Renormalization Group Analysis of Stochastic and Turbulent Systems

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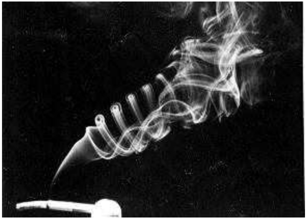
The plan is to give an overview of the area and then a little example of my own work.

In these problems we are concerning with three independent areas: hydrodynamics, statistical physics and high energy physics:

- Navier-Stokes equation describing the moving of the liquid;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

The problems under consideration are turbulent motion of the gas and liquid or turbulent advection of impirity fields.

Turbulence: pictures



Very complex trajectories of the particles, so huge difficulties in mathematical describing (Milleniun problem for 1 000 000$!).

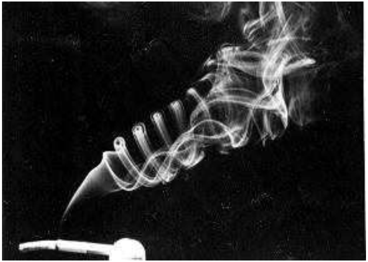
Turbulence

Incompressible version of the Navier-Stokes equation

$$\partial_t v_i + (v_k \partial_k) v_i = \nu \partial^2 v_i - \partial_i p$$

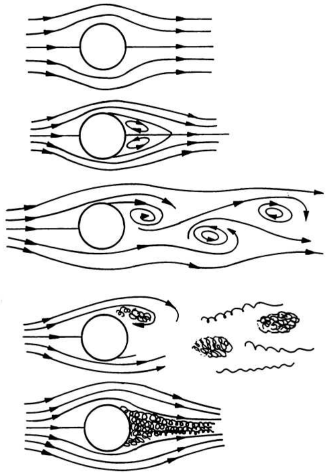
Relevant parameter is **Reynolds** number

$$\mathrm{Re} = \frac{L_0 V_0}{\nu} = \frac{\text{inertia}}{\text{dissipation}}$$



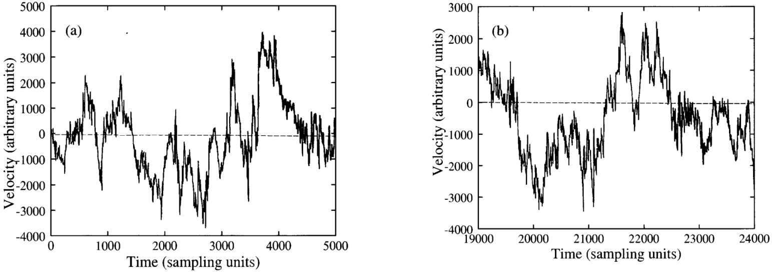
$\mathrm{Re} \to \infty$: **fully developed turbulence**, characterized by **statistical** restoration of symmetries (homogeneity in time and space, isotropy).

Turbulence: behavior of the system when $\mathrm{Re} \to \infty$



Probabilistic description of the turbulence

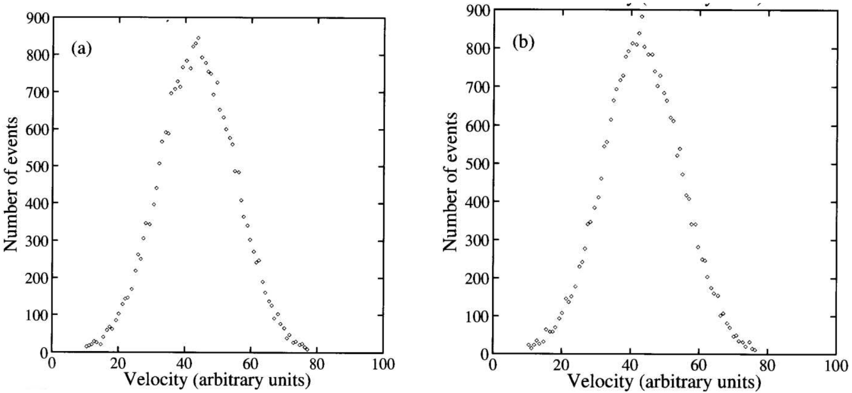
Fully developed turbulence is absolutely chaotic. The track from wind tunnel has the form



Properties:

- disorganization + structures with different scales;
- detailes of behavior are unpredictable;
- at the same time some properties are reproduced.

