Turbulent compressible fluid: Renormalisation group analysis near $d = 4$

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Fully developed turbrlence (1)

The turbulence is characterized by
- Cascades of energy;
- Scaling behaviour with universal "anomalous exponents";
- Intermittency.

The key parameters are
- W and L – power of the external source of energy and integral (external) scale; in troposphere $L \sim 1$ km;
- ν and l – viscosity coefficient and dissipation (internal) scale; in troposphere $l \sim 1$ cm.

Fully developed turbulence: Re $\gg 1 \Rightarrow L \gg l \Rightarrow$ the inertial range $l \ll r \ll L$ exists.

Definition of the model: Stochastic equation (3)

The stochastic Navier–Stokes equation for compressible fluid has the form

$$\rho\nabla_t v_i = \nu_0(\delta_{ik}\partial^2 - \partial_i\partial_k)v_k + \mu_0\partial_i\partial_k v_k - \partial_i p + \eta_i,$$

where $\nabla_t = \partial_t + v_k\partial_k$, ρ is a fluid density field, v_i is the velocity field, p is the pressure field, and η_i is the density of an external force per unit volume (supplied the energy W in our system). The constants ν_0 and μ_0 are two independent molecular viscosity coefficients.

The model must be augmented by a continuity equation and an equation of state between deviations δp and $\delta\rho$ from the equilibrium values and can be recast in terms of v_i and $\phi = c_0^2 \ln(\rho/\overline{\rho})$, where a parameter c_0 is the adiabatic speed of sound, $\overline{\rho}$ denotes the mean value of ρ.

The random force η_i is supposed to be Gaussian, with zero mean and correlation function

$$\langle \eta_i(t,\boldsymbol{x})\eta_j(t',\boldsymbol{x}')\rangle = \frac{\delta(t-t')}{(2\pi)^d}\int_{k>m} \mathrm{d}^d\boldsymbol{k}\, \widetilde{D}_{ij}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')},$$

where the argument is given by

$$\widetilde{D}_{ij}(\boldsymbol{k}) = g_{10}\nu_0^3 k^{4-d-y}\Big\{P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})\Big\}.$$

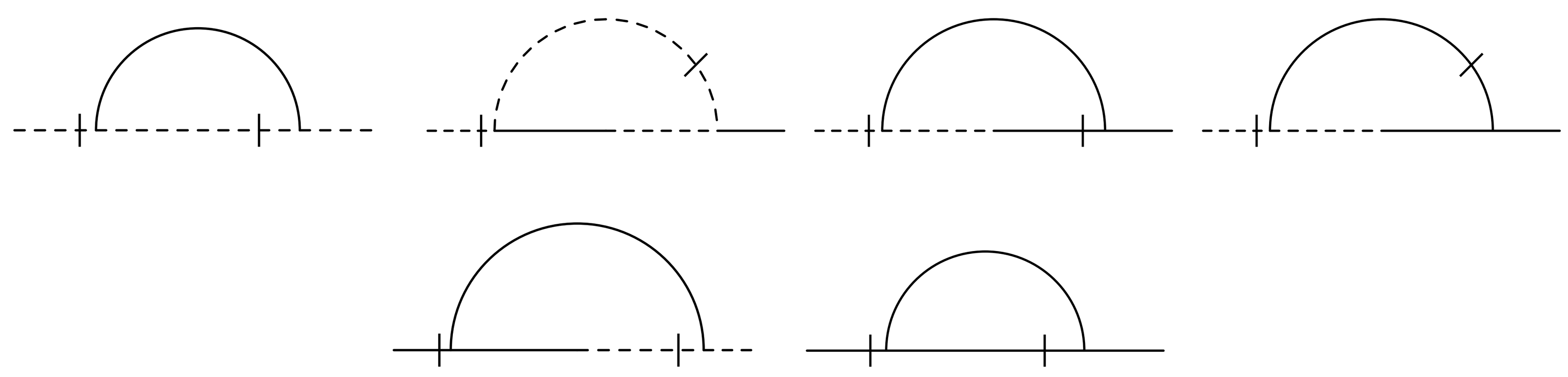
Here, $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$ and $Q_{ij}(\boldsymbol{k}) = k_i k_j/k^2$ are the transverse and longitudinal projectors, $k = |\boldsymbol{k}|$, the amplitude α is a free parameter, the amplitude g_{10} is a coupling constant.

