

with \( \xi'_(h) \equiv \frac{\xi(h) + 2}{\xi(h)} \) and \( \xi'_(B) \equiv -2 \frac{\xi(B) + 2}{\xi(B)} \), complemented by those for the ghosts, which we omit for brevity.
The Lagrangian is invariant under the BRST transformations

\[ Qh_{\mu\nu} = 2 \partial_{(\mu} c_{\nu)} \quad Qc_\mu = 0 \]

\[ Q\bar{c}_\mu = \left( \partial'^{\nu} h_{\mu\nu} - \frac{1}{2} \partial_\mu h \right) \frac{\xi(h)}{\xi(h)} \]

\[ QB_{\mu\nu} = 2 \partial_{[\mu} d_{\nu]} \quad Qd_\mu = \partial_\mu d \]

\[ Qd_\mu = \left( \partial'^{\nu} B_{\mu\nu} + \partial_\mu \eta \right) \frac{\xi(B)}{\xi(B)} \]

\[ Q\bar{d} = \frac{1}{\xi(d)} \partial_\mu \bar{d}_\mu \quad Q\eta = \frac{m(d)}{\xi(d)} \partial_\mu d_\mu \]

where our choice of Einstein frame implies the BRST invariance of the dilaton, \( Q\varphi = 0 \).
• The graviton

\[ h_{\mu \nu} = A(\mu \circ \tilde{A}_\nu) + a_1 \frac{\partial \mu \partial \nu}{\Box} A \circ \tilde{A} + a_2 \frac{\partial \mu \partial \nu}{\Box} c^\alpha \circ \tilde{c}_\alpha \]

\[ + \frac{a_3}{\Box} \left( \partial A \circ \partial (\mu \tilde{A}_\nu) + \partial (\mu A_\nu) \circ \partial \tilde{A} \right) \]

\[ + \eta_{\mu \nu} \left( b_1 A \circ \tilde{A} + b_2 c^\alpha \circ \tilde{c}_\alpha + \frac{b_3}{\Box} \partial A \circ \partial \tilde{A} \right) \]

• The Kalb-Ramond two-form

\[ B_{\mu \nu} = A[\mu \circ \tilde{A}_\nu] + \frac{2\xi - 1}{\Box} \left( \partial A \circ \partial[\mu \tilde{A}_\nu] - \partial[\mu A_\nu] \circ \partial \tilde{A} \right) \]

• The dilaton

\[ \varphi = A^\rho \circ \tilde{A}_\rho + \frac{1}{\xi} c^\alpha \circ \tilde{c}_\alpha + \left( \frac{1}{\xi^2} - 1 \right) \frac{1}{\Box} \partial A \circ \partial \tilde{A} \]

where

\[ a_1 = \frac{1}{1-\xi}, \quad a_2 = \frac{1+\xi}{2(\xi-1)}, \quad a_3 = -1/2, \]

\[ b_1 = \frac{\xi}{(2-D)(\xi-1)}, \quad b_2 = b_1/\xi, \quad b_3 = \left( \frac{1}{\xi^2} - 1 \right) b_1. \]
Division algebras

- DIVISION ALGEBRAS
Mathematicians deal with four kinds of numbers, called Division Algebras.

The Octonions occupy a privileged position:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Imaginary parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reals</td>
<td>$\mathbb{R}$</td>
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</tr>
<tr>
<td>Complexes</td>
<td>$\mathbb{C}$</td>
<td>1</td>
</tr>
<tr>
<td>Quaternions</td>
<td>$\mathbb{H}$</td>
<td>3</td>
</tr>
<tr>
<td>Octonions</td>
<td>$\mathbb{O}$</td>
<td>7</td>
</tr>
</tbody>
</table>

Table: Division Algebras
They provide an intuitive basis for the exceptional Lie algebras:

