Three-body problem in GR

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In collaboration with: Enrico Trincherini, Francesco Serra, Konstantin Leyde
INTRODUCTION

SOME HISTORY OF THE THREE-BODY PROBLEM

- Until end of 18th century, it was still unclear if Newton’s law could explain the orbits of Solar System

- Lagrange and Laplace inaugurated methods of celestial mechanics

\[
m = m_1 + m_2 \\
M = m_1 + m_2 + m_3
\]
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  - Motion: 2 slowly varying ellipses

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\[
H = H_{\text{inner}} + H_{\text{outer}} + H_{\text{int}}
\]

\[
H_{\text{int}} = \frac{Gm_1 m_2 m_3}{2ma_3} \left( \frac{a}{a_3} \right)^2 (3 \cos^2 \psi - 1)
\]
**INTRODUCTION**

**A SMALL CLOUD IN A BLUE SKY**

- Careful analysis showed supplementary precession for Mercury...

<table>
<thead>
<tr>
<th>Amount (&quot; cy(^{-1}))</th>
<th>Cause</th>
</tr>
</thead>
<tbody>
<tr>
<td>531.63 ± 0.69</td>
<td>gravitational tugs from the other planets</td>
</tr>
<tr>
<td>0.025 4</td>
<td>oblateness of the Sun</td>
</tr>
<tr>
<td>42.98 ± 0.04</td>
<td>general relativity</td>
</tr>
<tr>
<td>574.64 ± 0.69</td>
<td>total</td>
</tr>
<tr>
<td>574.10 ± 0.65</td>
<td>observed</td>
</tr>
</tbody>
</table>
Detection of GW so far beautifully corresponds to two-body systems

\[ \Phi(f) = \phi_0 + 2\pi ft_0 + \sum_{k=0}^{7} \alpha_k f^{(k-5)/3} \]

\[ m_1, m_2, \chi_1, \chi_2 \]
**Gravitational Waves**

**A Big Jump in History**

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If we ever detect a new feature in data, we have (as 19th century astronomers) two possible explanations:

- Modification of GR
- Perturbation by a third body (this talk)
Three-body systems are also quite common!

- 90% of low-mass binaries are expected to belong to a ‘hierarchical’ triple system

- ‘Migration traps’ around SMBH at $R \sim 20 - 600R_{sch}$

 Tokovinin et al. 2006

 Bellovary et al. 2015
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  \[ R \sim 20 - 600R_{\text{sch}} \]  
  \text{Bellovary et al. 2015}
- ‘Migration traps’ around SMBH

Can we detect and measure parameters of the third body from waveform?

⇒ As Lagrange and Laplace, we have to formulate the 3-body problem in GR and solve it perturbatively
RELATIVISTIC THREE-BODY PROBLEM
A COMMON MISCONCEPTION

\[ \mathcal{H}_{3\text{-body}} = \mathcal{H}_{1\leftrightarrow 2} + \mathcal{H}_{1\leftrightarrow 3} + \mathcal{H}_{2\leftrightarrow 3} \]
A COMMON MISCONCEPTION

\[ \mathcal{H}_{\text{8-body}} = \mathcal{H}_{1\leftrightarrow2} + \mathcal{H}_{1\leftrightarrow3} + \mathcal{H}_{2\leftrightarrow3} \]

GR IS A NONLINEAR THEORY!

\[ g_{\mu\nu} \neq g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} \quad (\text{would not solve } R_{\mu\nu} = 0) \]
GR CORRECTIONS AT 1PN

EOM in post-Newtonian

\[
a_a = -\sum_{b \neq a} \frac{Gm_b x_{ab}}{r_{ab}^3} + \frac{1}{c^2} \sum_{b \neq a} \frac{Gm_b x_{ab}}{r_{ab}^3} \left[ 4 \frac{Gm_b}{r_{ab}} + \frac{5}{r_{ab}} \sum_{c \neq a,b} \frac{Gm_c}{r_{bc}} + 4 \sum_{c \neq a,b} \frac{Gm_c}{r_{ac}} - \frac{1}{2} \sum_{c \neq a,b} \frac{Gm_c}{r_{bc}^3} (x_{ab} \cdot x_{bc}) - v_a^2 + 4v_a \cdot v_b \right.

- 2v_b^2 + \frac{3}{2} (v_b \cdot n_{ab})^2 \right] - \frac{7}{2c^2} \sum_{b \neq a} \frac{Gm_b}{r_{ab}} \sum_{c \neq a,b} \frac{Gm_c x_{bc}}{r_{bc}^3} + \frac{1}{c^2} \sum_{b \neq a} \frac{Gm_b}{r_{ab}^3} x_{ab} \cdot (4v_a - 3v_b)(v_a - v_b), \tag{3.1}
\]


GR CORRECTIONS AT 1PN

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- Lack of physical intuition
GR corrections at 1PN

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- Numerical evolution over long timescales difficult
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(3.1)

- Lack of physical intuition
- Numerical evolution over long timescales difficult
- Issues in the radiative sector
3-body motion = 2-body with spin!
Effective Two-Body

\[ \mathcal{L}_{\text{full}} = \sum_{A=1}^{3} -m_A \sqrt{-g_{\mu\nu} v_A^\mu v_A^\nu} \]