Divergences and diagrams: $d = 3$ (5)

For any $d \neq 2$ and $d \neq 4$ superficial divergences can be present only in the 1-irreducible functions of types

$$\langle v_i' v_i\rangle_{1-\mathrm{ir}}, \quad \langle v_i'\phi\rangle_{1-\mathrm{ir}}, \quad \langle \phi v_i\rangle_{1-\mathrm{ir}}, \quad \text{and} \quad \langle\phi'\phi'\rangle_{1-\mathrm{ir}}.$$

The one-loop approximation of these functions has the form



Renormalization group equation at $d = 3$ and $d = 4$ (7)

RG equation states that the large scale behavior is governed by the IR attractive fixed points g^* and evolution of invariant couplings is described by the set of flow equations

$$\mathcal{D}_s \overline{g}_i = \beta_i(\overline{g}_j).$$

From the analysis of β functions it follolws that the system possesses three fixed point: trivial one FPI, local one FPII and non-local one FPIII, which define anomalous exponents γ_n.

The fixed point FPII is invisible if we consider the system directly at $d = 3$ (Phys. Rev. E 90, 063016, 2014) and describes a macroscopic "shaking" of a fluid container.

Kolmogorov−Obukhov K41 theory (2)

The equal-time structure functions

$$S_n(\mathbf{r}) = \left\langle \left[v_r(t,\mathbf{x}) - v_r(t,\mathbf{x}')\right]^n\right\rangle,$$

where v_r is the component of the velocity field along the direction $\mathbf{r} = \mathbf{x} - \mathbf{x}'$.

From the two Kolmogorov's hypotesis (independence of L for $L \gg r$ and independence of l for $l \ll r$) it follows, that in the inertial range $l \ll r \ll L$

$$S_n(\mathbf{r}) = C_n\,(Wr)^{n/3}$$

with exact exponents and universal amplitudes C_n.

Due to the imtermittency statistical properties of the velocity are dominated by rare spatiotemporal configurations: the main contributions are given by infrequent, but strong events. This phenomenon leads to the violation of the classical K41 theory:

$$S_n(\mathbf{r}) = (Wr)^{n/3}\,(r/L)^{\gamma_n}$$

with an infinite set of "anomalous exponents" γ_n.

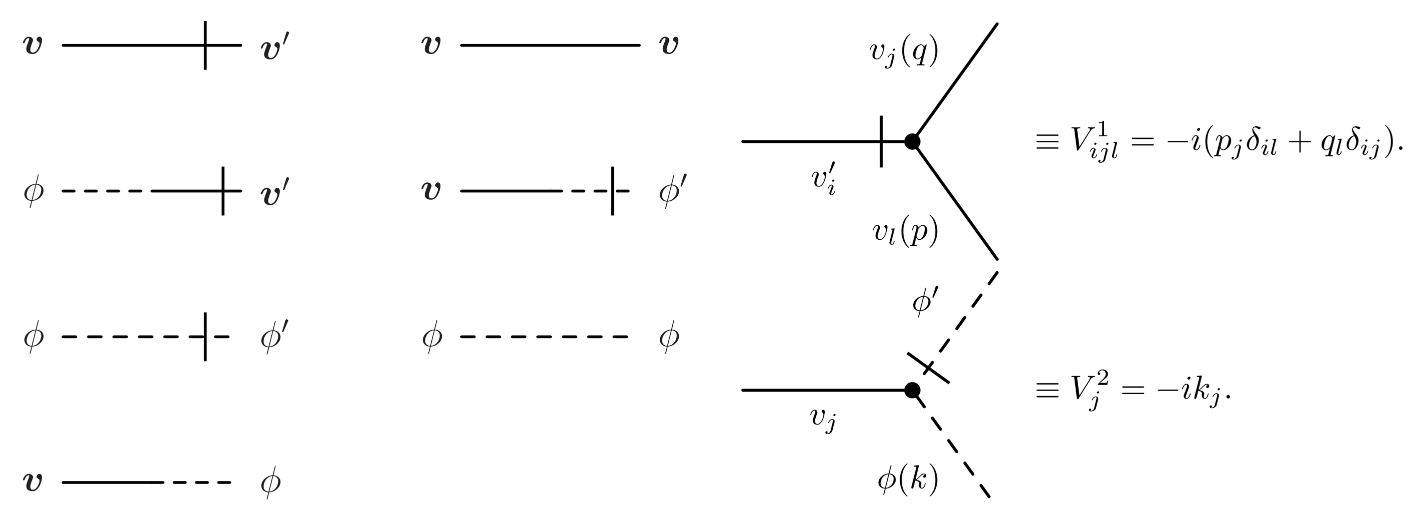
The goal is to calculate γ_n within a regular expansion.