<table>
<thead>
<tr>
<th>Classical algebras</th>
<th>Rank</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$n$</td>
<td>$(n + 1)^2 - 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$n$</td>
<td>$n(2n + 1)$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$n$</td>
<td>$n(2n + 1)$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$n$</td>
<td>$n(2n - 1)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exceptional algebras</th>
<th>Rank</th>
<th>Dimension</th>
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<tr>
<td>$E_6$</td>
<td>6</td>
<td>78</td>
</tr>
<tr>
<td>$E_7$</td>
<td>7</td>
<td>133</td>
</tr>
<tr>
<td>$E_8$</td>
<td>8</td>
<td>248</td>
</tr>
<tr>
<td>$F_4$</td>
<td>4</td>
<td>52</td>
</tr>
<tr>
<td>$G_2$</td>
<td>2</td>
<td>14</td>
</tr>
</tbody>
</table>

**Table:** Classical and exceptional Lie algebras
Freudenthal-Rozenfeld-Tits magic square

<table>
<thead>
<tr>
<th>$A_L/A_R$</th>
<th>$R$</th>
<th>$C$</th>
<th>$H$</th>
<th>$O$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$C$</td>
<td>$A_2$</td>
<td>$A_2 + A_2$</td>
<td>$A_5$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>$H$</td>
<td>$C_3$</td>
<td>$A_5$</td>
<td>$D_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$O$</td>
<td>$F_4$</td>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>

Table: Magic square

Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.
Octonions

- Octonion $x$ given by
  \[ x = x^0 e_0 + x^10 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5 + x^6 e_6 + x^7 e_7, \]
  
  One real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$ imaginary elements, where $e_0^* = e_0$ and $e_i^* = -e_i$.

- The imaginary octonionic multiplication rules are,
  \[ e_i e_j = -\delta_{ij} + C_{ijk} e_k \quad [e_i, e_j, e_k] = 2Q_{ijkl} e_l \]

  $C_{mnp}$ are the octonionic structure constants, the set of oriented lines of the Fano plane.

  \[ \{ijk\} = \{124, 235, 346, 457, 561, 672, 713\}. \]

- $Q_{ijkl}$ are the associators the set of oriented quadrangles in the Fano plane:
  \[ ijkl = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\}, \]
  
  \[ Q_{ijkl} = -\frac{1}{3!} C_{mnp} \epsilon_{mnpijkl}. \]
The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point.
Division algebras

- Division: $ax+b=0$ has a unique solution
- Associative: $a(bc)=(ab)c$
- Commutative: $ab=ba$

<table>
<thead>
<tr>
<th>A</th>
<th>construction</th>
<th>division?</th>
<th>associative?</th>
<th>commutative?</th>
<th>ordered?</th>
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</thead>
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<td>$R$</td>
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<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>C</td>
<td>$R + e_1 R$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>H</td>
<td>$C + e_2 C$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>O</td>
<td>$H + e_3 H$</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>S</td>
<td>$O + e_4 O$</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

As we shall see, the mathematical cut-off of division algebras at octonions corresponds to a physical cutoff at 16 component spinors in super Yang-Mills.
Cayley-Dickson

- Octonion
  \[ O = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^3 e_3 + O^4 e_4 + O^5 e_5 + O^6 e_6 + O^7 e_7 = H(0) + e_3 H(1) \]

- Quaternion
  \[ H(0) = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^4 e_4 \quad H(1) = O^3 e_0 - O^7 e_1 - O^5 e_2 + O^6 e_4 \]
  \[ H(0) = C(00) + e_2 C(10) \quad H(1) = C(01) + e_2 C(11) \]

- Complex
  \[ C(00) = O^0 e_0 + O^1 e_1 \quad C(01) = O^3 e_0 - O^7 e_1 \]
  \[ C(10) = O^2 e_0 - O^4 e_1 \quad C(11) = -O^5 e_0 - O^6 e_1 \]
  \[ C(00) = R(000) + e_1 R(100) \quad C(01) = R(001) + e_1 R(101) \]
  \[ C(10) = R(010) + e_1(110) \quad C(11) = R(011) + e_1 R(111) \]

- Real
  \[ R(000) = O^0 \quad R(100) = O^1 \quad R(001) = O^3 \quad R(101) = -O^7 \]
  \[ R(010) = O^2 \quad R(110) = -O^4 \quad R(011) = -O^5 \quad R(111) = -O^6 \]
To understand the symmetries of the magic square and its relation to YM we shall need in particular two algebras defined on $A$.

