Inhomogeneous modes in the pseudo-Hermitian quantum cosmology

Oleg O. Novikov

Saint Petersburg State University

o.novikov@spbu.ru, oonovikov@gmail.com

1. Motivation

The average energy density ϵ and pressure p of the matter are dominated by something called *dark energy* with equation of state $w = p/\epsilon \sim -1$. The simplest explanation is the cosmological constant with $w = -1$. However its value looks fine-tuned. One may suppose that the dark energy is actually a dynamical field. However the observations require it to be the so-called *phantom matter* with $w < -1$.

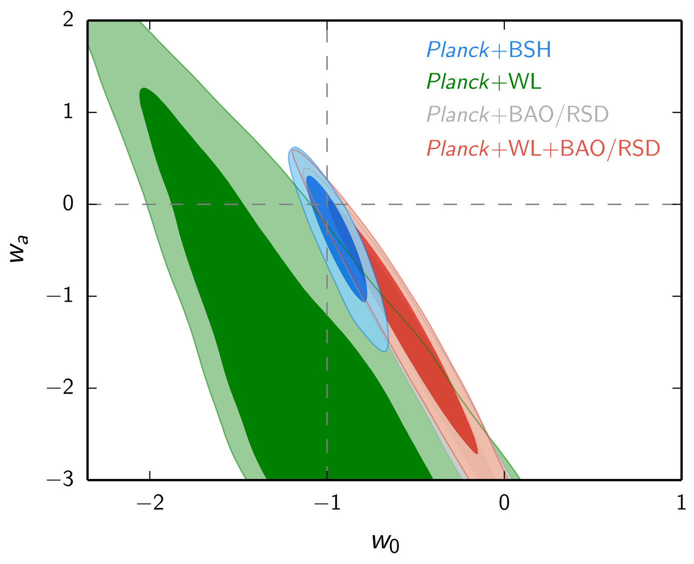


Figure 1: *PLANCK [1] results for dark energy equation of state* $w = w_0 + w_a(1-a) + O((1-a)^2)$.

The phantom matter can be e.g. a scalar field ξ possessing negative kinetic energy,

$$L_{phantom} = -\frac{1}{2}\partial_\mu\xi\partial^\mu\xi - V(\tilde{\xi})$$

However this leads to severe instabilities because its energy is not bounded from below and cosmological evolution may end up in the Big Rip. We propose to describe them with a classically equivalent model,

$$L_{PTom} = \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} - V(i\tilde{\Phi})$$

with PT symmetry $\tilde{\Phi} \mapsto -\tilde{\Phi}$, $i \mapsto -i$. The perturbations should be considered along the real axis,

$$\tilde{\Phi} = i\xi_{class} + \delta\Phi$$

Such perturbations near classical trajectory happen to possess positively definite effective Hamiltonian [2]. To separate the fields with PT-symmetry from usual phantoms we coin a new name - PTom. The aim of this work is to explore the possibility to construct quantum model of PToms.

2. Pseudo-Hermitian models

Consider the family of the Hamiltonians,

$$H = p^2 + x^2(ix)^{i\varepsilon}$$

For $\varepsilon > 0$ those Hamiltonians are not Hermitian but possess $\mathcal{PT}$-symmetry

$$x \mapsto -x, \quad p \mapsto p, \quad i \mapsto -i$$

Those Hamiltonians happen to possess purely real and positive spectrum (with some modification of the boundary conditions for $\varepsilon \geq 2$) [5]. Note that for $\varepsilon = 2$ it's naively unstable. Its eigenfunctions of course are not orthogonal in terms of the initial norm but rather some new norm (certain $\mathcal{C}$ is required for positivity)

$$\langle\phi_n|\mathcal{CPT}|\phi_m\rangle = \delta_{nm}, \ \mathcal{C}^\dagger = -\mathcal{C}, \ [\mathcal{C}, H] = 0, \ [\mathcal{C}, \mathcal{PT}] = 0$$

From the point of view of this new norm the Hamiltonian can be considered Hermitian and generates unitary evolution. PT-symmetric Hamiltonians are a particular case of the Pseudo-Hermitian quantum mechanics.

$$H = \eta^{-1}h\eta, \quad h = h^\dagger, \quad \eta^\dagger\eta \neq 1$$

One may either use the Hermitian Hamiltonian h and the corresponding norm $\langle\psi|\phi\rangle$ to compute probabilities or the equivalent description with non-Hermitian Hamiltonian H and new norm $\langle\psi|\eta^\dagger\eta|\phi\rangle$.