Histograms of the velocity field:



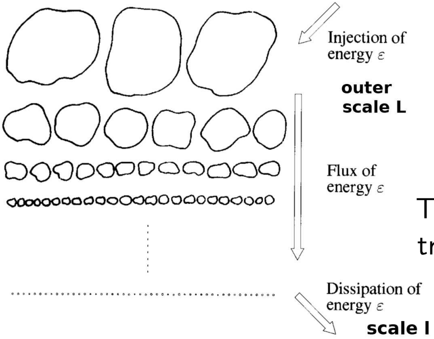
Almost equal!

Fully developed turbulence

Fully developed turbulence is characterized by

- velocity field $V_i(\mathbf{x}, t) = v_i(\mathbf{x}, t) + u_i(\mathbf{x}, t)$, where $u_i(\mathbf{x}, t)$ is laminar component, $v_i(\mathbf{x}, t)$ is small stochastic (unregular) component;
- statistical characteristics (objects of interest) are correlation and response functions (Green functions in quantum field theory);
- fully developed turbulence is similar for liquids and gases.

Fully developed turbulence: Richardson's cascade



Navier-Stokes equation:

$$\partial_t v_i + (v_k \partial_k) v_i = \nu \partial^2 v_i - \partial_i p$$

There are two non-local terms which transfer energy across the spectrum!

Fully developed turbulence

According to the Kolmogorov's hypotesis the key objects are

- W and L: the power of external energy and related with it large scale; for trophosphere $L \sim 5$ km.
- ν и l: viscosity and related with it small scale; for trophosphere $l \sim 1$ cm.

Fully developed turbulence: $\mathrm{Re} \gg 1 \Rightarrow L \gg l \Rightarrow$

we may deal with inertial range $l \ll r \ll L$.

Kolmogorov's theory "K41"

Phenomenological theory: consider structure functions

$$S_n(\mathbf{r}) = \left\langle \left[v_r(t,\mathbf{x}) - v_r(t,\mathbf{x}') \right]^n \right\rangle .$$

Kolmogorov's hypotesis No.1: in the region $r \ll L$ distribution depends on power of external energy W and **does not** depend on any detailes, in particular it does not depend on L.

Kolmogorov's hypotesis No.2: in the region $r \gg l$ distribution **does not** depends on viscosity ν and small scale l.

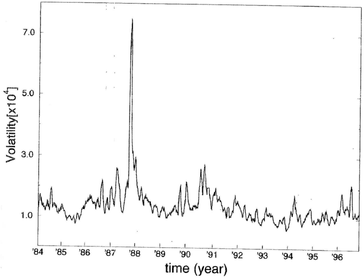
From two hypotesis it follows that in the inertial range $l \ll r \ll L$

$$S_p(\mathbf{r}) = C_p \, (Wr)^{\zeta_p}$$

with exacts exponents $\zeta_p = p/3$ and universal amplitudes (Kolmogorov's constants) C_p.

Intermittency and anomalous scaling

Intermittency: rare configurations may give nonzero contribution to the statistics.



This phenomena is connected with strong fluctuations of energy flux and leads to violation of classical theory K41:

$$S_p(\mathbf{r}) \cong (Wr)^{p/3} \, (r/L)^{\gamma_p}$$

with singular dependence of L and infinite set of independent exponents γ_p.

The goal is to calculate γ_p within a controllable scheme.