First, the algebra $\text{norm}(A)$ that preserves the norm

$$\langle x|y \rangle := \frac{1}{2}(xy + yx) = x^a y^b \delta_{ab}$$

- $\text{norm}(\mathbb{R}) = 0$
- $\text{norm}(\mathbb{C}) = \text{so}(2)$
- $\text{norm}(\mathbb{H}) = \text{so}(4)$
- $\text{norm}(\mathbb{O}) = \text{so}(8)$
Triality Algebra

Second, the triality algebra $\text{tri}(A)$

$$\text{tri}(A) \equiv \{(A, B, C) | A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in A.$$ 

$$\text{tri}(\mathbb{R}) = 0$$
$$\text{tri}(\mathbb{C}) = \mathfrak{so}(2) + \mathfrak{so}(2)$$
$$\text{tri}(\mathbb{H}) = \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3)$$
$$\text{tri}(\mathbb{O}) = \mathfrak{so}(8)$$

[Barton and Sudbery:2003]:
Yang-Mills spin-off: interesting in its own right

- We give a unified description of
  \(D = 3\) Yang-Mills with \(\mathcal{N} = 1, 2, 4, 8\)
  \(D = 4\) Yang-Mills with \(\mathcal{N} = 1, 2, 4\)
  \(D = 6\) Yang-Mills with \(\mathcal{N} = 1, 2\)
  \(D = 10\) Yang-Mills with \(\mathcal{N} = 1\)

in terms of a pair of division algebras \((\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})\), \(n = D - 2\)


- We present a master Lagrangian, defined over \(\mathbb{A}_{n\mathcal{N}}\)-valued fields, which encapsulates all cases.

- The overall (spacetime plus internal) on-shell symmetries are given by the corresponding *triality* algebras.

- We use imaginary \(\mathbb{A}_{n\mathcal{N}}\)-valued auxiliary fields to close the non-maximal supersymmetry algebra off-shell. The failure to close off-shell for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.

[arXiv:1309.0546
A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]
Global symmetries of supergravity in D=3

- MATHEMATICS: Division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

\[(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS\]

- PHYSICS: $N = 1, 2, 4, 8$ $D = 3$ Yang – Mills

\[(YANG – MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES\]

- CONNECTION: $N = 1, 2, 4, 8 \sim \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

\[MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE\]

The $D = 3$ $G/H$ grav symmetries are given by $ym$ symmetries

\[G(grav) = tri\ ym(L) + tri\ ym(R) + 3[ym(L) \times ym(R)].\]

eg

\[E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{O} \times \mathbb{O})\]

\[248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)\]
The $\mathcal{N} > 8$ supergravities in $D = 3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N} \leq 8$ may be coupled to $k$ additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N} = 2, 3, 4, 5, 6, 8$ supergravity with $k = 1, 1, 2, 1, 2, 4$: just the right matter content to produce the U-duality groups appearing in the magic square.