However e.g. for $H = p^2 + x^2(ix)^\epsilon$ the similarity transformation η is known only in perturbation theory and the corresponding h is highly nonlocal.

Thus simple naively non-Hermitian or unbounded from below Hamiltonian may describe unitary evolution of some stable but very complicated system.

For time-dependent Hamiltonian $h(t)$ and similarity operator $\eta(t)$ the equivalent non-Hermitian Hamiltonian no longer is Pseudo-Hermitian and may have imaginary eigenvalues,

$$H = \eta^{-1}h\eta - i\eta^{-1}\dot{\eta}$$

In QFT we may introduce special non-Hermitian (and thus directly unobservable) fields and hope describe some very complicated nonlinear interactions with simple models.

3. Quintessence+PTom

In order to fit observations we need a composition of two scalar fields: quintessence and PTom ones. Let's consider the following model,

$$S = \int d^4x\sqrt{-g}\Big(-\frac{1}{2\kappa^2}R + \frac{1}{2}M_{\Phi\Phi}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}M_{\tilde{\Phi}\tilde{\Phi}}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + iM_{\Phi\tilde{\Phi}}\partial_\mu\Phi\partial^\mu\tilde{\Phi} - V(\Phi) + \tilde{V}(\tilde{\Phi})\Big), \qquad (1)$$

where all parameters are real, $V(\Phi)^* = V(\Phi)$ and $\tilde{V}(\tilde{\Phi})^* = \tilde{V}(-\tilde{\Phi})$ to preserve the following symmetry,

$$t \mapsto -t, \quad i \mapsto -i, \quad \Phi \mapsto \Phi, \quad \tilde{\Phi} \mapsto -\tilde{\Phi}.$$

Easy to study example,

$$V = V_0 e^{\lambda\Phi}, \quad \tilde{V} = -\tilde{V}_0 e^{i\tilde{\lambda}\tilde{\Phi}}$$

with special form of kinetic matrix after the following transformation:

$$\chi = \lambda\Phi + 6\rho, \quad \pi = \frac{1}{\lambda}p_\Phi, \quad \tilde{\chi} = \tilde{\lambda}\tilde{\Phi} - 6i\rho, \quad \tilde{\pi} = \frac{1}{\tilde{\lambda}}p_{\tilde{\Phi}}, \qquad (2)$$

$$\omega = p_\rho - \frac{6}{\lambda}p_\Phi + \frac{6i}{\tilde{\lambda}}p_{\tilde{\Phi}}, \qquad (3)$$

the variables separate and exact classical solutions may be obtained that are symmetric under $\mathcal{PT}$ transformation affecting $\tilde{\chi}$.

4. Homogeneous and inhomogeneous part

We will demonstrate our approach on the longitudinal (3d scalar) modes that decouple from transverse (tensor and vector) ones at quadratic order. The tensor and vector fluctuations are added in the same fashion,

$$g_{\mu\nu} = (N^2(t) + s(t,x))dt^2 + 2(\partial_k v(t,x))dtdx^k - e^{2\rho}\Big(\delta_{ij} + h(t,x)\delta_{ij} + \partial_i\partial_j E(t,x)\Big)dx^i dx^j \qquad (4)$$

$$\Phi = \Phi(t) + \phi(t,x), \quad \tilde{\Phi} = \tilde{\Phi}(t) + \tilde{\phi}(t,x)$$

Let us choose the partial gauge $h = E = 0$ and decompose into eigenfunctions of Laplace operator on the space with IR regulator L.

$$\phi(t,x) = \sum_\Omega \phi(t,\Omega)f(\Omega,x), \quad -\Delta f(\Omega,x) = \Omega^2 f(\Omega,x)$$

We get the system of constraints,

$$p_N = 0, \quad L^3H_0 + \sum_\Omega H_2(\Omega) = 0, \quad p_s = 0, \quad p_v = 0$$

Thus s and v are undynamical variables that can be get rid of after solving secondary constraints.