PROPER TIME

\[ \mathcal{L}_{\text{EFT}} = -\mathcal{E} \sqrt{-g_{\mu\nu} V_{\text{CM}}^\mu V_{\text{CM}}^\nu} + \frac{1}{2} J_{\mu\nu} \Omega^{\mu\nu} - m_3 \sqrt{-g_{\mu\nu} v_3^\mu v_3^\nu} \]

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The equivalence principle fixes nearly everything!

\[ \mathcal{E} = m - \frac{G_N m \mu}{2a} , \quad J_{ij} = \epsilon_{ijk} J^k , \quad \Omega_{ij} = \epsilon_{ijk} \Omega^k , \quad J = \sqrt{G_N m a (1 - e^2)} \hat{j} , \quad \Omega = \hat{e} \times \hat{e} \]

\( \hat{e} \equiv \text{Unit Runge-Lenz vector} \)
A SYSTEMATIC EXPANSION

\[ \mathcal{L}_{\text{EFT}} = -\sqrt{-g_{\mu\nu}} V_{\text{CM}}^\mu V_{\text{CM}}^\nu + \frac{1}{2} J_{\mu\nu} \Omega^{\mu\nu} - m_3 \sqrt{-g_{\mu\nu}} v_3^\mu v_3^\nu \]

As in any EFT, the Lagrangian is organised with power-counting rules:

\[ v^2 \equiv \frac{Gm}{a} \quad \text{and} \quad \epsilon \equiv \frac{a}{a_3} \]
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To get the EOM for the point-particles, one should ‘integrate out’ the gravitational field

\[ \mathcal{H} = -\frac{Gm_1m_2}{2a} - 3m \frac{G^2m_1m_2}{a^2 \sqrt{1 - e^2}} \quad \text{Hamiltonian of inner orbit} \quad e^{-1}v^0 + e^{-1}v^2 \]
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\[ -\frac{Gmm_3}{2a_3} - 3M\frac{G^2mm_3}{a_3^2\sqrt{1 - e_3^2}} \quad \text{Hamiltonian of outer orbit} \quad \epsilon^0v^0 + \epsilon^1v^2 \]

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- Hamiltonian of inner orbit \( \varepsilon^{-1}v^0 + \varepsilon^{-1}v^2 \)
- Hamiltonian of outer orbit \( \varepsilon^0 v^0 + \varepsilon^1 v^2 \)
- Spin-orbit coupling \( \varepsilon^{3/2} v^2 \)
**A Systematic Expansion**

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\]

Hamiltonian of inner orbit \(\mathcal{E}^{-1}v_0 + \mathcal{E}^{-1}v^2\)

Hamiltonian of outer orbit \(\mathcal{E}^0v_0 + \mathcal{E}^1v^2\)

Spin-orbit coupling \(\mathcal{E}^{3/2}v^2\)

Quadrupolar coupling \(\mathcal{E}^2v_0 + \mathcal{E}^2v^2\)
APPLICATION #1

LONG-TERM EVOLUTION OF RELATIVISTIC 3-BODY SYSTEMS

\[ \mathcal{H} = \mathcal{H}_{\text{inner}} + \mathcal{H}_{\text{outer}} + \mathcal{H}_{\varepsilon^{3/2}v^2} + \mathcal{H}_{\varepsilon^2v^0} + \mathcal{H}_{\varepsilon^2v^2} + \ldots \]
APPLICATION #2

RELATIVISTIC DOPPLER EFFECT IN WAVEFORMS

- Longitudinal Doppler effect:  Randall Xianyu ‘18  Inayoshi et al. ‘17  Strokov et al. ’17...

\[ M \]

\[ V_\parallel = V \cdot n \]

\[ f_r = \frac{f_s}{1 + V_\parallel} \]
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- Transverse Doppler effect: break degeneracies

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AK, K. Leyde (In prep.)
**APPLICATION #2**

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- Transverse Doppler effect: break degeneracies

\[
f_r = \frac{f_s \sqrt{1 - V^2}}{1 + V_{||}}
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- Higher order effects in waveforms like spin-orbit coupling…

\[
M \\
\bullet \\
J \\
m = m_1 + m_2
\]

AK, K. Leyde (In prep.)
APPLICATION #3

NEW RESONANCES

- Resonances are a fascinating phenomenon of the 3-body problem
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NEW RESONANCES

- Resonances are a fascinating phenomenon of the 3-body problem

- When relativistic effects are included, there are other kinds of resonances

\[
p \, \dot{\omega} + q \, \sqrt{\frac{GM}{a^3}} = 0
\]

\[
p, q \in \mathbb{Z}
\]
APPLICATION #3

NEW RESONANCES

\[ a(t) = a_0 \left( 1 - \frac{t}{t_{RR}} \right)^{1/4} \]
Conclusions

- Very rich phenomenology in the Newtonian 3-body problem, even more in the relativistic one...

- EFT formulation suited to precision computations

- Future work: more precise waveforms for 3-body problem