Topological Invariant For Star-Triangle Relation

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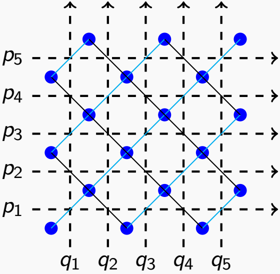
Introduction For Models in Statistical Mechanics

The aim of statistical mechanics is to predict the relations between the observable macroscopic properties of the system,given only a knowledge of the microscopic forces between the components.

- What do we want to understand?
- What do we want to calculate?

Two Dimensional Exactly Solved Models



- On each site i we place a 'spin'σ_i which takes some set of discrete or continuous values.
- For Ising Model:
  $\sigma_i = \pm 1$
  Periodic boundary condition $\sigma_{n+1} = \sigma_1$ and no external field.
  Spin interaction occurs only between neighbour spins.

Boltzmann Weights

The Boltzmann weight for the Ising model of the edge $< i, j >$

$$\mathcal{W}(\sigma_i, \sigma_j) = \exp\{-\varepsilon(\sigma_i, \sigma_j)/k_B T\}$$

Boltzmann weights for two-dimensional models, are characterized according to the spin variables and "rapidity" variables p and q.

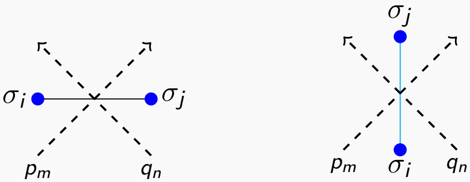


Figure: $\mathcal{W}_{p_m q_n}(\sigma_i, \sigma_j)$ (left) and $\overline{\mathcal{W}}_{p_m q_n}(\sigma_i, \sigma_j)$.

Transfer Matrices Commutation

Partion Function $= \mathrm{Tr}(\mathcal{W}\overline{\mathcal{W}})$
Product $(\mathcal{W}\overline{\mathcal{W}})$ is not symmetric.

- Transfer matrices diagonalization

The Free energy of two dimensional Ising model in zero field was first obtained by Onsager in 1944.He diagonalized the transfer matrix by using matrix algebra.

Star-Triangle Relation

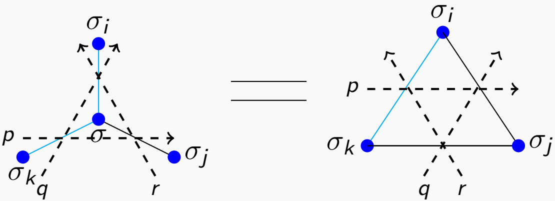


Figure: The star-triangle relation

Onsager(1971) noted that the star-triangle relation implies that their diagonal-to-diagonal transfer matrices commutation. Baxter(1978) has shown that 2D Ising model can be solved quite directly by star-triangle relation.

$$\sum_{\sigma} \mathcal{S}(\sigma)\overline{\mathcal{W}}_{qr}(\sigma,\sigma_i)\mathcal{W}_{pr}(\sigma_j,\sigma)\overline{\mathcal{W}}_{pq}(\sigma_k,\sigma)$$

$$= \mathcal{R}_{(pqr)}\mathcal{W}_{pq}(\sigma_i,\sigma_j)\overline{\mathcal{W}}_{pr}(\sigma_k,\sigma_i)\mathcal{W}_{qr}(\sigma_k,\sigma_j)$$

$$\sum_{\sigma} \mathcal{S}(\sigma)\overline{\mathcal{W}}_{qr}(\sigma,\sigma_i)\mathcal{W}_{pr}(\sigma_j,\sigma)\overline{\mathcal{W}}_{pq}(\sigma_k,\sigma) = \mathcal{R}_{(pqr)}\mathcal{W}_{pq}(\sigma_i,\sigma_j)\overline{\mathcal{W}}_{pr}(\sigma_k,\sigma_i)\mathcal{W}_{qr}(\sigma_k,\sigma_j)$$

The R_{pqr} is some factor independent of the spins.

- Spins can be interchanged.

There exist spin-independent quantities:

$$\prod_{\sigma_i}\mathcal{W}_{pq}(\sigma_i,\sigma_j) = \prod_{\sigma_i}\mathcal{W}_{pq}(\sigma_j,\sigma_i) = P_{pq}$$

$$\prod_{\sigma_i}\overline{\mathcal{W}}_{pq}(\sigma_i,\sigma_j) = \prod_{\sigma_i}\overline{\mathcal{W}}_{pq}(\sigma_j,\sigma_i) = \overline{P}_{pq}$$

for all values of the spin σ_i let Y_{pq} be the NxN matrix with entry $\mathcal{W}_{pq}(\sigma_i,\sigma_j)$ in position (σ_i,σ_j), $\overline{Y}_{pq}$ the matrix corresponding entry $\overline{\mathcal{W}}_{pq}(\sigma_i,\sigma_j)$.