<table>
<thead>
<tr>
<th>$\mathcal{N}$</th>
<th>$G$</th>
<th>$H$</th>
<th>$\mathcal{N}$</th>
<th>$G$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, $f = 4$</td>
<td>SL(2, R), dim 3</td>
<td>$SU(2) \times SO(2)$, dim 4</td>
<td>5, $f = 16$</td>
<td>USp(4, 2), dim 21</td>
<td>SO(9), dim 36</td>
</tr>
<tr>
<td>3, $f = 8$</td>
<td>$SU(2, 1)$, dim 8</td>
<td>$SU(2) \times SU(2)$, dim 8</td>
<td>6, $f = 32$</td>
<td>$SU(4, 2)$, dim 35</td>
<td>SO(10), dim 46</td>
</tr>
<tr>
<td>5, $f = 16$</td>
<td>$SU(2) \times SO(2)$, dim 4</td>
<td>$SU(2)^2 \times SO(2)$, dim 8</td>
<td>8, $f = 64$</td>
<td>$SO(8) \times SO(2)$, dim 34</td>
<td>SO(12), dim 69</td>
</tr>
<tr>
<td>9, $f = 32$</td>
<td>$F_4(-20)$, dim 52</td>
<td>$E_6(-14)$, dim 78</td>
<td>12, $f = 128$</td>
<td>$E_7(-5)$, dim 133</td>
<td>SO(16), dim 120</td>
</tr>
</tbody>
</table>
Magic Pyramid: G symmetries
TWIN SUPERGRAVITIES
We consider so-called ‘twin supergravities’ - pairs of supergravities with $\mathcal{N}_+$ and $\mathcal{N}_-$ supersymmetries, $\mathcal{N}_+ > \mathcal{N}_-$, with identical bosonic sectors - in the context of tensoring super Yang-Mills multiplets.

[Gunaydin, Sierra and Townsend Dolivet, Julia and Kounnas Bianchi and Ferrara]

Classified in [Roest and Samtleben Duff and Ferrara]

Related work in [Chiodaroli, Gunaydin, Johansson, Roiban, 2015]
Pyramid of twins
The $D=4, \mathcal{N}=6$ supergravity theory is unique and determined by supersymmetry. The multiplet consists of

$$G_6 = \{g_{\mu\nu}, 16A_\mu, 30\phi; 6\psi_\mu, 26\chi\}$$

Its twin theory is the magic $\mathcal{N}=2$ supergravity coupled to 15 vector multiplets based on the Jordan algebra of $3 \times 3$ Hermitian quaternionic matrices $\mathcal{J}_3(\mathbb{H})$. The multiplet consists of

$$G_2 \oplus 15V_2 = \{g_{\mu\nu}, 2\psi_\mu, A_\mu\} \oplus 15\{A_\mu, 2\chi, 2\phi\}$$

In both cases the 30 scalars parametrise the coset manifold

$$\frac{SO^*(12)}{U(6)}$$

and the 16 Maxwell field strengths and their duals transform as the $32$ of $SO^*(12)$ where $SO^*(2n) = O(n, \mathbb{H})$. 

Example: $\mathcal{N}_+ = 6$ and $\mathcal{N}_- = 2$ twin supergravities
Key idea: reduce the degree of supersymmetry by using ‘fundamental’ matter multiplets

\[ \chi^{\text{adj}} \rightarrow \chi^{\text{fund}} \]

Twin supergravities are systematically related through this process

Generates new from old (supergravities that previously did not have a Yang-Mills origin)
Yang-Mills origin of $(6, 2)$ twin supergravities

$\mathcal{N} = 6$

- The $\mathcal{N} = 6$ multiplet is the product of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ vector multiplets,

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes \tilde{\mathbf{V}}_4 = G_6,$$

- $G_{\mathcal{N}}$, $\mathbf{V}_{\mathcal{N}}$ and $\mathbf{C}_{\mathcal{N}}$ denote the $\mathcal{N}$-extended gravity, vector, and spinor multiplets

- The hypermultiplet $\mathbf{C}_2^\rho$ carries a non-adjoint representation $\rho$ of $G$

- $\mathbf{C}_2^\rho$ does not ‘talk’ to the right adjoint valued multiplet $\tilde{\mathbf{V}}_4$
To generate the twin $\mathcal{N} = 2$ theory:

- Replace the right $\mathcal{N} = 4$ Yang-Mills by an $\mathcal{N} = 0$ multiplet
  
  $$[V_2 \oplus C^\rho_2] \otimes \tilde{V}_4 \rightarrow [V_2 \oplus C^\rho_2] \otimes [\tilde{A} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}]$$

- Here $\tilde{\chi}^{\alpha}$ in the adjoint of $\tilde{G}$ and 4 of $SU(4)$ is replaced by $\tilde{\chi}^{\rho\alpha}$ in a non-adjoint representation of $\tilde{G}$