Field theoretic formulation (4)

This stochastic problem is equivalent to the field theoretic model with a doubled set of fields $\Phi = \{v_i, v_i', \phi, \phi'\}$ and action functional

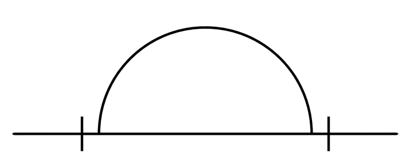
$$\mathcal{S}_{\boldsymbol{v}}(\Phi) = \frac{v_i'\widetilde{D}_{ij}v_j'}{2} + v_i'\Big[\nabla_t v_i + \nu_0(\delta_{ij}\partial^2 - \partial_i\partial_j)v_j + u_0\nu_0\partial_i\partial_j v_j - \partial_i\phi\Big] + \phi'[-\nabla_t\phi + v_0\nu_0\partial^2\phi - c_0^2(\partial_i v_i)], \quad (1)$$

This model corresponds to a standard Feynman diagrammatic technique with two the triple vertices and seven bare propagators:



Divergences and diagrams: $d = 4$ (6)

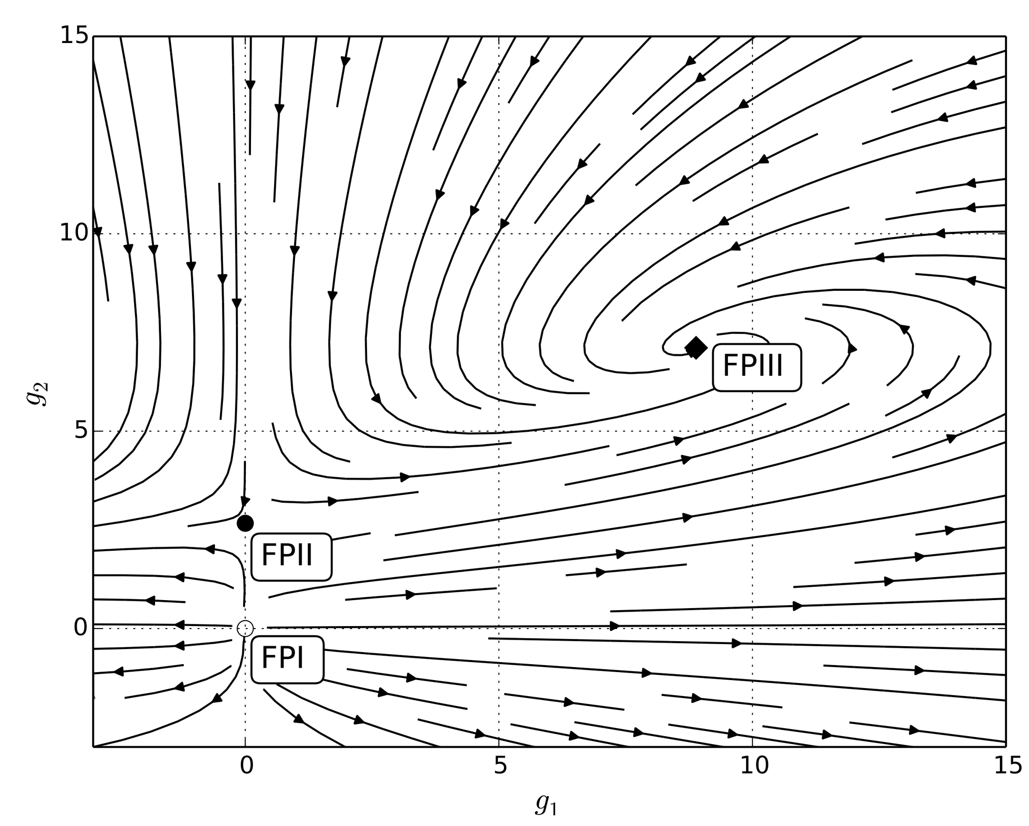
If $d = 4$, one more divergent function is presented: $\langle v_i' v_j'\rangle_{1-\mathrm{ir}}$.
The one-loop approximation for it has the form



The corresponding index of divergence reads $\delta_\Gamma = -d + 4$, therefore, it becomes UV divergent in $d = 2, 3,$ and 4 and requires a presence of specific counterterms. But for the physical case $d = 3$ ($\delta_\Gamma = 1$) it is impossible to construct a scalar counterterm containing two vector fields and one derivative.

Numerical simulation of RG flow and Conclusion (8)

RG flow diagram at some values of parameters.



Three fixed points FPI, FPII, and FPIII are marked by an empty circle, a filled circle, and a filled rhombus, respectively. Newly found fixed point FPII exists but is unstable at these values of parameters.

for details, see Phys. Rev. E 95, 033120 (2017)