Diagonal Matrices:

$$S = S(\sigma)\delta(\sigma_i, \sigma_j)$$

$$X_{pq|\sigma} = \mathcal{W}_{pq}(\sigma, \sigma_i)\delta(\sigma_i, \sigma_j), \quad X'_{pq|\sigma} = \mathcal{W}_{pq}(\sigma_i, \sigma)\delta(\sigma_i, \sigma_j)$$
$$\overline{X}_{pq|\sigma} = \overline{\mathcal{W}}_{pq}(\sigma, \sigma_i)\delta(\sigma_i, \sigma_j), \quad \overline{X'}_{pq|\sigma} = \overline{\mathcal{W}}_{pq}(\sigma_i, \sigma)\delta(\sigma_i, \sigma_j)$$

in position (σ_i, σ_j). All diagonal matrices depend on the spin σ.

$$\sum_{\sigma}\mathcal{S}(\sigma)\overline{\mathcal{W}}_{qr}(\sigma,\sigma_i)\mathcal{W}_{pr}(\sigma_j,\sigma)\overline{\mathcal{W}}_{pq}(\sigma_k,\sigma)$$
$$=\mathcal{R}_{(pqr)}\mathcal{W}_{pq}(\sigma_i,\sigma_j)\overline{\mathcal{W}}_{pr}(\sigma_k,\sigma_i)\mathcal{W}_{qr}(\sigma_k,\sigma_j)$$

Regard the σ_j as fixed and think of each side of the Star-Triangle Equation as the element of σ_i,σ_k of some matrix.
Matrix form of Star-Triangle relation:

$$Y_{qr}SX_{pr|\sigma_j}\overline{Y}_{pq}=\mathcal{R}_{pqr}X_{pq|\sigma_j}\overline{Y}_{pr}X_{qr|\sigma j}$$

One can obtain $\mathcal{R}_{pqr}$ by taking determinants.

$$\mathcal{R}_{pqr}=f_{qr}f_{pq}/f_{pr}$$

$$f_{pq}=P_{pq}^{-1}det(S\overline{Y}_{pq})$$

The Invariant $\mathcal{I}$

Hold σ_i fixed and think them as elements (σ_j, σ_k)

$$Y_{pr} S X_{qr|\sigma_i} \overline{Y}_{pq} = \mathcal{R}_{pqr} X'_{pq|\sigma_i} Y_{qr} \overline{X}_{pr|\sigma_i}$$

$$\mathcal{R}_{pqr} = f_{pq} \overline{f}_{pr} / \overline{f}_{qr}$$

Hold σ_k fixed and regarding the relations as elements (σ_i, σ_j) of a matrix,

$$Y_{pr} S \overline{X'}_{pq|\sigma_k} \overline{Y}^{T}_{qr} = \mathcal{R}_{pqr} X'_{qr|\sigma_k} Y_{pq} \overline{X'}_{pr|\sigma_k}$$

$$\mathcal{R}_{pqr} = f_{qr} \overline{f}_{pr} / \overline{f}_{pq}$$

The relations of the $\mathcal{R}$'s are mutually consistent if and only if:

$$f_{qr}\overline{f}_{qr} = f_{pr}\overline{f}_{pr} = f_{pq}\overline{f}_{pq}$$

The only way this can happen is for the quantity

$$\mathcal{I} = N^{-1} f_{pq}\overline{f}_{pq}$$

to be independent of p and q!

Invariant $\mathcal{I}$ for exactly solved models.

Ising model: $\mathcal{I}^2 = sinh2Ksinh2\overline{K} = 1$ [R.J.Baxter.Exactly Solved Models in Statistical Mechanics. Academic, London, 1982.]

Ashkin-Teller model: $\mathcal{I} = 1$

Self-dual Potts model: $\mathcal{I} = 1$

Fateev-Zamoldchikov model : $\mathcal{I} = 1$

[Baxter 0108363]

Summary

- What we have learned?

Solutions to the star-triangle relation leads to two-dimensional exactly solvable models.

For all previously solved models $\mathcal{I}$ is topological invariant,it does not depend on rapidity variables.

- Some Of The New Models

[Spiridonov 1011.3798]

[Kels 1504.07074]

[Gahramanov, Spiridonov 1505.00765]

Thank you for your attention!