Let us introduce the wavefunctional $\Psi\Big[\rho, \Phi, \tilde{\Phi}, N\Big|\{\phi, \tilde{\phi}, s, v\}\Big]$ and perform the canonical quantization of the constraints,

$$\Big[L^3\hat{H}_0 + \sum_\Omega \hat{H}_2(\Omega)\Big]\Psi = 0, \quad \hat{p}_N\Psi = 0 \quad \hat{p}_s\Psi = 0, \quad \hat{p}_v\Psi = 0,$$

After quantization we assume Born-Oppenheimer approximation,

$$\Psi = \Psi_0(\rho, \Phi, \tilde{\Phi})\psi_2[\phi, \tilde{\phi}|\rho, \Phi, \tilde{\Phi}]$$

Ψ_0 satisfies the minisuperspace WdW equation, $\{\Phi_a\} = (\Phi, \tilde{\Phi})$

$$\left[\frac{\kappa^2}{12L^6}\partial_\rho^2 - \frac{1}{2L^6}(M^{-1})_{ab}\partial_a^2 + \Big(V(\Phi) - \tilde{V}(\tilde{\Phi})\Big)e^{6\rho}\right]\Psi_0 = 0$$

This allows us to take into account some contribution from nonlinearities in the homogeneous sector that may play crucial role in the pseudo-Hermitian construction. However then we are faced with all the issues of defining observables and probabilities in the quantum gravity.

5. Probability current

The Wheeler-DeWitt equation with the ordinary fields admit the conserved Klein-Gordon current, however it is not positively definite. But consider the WKB-wavepacket in the minisuperspace $Q_A = (\rho, \Phi_a)$

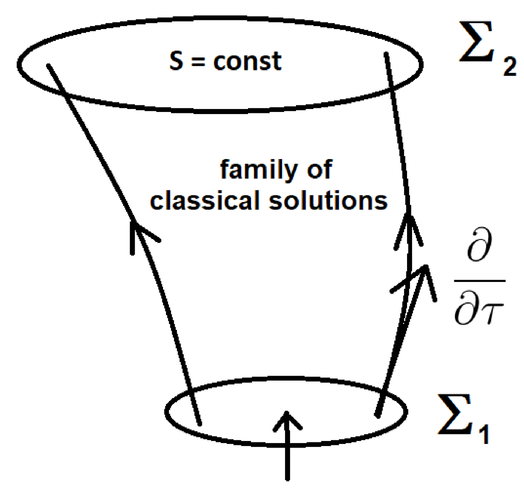
$$\Psi_0 \simeq \mathcal{A}e^{iL^3S - L^3R}$$

In the center of this wavepacket $|\partial_A S| \gg |\partial_A R|$. Different R may allow different $\mathcal{A}$. This way we can treat all wavepackets with similar S as having a probability amplitude $\mathcal{A}$ defined on the surface of constant S. Introducting the WKB-time

$$\frac{i}{L^3}\frac{\partial}{\partial_\tau} = \frac{i\kappa^2}{6}(\partial_\rho S)(\partial_\rho\psi_2) - i(M^{-1})_{ab}(\partial_{\Phi_a}S)(\partial_{\Phi_b}\psi_b)$$

for fluctuation part we get the Schrodinger-like equation that admits some conserved inner product,

$$\frac{i}{L^3}\frac{\partial}{\partial\tau}\psi_2 = \sum_\Omega H_2(\Omega)\psi_2,$$



$$\mathcal{A}(Q_\perp, \tau)\psi_2(Q_\perp, \tau|\{v_k\})$$

Integrating over some surface Σ of constant S in the center of the wavepackets allows to introduce conditional probabilities using the following inner product for two wavepackets with the same S,

$$\langle\Psi^{(1)}|\Psi^{(2)}\rangle = \int_\Sigma d\Sigma\Big[\mathcal{A}^{(1)}(Q^\perp)\Big]^*\mathcal{A}^{(2)}(Q^\perp)\langle\psi_2^{(1)}|\psi_2^{(2)}\rangle_{Q^\perp}$$

This inner product may actually be obtained by using usual KG current for this particular ansatz.

We can easily generalize this to PToms when classical trajectories are $\mathcal{PT}$-symmetric (and therefore we can define τ in $\mathcal{PT}$-symmetric fashion),

$$H_{2,\Omega}(\tau, Q^\perp) = \eta^{-1}(\tau, Q^\perp)h_{2,\Omega}(\tau, Q^\perp)\eta - i\eta^{-1}(\tau, Q^\perp)\partial_\tau\eta(\tau, Q^\perp)$$

The $\mathcal{PT}$ symmetry under the minisuperspace variables implies the symmetry of η.

