Pentagon Identities arising from Supersymmetric Gauge Theories

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Motivation

$$T_{12} T_{13} T_{23} = T_{23} T_{12}$$

In physics

1. Biedenharn-Elliott identity in representation theory of rotation groups (Biedenharn, Louck)
2. Identity for fusion matrices in CFT (Moore, Sieberg)
3. Consistency condition for associator in quasi-Hopf algebras (Drinfeld)
4. Quantum dilogarithm

In mathematics

1. Durfee's square identity in combinatorics (Rimanyi, Weigandt, Yong)
2. Donaldson-Thomas invariant in geometry (Kontsevich, Soibelman)
3. Counting Betti number in topology (Rimanyi, Kontsevich, Soibelman)

Introduction

[Bozkurt, Gahramanov 1803.00855]

My talk is based on this paper.

Pentagon Identities arising from Supersymmetric Gauge Theories

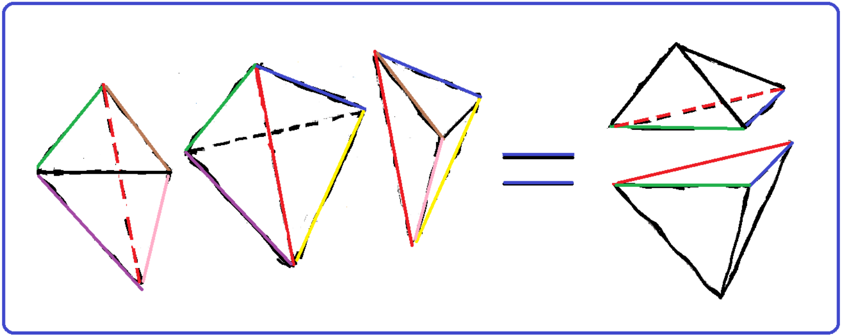
Integral pentagon relations are

$$\int d\mu B_c B_c = B_c B_c B_c$$

Introduction

There is the relation between 3d N=2 supersymmetric gauge theories and 3-manifolds known as 3d-3d correspondence. In this context the pentagon identity encodes information about the geometry of the corresponding 3-manifolds and it would be interesting to understand the meaning of the pentagon identity from this point of view.

The superconformal index of a 3-dimensional supersymmetric field theory can be expressed in terms of basic hypergeometric integrals. By comparing the indices of dual theories, one can find new integral identities for basic hypergeometric integrals. Some of these integral identities have a form of the pentagon identity which can be interpreted as the 2-3 Pachner move for triangulated 3-manifolds.



Dividing a polytope in two different ways

Q-Calculus

$$xy = qyx$$

Notations

$$(z, q)_\infty = \prod_{i=0}^{\infty}(1 - zq^i)$$

Faddeev's Quantum Dilogarithm

[Faddeev, Kashaev 9310070]

According to Faddeev and Kashaev, From the Euler's and Roger's definitions of dilogarithm

$$L_2(x) = -\int_0^x \frac{log(1-z)}{z} dz$$

$$L(x) = L_2(x) + log(1-x)log(x)/2$$

one can obtain following pentagon identity, called Roger's identity,

$$L(x) + L(y) + L(xy) = L(\frac{x-xy}{1-xy}) + L(\frac{y-xy}{1-xy}) \ .$$

By using the following function,

$$\mathcal{L}(x) = (x, q)_\infty$$

we can write the quantum version of the Roger's identity.

$$\mathcal{L}(v)\mathcal{L}(u) = \mathcal{L}(u)\mathcal{L}(-qvu)\mathcal{L}(v)$$

New Pentagon Identity

[Mori, Tanaka 1512.02835]

Another simple but more interesting example of the pentagon identity is provided by the equality of $\mathbb{RP}^2 \times \mathbb{S}^1$ partition functions. According to the mirror symmetry we have the following integral identity

$$q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\oint_{C_0}\frac{dz}{2\pi i z}z^s\sum_{m=0}^{1}a^{-1/2+m}q^{-1/4+m/2}\frac{(z^{-1}aq^{1+m};q^2)_\infty(zaq^{1+m};q^2)_\infty}{(za^{-1}q^m;q^2)_\infty(z^{-1}a^{-1}q^m;q^2)_\infty}$$

$$=q^{-\frac{1}{8}}a^{-1/2-|\tilde{s}|}\frac{(a^{-1}q^{1/2+|\tilde{s}|};q^2)_\infty}{(aq^{1/2+|\tilde{s}|};q^2)_\infty}\frac{(a^{-1}q^{3/2+|\tilde{s}|};q^2)_\infty}{(aq^{3/2+|\tilde{s}|};q^2)_\infty}\frac{(a^2q;q^2)_\infty}{(a^{-2};q^2)_\infty}\ .$$

By introducing the following function

$$\mathcal{B}(z,m,q^2) = z^{-\frac{1}{4}+\frac{m}{2}} q^{-\frac{1}{8}+\frac{m}{4}} \frac{(zq^{m+1};q^2)_\infty}{(z^{-1}q^m;q^2)_\infty} \ ,$$

one finds a non-trivial pentagon relation

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \oint_{C_0} \frac{dz}{2\pi i z} z^s \sum_{m=0}^{1} \mathcal{B}(z^{-1}a,m;q^2)\mathcal{B}(za,m;q^2)$$
$$= \mathcal{B}(a^{-1}q^{-\frac{1}{2}},|\tilde{s}|;q^2)\mathcal{B}(a^{-1}q^{-\frac{1}{2}},|\tilde{s}|+1;q^2)\mathcal{B}(a^2,0;q^2) \ .$$

Concluding Remarks

The identities in terms of hyperbolic hypergeometric functions are related to the knot invariants which are connected also to the volumes of hyperbolic 3-manifolds. It would be interesting to make a connection between knot invariants (as Hikami invariant) and results presented here.

From the physics perspective, another important application of pentagon identities is that they may be used to construct new solutions to the quantum Yang-Baxter equation.

Thank you for your attention!