Stochastic formulation of the problem

Turbulence is modelled by an external force (random variable) f_i which mimics the input of energy into the system:

$$\partial_t v_i + (v_k \partial_k) v_i = \nu \partial^2 v_i - \partial_i p + f_i.$$

The force f_i supposed to be Gaussian with zero mean and correlation function

$$\langle f_i(t, \boldsymbol{x}) f_j(t', \boldsymbol{x}') = \frac{\delta(t-t')}{(2\pi)^d} \int_{k>m} d\boldsymbol{k} P_{ij}(\boldsymbol{k}) d(k) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')},$$

$$d(k) = g_0 \nu_0^3 k^{4-d-\varepsilon}.$$

Here $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector,
g_0 is a coupling constant,
ε is an ultraviolet regularizator (free parameter).

Quantum Field Theory

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t-local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2}\varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}}.$$

Quantum Field Theory

For the incompressible Navier-Stokes equation this means that

$$S(\varphi) = \frac{v_i' D_{ik} v_k'}{2} + v_i' \left[-\partial_t v_i - v_j \partial_j v_i + \nu_0 \partial^2 v_i \right].$$

What does it mean:

- statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- classical random field $\rightarrow$ quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator expansion, ...

Navier-Stokes equation for compressible fluid

Mine task is compressible fluid. Equation of motion for it is

$$\rho(\partial_t + v_k\partial_k)v_i = \nu_0(\delta_{ik}\partial^2 - \partial_i\partial_k)v_k + \mu_0\partial_i\partial_k v_k - \partial_i p + f_i,$$

for random force f_i we choose

$$\langle f_i(t,\boldsymbol{x})f_j(t',\boldsymbol{x}') = \frac{\delta(t-t')}{(2\pi)^d}\int_{k>m} d\boldsymbol{k}\, D_{ij}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')},$$

$$D_{ij}(\boldsymbol{k}) = g_{10}\nu_0^3 k^{4-d-y}\Big\{P_{ij}(\boldsymbol{k}) + \alpha Q_{ij}(\boldsymbol{k})\Big\} + g_{20}\nu_0^3.$$

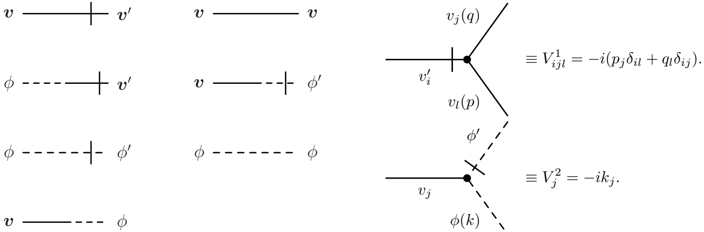
Here $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$ is the transverse projector;
$Q_{ij}(\boldsymbol{k}) = k_i k_j/k^2$ is the longitudinal projector;
g_{10} and g_{20} are two coupling constants;
y is an UV regularizator.

Action functional for compressible fluid

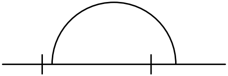
The actional functional has the form

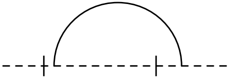
$$S(\varphi) = v_i'\Big\{-\partial_t v_i - v_j\partial_j v_i + \nu_0(\delta_{ik}\partial^2 - \partial_i\partial_k)v_k + u_0\nu_0\partial_i\partial_k v_k - \partial_i\phi\Big\}$$
$$+\phi'\Big\{-\partial_t\phi + v_j\partial_j\phi + v_0\nu_0\partial^2\phi - c_o^2(\partial_i v_i)\Big\} + \frac{v_i' D_{ik}^f v_k'}{2}.$$

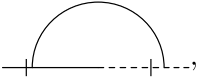
This model corresponds to a standard Feynman diagrammatic technique with two the triple vertices and seven bare propagators:

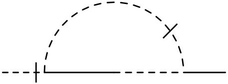
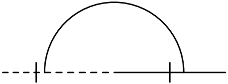


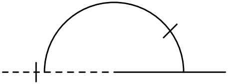
Renormzlization procedure: Feynman graphs

$$\Gamma_{v'v} = i\omega - (\delta_{ij}p^2 - p_i p_j)Z_1\nu - p_i p_j Z_2 u\nu +$$  ,

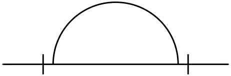
$$\Gamma_{\phi\phi'} = i\omega - p^2 Z_3 v\nu +$$  ,

$$\Gamma_{v'\phi} = -iZ_4 p_i +$$  ,

$$\Gamma_{\phi'v} = -iZ_5 p_i c^2 +$$  $+$  $+$

$+$  ,

$$\Gamma_{v'v'} = g_1\nu^3 p^{4-d-y} Z_6 \left\{ P_{ij}(\mathbf{p}) + \alpha Q_{ij}(\mathbf{p}) \right\} + g_2 \nu^3 \delta_{ij} Z_7 +$$

$+ \frac{1}{2}$  .