In the minisuperspace we can generalize the inner product density using $\mathcal{PT}$ transformation on the surface of constant S.

$$\langle\Psi^{(1)}|\Psi^{(2)}\rangle = \int_\Sigma d\Sigma\Big[\mathcal{A}^{(1)}(\mathcal{PT}Q^\perp)\Big]\mathcal{A}^{(2)}(Q^\perp)(\psi_2^{(1)}|\psi_2^{(2)})_{Q^\perp}$$

where,

$$(\psi_2^{(1)}|\psi_2^{(2)})_{Q^\perp} = \int_{\Sigma=\mathcal{PT}\Sigma}\prod_a dQ_a^\perp\psi_2^{(2)}(\tau, \mathcal{PT}Q^\perp)\eta^\dagger(\tau, \mathcal{PT}Q^\perp)\eta(\tau, Q^\perp)\psi_2^{(1)}(Q^\perp)$$

For exponential potential the fluctuation Hamiltonian is greatly simplified when $\kappa \to 0$,

$$H_{2,\Omega} = \Omega^2(\phi^2 + \tilde{\phi}^2) + \frac{D}{2}p_\phi p_\phi + \frac{\tilde{D}}{2}p_{\tilde{\phi}}p_{\tilde{\phi}} + \frac{V}{2}e^\chi\phi^2 - \frac{\tilde{V}}{2}e^{i\tilde{\chi}}\phi^2 + O(\kappa)$$

The Hamiltonian is Hermitian on the purely imaginary classical trajectory $\tilde{\chi} = i\xi_{class}$ and we use it as h. For $\mathrm{Re}\tilde{\chi} \neq 0$ the similarity operator η should be introduced,

$$\eta = \exp\Big[\alpha(t)p_{\tilde{\phi}}^2 + \beta(t)\tilde{\phi}^2 + \gamma(t)(p_{\tilde{\phi}}\tilde{\phi} + \tilde{\phi}p_{\tilde{\phi}})\Big]$$

We note that the operators in the exponent form the finite dimensional algebra. Then we use,

$$e^{-X_n}\hat{h}_{\tilde{\phi}}^{(n)}e^{X_n} = e^{-\mathrm{ad}_{X_n}}\hat{h}_{\tilde{\phi}}^{(n)}, \quad e^{-X_n}\partial_\tau\Big(e^{X_n}\Big) = \frac{1 - e^{-\mathrm{ad}_{X_n}}}{\mathrm{ad}_{X_n}}\partial_\tau X_n,$$

where the adjoint operator $\mathrm{ad}_X Y \equiv [X, Y]$ may be represented as a finite-dimensional matrix acting on the algebra. In perturbation theory of the small $\mathrm{Re}\tilde{\chi}$ this gives equations,

$$\partial_\tau\alpha_n = 2\gamma_n\tilde{D}, \quad \partial_\tau\gamma_n = 2\beta\tilde{D} - 2\alpha(\tilde{V}e^{6\rho-\xi_{class}} + \Omega^2), \qquad (5)$$

$$\partial_\tau\beta_n = \frac{\tilde{V}}{2}e^{6\rho-\xi_{class}}\delta\tilde{\Phi} - 4\gamma_n(\tilde{V}e^{6\rho-\xi_{class}} + \Omega^2) \qquad (6)$$

Which can be solved e.g. numerically near the exact solution $i\xi_{class}$

This result can be used to construct the probability distributions and check whether the observables for this model show any pathologies. This hopefully will allow us to construct the viable phantom model.

References

[1] P. A. R. Ade et al. Planck 2015 results. XIV. Dark energy and modified gravity. *Astron. Astrophys.*, 594:A14, 2016.

[2] Alexander A. Andrianov, Francesco Cannata, Alexander Y. Kamenshchik, and Daniele Regoli. Phantom without phantom or how the PT symmetry saves us from the Big Rip. *Int. J. Mod. Phys. D*, 19:97–111, 2010.

[3] Alexander A. Andrianov, Chen Lan, and Oleg O. Novikov. PT Symmetric Classical and Quantum Cosmology. *Springer Proc. Phys.*, 184:29–44, 2016.

[4] Alexander A. Andrianov, Oleg O. Novikov, and Chen Lan. Quantum cosmology of multifield scalar matter: Some exact solutions. *Theor. Math. Phys.*, 184:1224–1233, 2015. [Teor. Mat. Fiz.184,no.3,380(2015)].

[5] Carl M. Bender. Making sense of non-Hermitian Hamiltonians. *Rept. Prog. Phys.*, 70:947, 2007.

[6] O. O. Novikov. Inhomogeneous modes in the PT-symmetric quantum cosmology [arXiv:1712.03939].