- $\tilde{\chi}^{\rho\alpha}$ does not ‘talk’ to the right adjoint valued multiplet $V_2$, but does with $C^\rho_2$

- Gives a “sum of squares”
  
  $$[V_2 \oplus C^\rho_2] \otimes [\tilde{A} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}] = V_2 \otimes [\tilde{A} \oplus \tilde{\phi}^{[\alpha\beta]}] \oplus [C^\rho_2 \otimes \tilde{\chi}^{\rho\alpha}] = G_2 \oplus 15V_2$$
Introduce bi-fundamental scalar $\Phi^{a\bar{a}}$ to obtain sum of squares off-shell:

- Block-diagonal spectator field $\Phi$ with bi-adjoint and bi-fundamental sectors
  \[ \Phi = \begin{pmatrix} \phi^{i\bar{i}} & 0 \\ 0 & \phi^{a\bar{a}} \end{pmatrix}. \]

- The off-shell dictionary correctly captures the sum-of-squares rule:
  \[ [V_{N_L} \oplus C_{N_L}^\rho] \circ \Phi \circ [\tilde{V}_{N_R} \oplus \tilde{C}_{N_R}^\tilde{\rho}] = V_{N_L}^i \circ \Phi_{i\bar{i}} \circ \tilde{V}_{N_R}^i \oplus C_{N_L}^a \circ \Phi_{a\bar{a}} \circ \tilde{C}_{N_R}^a. \]

- Crucially, the gravitational symmetries are correctly generated by those of the Yang-Mills-matter factors via $\star$ and $\Phi$. 
Universal rule

- This construction generalises: all pairs of twin supergravity theories in the pyramid are related in this way

Super Yang-Mills factors

\[
\left[ V_{N_L} \oplus C_{N_L} \right] \otimes \tilde{V}_{N_R} \quad \longrightarrow \quad \left[ V_{N_L} \oplus C_{N_L} \right] \otimes \left[ \tilde{A} \oplus \tilde{\chi} \oplus \tilde{\phi} \right]
\]

\[
\downarrow \quad \downarrow
\]

\[
G_{N_+} + \text{matter} \quad \longrightarrow \quad G_{N_-} + \text{matter}
\]

Twin supergravities
Remarks

- Twin relations gives new from old

- Raises the question: what class of gravitational theories are double-copy constructible?

- What about supergravity coupled to the MSSM: is it a double-copy?
Are All Supergravity Theories the Square of Yang-Mills?

- All $N \geq 2$ supergravities with arbitrary matter couplings, with scalars parametrising a symmetric manifold
  [Anastasiou, Borsten, Duff, Marrani, Nagy, Zoccali ’17]

- Exceptions: $\mathcal{N} = 2$ pure sugra and the $T^3$ model, but see
  [Anastasiou, LB, Johansson to appear]

- Can extend to all homogenous (not necessarily symmetric) matter couplings although the BCJ compatibility remains unclear
  [Anastasiou, Borsten, Duff, Marrani, Nagy, Zoccali ’17]
Spin-four $N = 7$ W-Supergravity, [arXiv:1805.10022] [S. Ferrara and D, Lust] Obtain a new $N = 7$ (corresponding to 28 supercharges) W-supergravity, which does not exist as a four-dimensional perturbative model with an (effective) Lagrangian description. The resulting theory does not contain any massless states, but instead a massive higher spin-four supermultiplet of the $N = 7$ supersymmetry algebra.
“Supergravity is very compelling but it has yet to prove its worth by experiment”

MJD "What’s up with gravity?"
New Scientist 1977

“...a remark still unfortunately true at Supergravity@25. Let us hope that by the Supergravity@50 conference we can say something different.”

MJD "M-theory on manifolds of $G_2$ holonomy"
Supergravity@25 2001

“I’m glad I said 50 and not 40”
MJD “Twin supergravities from Yang-Mills squared”
Supergravity@40 2016