From remormalization group (RG) it follows, that in the case of one charge the asymptotic behaviour of the invariant charge $\bar{g}$ is

$$\bar{g}\,(s) \cong g^* + const \cdot s^{\omega},$$

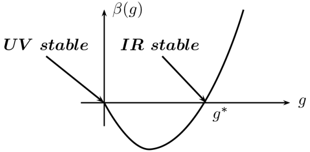
where $s = 1/\mu r$, μ is the renormalization mass, g^* is fixed point:

$$\beta_g(g^*) = 0.$$

IR asymptotic behaviour ($s \to 0 \Leftrightarrow r \to \infty$): $\omega = \beta'(g^*) > 0$.

In the case of many charges $\beta_i(g_j^*) = 0$ and $\Omega_{ik} = \partial\beta_i/\partial g_k$ at the point $g_j = g_j^*$ has to be positive.

Def $\beta(\bar{g})$: $\dfrac{d\bar{g}}{d\ln\mu} = -\beta(\bar{g})$;



$\beta(\bar{g})$ is responsible for evolution of the coupling constant $\bar{g}$.

Compressible fluid: Fixed points and asymptotic

Depending of the values of parameters the model possesses three different fixed points:

- Gaussian,

$$g_1^* = 0, \quad g_2^* = 0.$$

- Local regime,

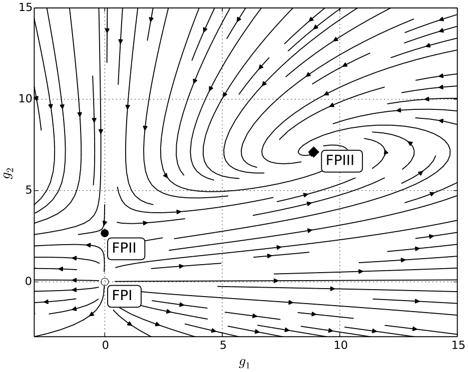
$$g_1^* = 0, \quad g_2^* = \frac{8\varepsilon}{3}.$$

- Non-local regime,

$$g_1^* = \frac{16y(2y-3\varepsilon)}{9[y(2+\alpha)-3\varepsilon]}, \quad g_2^* = \frac{16\alpha y^2}{9[y(2+\alpha)-3\varepsilon]}.$$

These fixed points define possible types of inertial range behavior, i.e., possible values of exponents γ_p.

Numerical simulation of the RG flow in the plane g_1 and g_2; three fixed points exist; stable (IR-attractive) fixed point which defines the inertial range asymptotic behavior (at some concrete values of parameters) is non-local point FPIII.



Conclusion

This techniques well works for any stochastic equations of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x),$$

where $U(x, \phi)$ is a t-local functional, so we may use them to explore different systems:

- turbulent advection of impurity fields;
- systems with anisotropy and helicity;
- magnetic hydrodynamics;
- Kardar-Parisi-Zhang or similar models describing erosion of landscapes;
- percolation model;
- etc.

Conclusion

In this type of problems methods of **quantum field theory** (functional integration, calculation of Feynman graphs and renormalization group) are applied to the **models of turbulent motion**.

- The goal is to justify the anomalous scaling, i.e., deviations from the classical K41 theory, using a controllable scheme.
- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- Feynman graphs are divergent. Renormalization group allows us to work with these objects and, moreover, provides the leading term of inertial range asymptotic behavior.
- The similar techniques can be applied to different models describing large variety of phenomena.

Thank you for your attention!